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Surface Displacements due to a Steadily Moving Point Force

Closed-form expressions are obtained for the normal surface displacements due to a normal point force moving at constant speed over the surface of an elastic half-space. The Smirnov-Sobolev technique is used to reduce the problem to a linear superposition of two-dimensional stress and displacement fields.

1 Introduction

In this paper we shall investigate the normal surface displacement of the isotropic elastic half-space $z > 0$, due to a normal point force, F , which moves with constant speed, V , in the x -direction over the surface $z = 0$. This problem defines the Green's function for a class of elastodynamic contact problems for the half-space in which a frictionless rigid indenter moves at constant speed over the surface.

We assume that the force has been moving for a long time, so that the stress and displacement fields have achieved a steady state when viewed in a frame of reference that moves with the force. The more challenging transient problem in which the half-space is initially quiescent and the force moves for a finite time was investigated by Payton (1964), who gave results for the special case where Poisson's ratio $\nu = \frac{1}{4}$. A more convenient solution to the steady-state problem for general Poisson's ratio was given by Churilov (1977), who also extended the argument to give simple solutions to some elastodynamic contact problems (Churilov, 1978).

The steady-state problem is self-similar and we can therefore use equilibrium and dimensional arguments to demonstrate that the stresses must decay with R^{-2} and the displacements with R^{-1} , where R is the distance from the moving force (see, for example, Willis, 1967; Barber and Sturla, 1992). The problem therefore reduces to the determination of the ϕ -dependence of the normal surface displacements in a cylindrical polar coordinate system (r, ϕ, z) moving with the force.

Churilov's solution is based on his observation that the governing elastodynamic equations in the moving frame of reference have the same form as the elastostatic equations for a fictitious anisotropic material. The problem is thereby reduced to that of the point force acting on a half-space of this fictitious material, for which Churilov uses the method of Sveklo (1964), which is in turn based on a technique due to Smirnov and Sobolev (see for example Eringen and Suhubi, 1975, Section 8.8). However, it can be shown that Churilov's fictitious anisotropic material has physically reasonable (i.e., positive definite) elastic constants if and only if the speed V in the real problem is less than the shear wave speed c_2 .

In the present paper, we shall demonstrate that the Smirnov-Sobolev technique can be applied directly to the elastodynamic problem, without invoking the analogy to anisotropic elastostatics. This will permit us to obtain simple closed-form expressions for the normal surface displacements over the entire speed range and also points up some interesting relationships between the

present three-dimensional problem and the two-dimensional problem of a line load moving on the surface of a half-plane solved by Cole and Huth (1958).

2 Solution of the Problem

2.1 The Smirnov-Sobolev Technique. The basis of the Smirnov-Sobolev technique is to represent a fairly general three-dimensional field by the superposition of two-dimensional fields of differing orientation. We define a fixed Cartesian coordinate system x, y, z in which the plane $z = 0$ is the surface of the half-space $z > 0$. We then define a second system x_1, y_1, z rotated through angle θ about the z -axis, through the relations

$$x_1 = x \cos \theta + y \sin \theta; \quad (1)$$

$$y_1 = y \cos \theta - x \sin \theta. \quad (2)$$

We can envisage a two-dimensional solution in the rotated system in which a scalar field quantity $f(x_1, z)$ is independent of y_1 . A more general three-dimensional solution can then be obtained by superposing such functions with different values of θ . The most general such function can be written

$$g(x, y, z) = \int_{-\pi/2}^{\pi/2} f(x_1, z, \theta) d\theta, \quad (3)$$

where we remark that θ appears in the function $f(x_1, z, \theta)$ only as a parameter. In other words, f is any two-dimensional solution of the field equations expressed in the rotated system (x_1, y_1, z) . Notice that it is not necessary to include a full 2π range in θ , since a rotation of π maps the coordinate system into itself except for a sign change.

We emphasise the superpositional nature of this representation because it is then clear that the method can be applied to any linear system. In particular, it is not necessary to resort to Churilov's device of reducing the elastodynamic problem to an equivalent anisotropic elastostatic problem. Instead, we shall develop the solution by direct superposition of relatively simple two-dimensional elastodynamic results.

Equation (3) can also be written in terms of polar coordinates (r, ϕ, z) , where $x = r \cos \phi$, $y = r \sin \phi$. Using (1), we then have

$$g(r, \phi, z) = \int_{-\pi/2}^{\pi/2} f(r \cos(\theta - \phi), z, \theta) d\theta. \quad (4)$$

The special case where f is independent of the parameter θ and even in x_1 leads to an axisymmetric function g . To prove this, note that Eq. (4) then reduces to

$$g(r, \phi, z) = \int_{-\pi/2}^{\pi/2} f(r \cos(\theta - \phi), z) d\theta. \quad (5)$$

A change of variable $\psi = \theta - \phi$ and the evenness of the function f then require that this integral be the same as

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$$g(r, \phi, z) = \int_{-\pi/2}^{\pi/2} f(r \cos \psi, z) d\psi, \quad (6)$$

which is clearly independent of ϕ .

2.2 The Sveklo Transformation. The function $f(x_1, z, \theta)$ can be regarded as a transform of $g(r, \phi, z)$. For it to be useful, it is desirable that we be able to invert the transform—i.e., to determine the two-dimensional function f that will generate a given three-dimensional function g . Sveklo (1964) has shown that this inversion can be reduced to the solution of an Abel integral equation when the function g is axisymmetric. Consider the equation

$$g(r) = \int_{-\pi/2}^{\pi/2} f(r \cos \theta) d\theta. \quad (7)$$

Writing $s = r \cos \theta$, we then have

$$g(r) = 2 \int_0^r \frac{f(s) ds}{\sqrt{r^2 - s^2}}, \quad (8)$$

which is an Abel integral equation for f with solution

$$f(s) = \frac{1}{\pi} \frac{d}{ds} \int_0^s \frac{rg(r) dr}{\sqrt{s^2 - r^2}}. \quad (9)$$

The technique will generally be applied to boundary-value problems in which some quantity $g(r, \phi, 0) \equiv g(r)$ is defined to be axisymmetric on the surface $z = 0$. We note however that z appears only as a parameter in Eq. (5) and hence the same inversion would permit us to determine the two-dimensional function $f(x_1, z)$ from the axisymmetric function $g(r, z)$.

2.3 Uniform Loading Over the Circle. To avoid the singularity associated with the concentrated normal force, it is convenient to consider the case where the half-space is loaded by a uniform pressure

$$p(r) = \frac{F}{\pi a^2}; \quad 0 < r < a, \quad (10)$$

the rest of the surface, $r > a$, being traction-free. We can later recover the point force result by allowing the radius a of the loaded region to tend to zero. Replacing g in (9) by p and evaluating the integral, we find

$$f(x_1) = \frac{F}{\pi^2 a^2}; \quad |x_1| < a \quad (11)$$

$$= \frac{F}{\partial^2 a^2} \left(1 - \frac{|x_1|}{\sqrt{x_1^2 - a^2}} \right); \quad |x_1| > a. \quad (12)$$

The function $f(x_1)$ describes a two-dimensional distribution of pressure on the surface that is uniform in the strip $-a < x_1 < a$ and negative outside this strip. Readers preferring a physical interpretation of Sveklo's transformation might note that, if such a distribution of pressure is superposed under rotation about the z -axis, the resulting axisymmetric pressure distribution is that defined by Eq. (10). Thus, if we can find the surface displacement due to the two-dimensional pressure distribution (11), (12), a similar superposition will enable us to find the same quantity for the axisymmetric pressure distribution (10).

Notice that the final displacement field will only exhibit the same axisymmetry as the load if the displacement associated with the two-dimensional problem in the transformed domain is the same for all θ . This will clearly not be the case if the material is generally anisotropic, since the elastic constants will depend on orientation.

When the load (10) moves at constant speed V in the x -direction over the surface, the corresponding transformed problem involves a two-dimensional field defined by (11), (12) also moving at speed V in the x -direction. Denoting the velocity

vector $\mathbf{V} = V\mathbf{i}$, where $\mathbf{i}$ is the unit vector in the x -direction, we note that the component of $\mathbf{V}$ in the y_1 -direction causes no change in the pressure distribution with time, so the transformed problem is the same as if the distribution (11), (12) were to move at speed $V \cos \phi$ in the direction x_1 . We now proceed to the solution of this problem.

2.4 Solution of the Two-Dimensional Problem. The surface displacement due to the distribution (11), (12) moving at constant speed $V \cos \phi$ in direction x_1 could be obtained by convolution on the two-dimensional elastodynamic Green's function due to Cole and Huth (1958) and corrected by Rahman (1990) and Georgiadis and Barber (1993a,b). These authors showed that the normal surface displacement due to a line load P moving at constant speed V is

$$u_z(x, 0) = \frac{P}{\mu} [F_1(V) \log(r) + F_2(V)H(-x)] \quad (13)$$

(omitting an arbitrary rigid-body displacement), where $r = |x|$ is the distance from the moving load and

$$F_1(V) = \frac{M_2^2 \sqrt{1 - M_1^2}}{\pi R(V)}; \quad V < c_2$$

$$= \frac{M_2^2 (2 - M_2^2) \sqrt{1 - M_1^2}}{\pi [(2 - M_2^2)^4 + 16(1 - M_1^2)(M_2^2 - 1)]}; \quad c_2 < V < c_1$$

$$= 0; \quad V > c_1 \quad (14)$$

$$F_2(V) = 0; \quad V < c_2$$

$$= \frac{4M_2^2 (1 - M_1^2) \sqrt{M_2^2 - 1}}{(2 - M_2^2)^4 + 16(1 - M_1^2)(M_2^2 - 1)}; \quad c_2 < V < c_1$$

$$= \frac{M_2^2 \sqrt{M_1^2 - 1}}{(2 - M_2^2)^2 + 4\sqrt{(M_1^2 - 1)(M_2^2 - 1)}}; \quad V > c_1. \quad (15)$$

In these equations,

$$M_i = \frac{V}{c_i}; \quad i = 1, 2, \quad (16)$$

$c_1 = \sqrt{(\lambda + 2\mu)/\rho}$, $c_2 = \sqrt{\mu/\rho}$ are the dilatational and shear wave speeds, respectively, λ , μ are Lamé's constants, ρ is the density, and

$$R(V) = (2 - M_2^2)^2 - 4\sqrt{(1 - M_1^2)(1 - M_2^2)} \quad (17)$$

is the Rayleigh function, which passes through zero at the Rayleigh wave speed c_R .

Readers familiar with the fracture mechanics literature will immediately recognize Eq. (11), (12) as defining the traction distribution on the crack plane for a Griffith crack $-a < x_1 < a$ opened by a uniform pressure $F/(\pi^2 a^2)$ —see, for example, Barber (1992), Eq. (13.37). Thus, for the static case ($V = 0$), the normal displacement on the plane $z = 0$ is given by Eq. (13.41) of Barber (1992) as

$$u_z(x_1, 0) = \frac{F(1 - \nu)}{\pi^2 a^2 \mu} \sqrt{a^2 - x_1^2} \quad -a < x_1 < a \quad (18)$$

$$= 0 \quad |x_1| > a. \quad (19)$$

Furthermore, we notice from (13)–(15), that for $V < c_2$ the Green's function reduces to $u_z(x, 0) = [PF_1(V)/\mu] \log(r)$, which is identical with the static Green's function for a material with a reduced elastic modulus

$$\mu^* = \frac{\mu F_1(0)}{F_1(V)} = -\frac{\mu(1 - \nu)}{\pi F_1(V)}. \quad (20)$$

Notice that $F_1(V) < 0$ for $0 < V < c_R$ and also that $\mu^* \rightarrow 0$

as $V \rightarrow c_R$ because of the factor $R(V)$ in the denominator of $F_1(V)$.

The normal surface displacement due to the distribution (11), (12) moving at speed $V \cos \theta < c_2$ can therefore be obtained immediately from (18), (19), by replacing μ with the effective modulus corresponding to the speed $V \cos \theta$, with the result

$$u_z(x_1, 0) = -\frac{FF_1(V \cos \theta)}{\pi a^2 \mu} \sqrt{a^2 - x_1^2} \quad -a < x_1 < a \quad (21)$$

$$= 0 \quad |x_1| > a. \quad (22)$$

It is convenient at this stage to proceed to the limit in which the radius a of the loaded circle tends to zero, while the total force F remains constant. In this limit, it can be shown by evaluating the integral $\int_{-\infty}^{\infty} u_z(x_1, 0) dx_1$ that Eqs. (21), (22) define the delta function

$$u_z(x_1, 0) = -\frac{FF_1(V \cos \theta)}{2\mu} \delta(x_1). \quad (23)$$

Transonic and Supersonic Cases. When $V \cos \theta > c_2$, the above argument still applies to the logarithmic term in Eq. (13), but additional displacements are obtained from the second step-function term. These are conveniently found by convolution. We obtain

$$u_z(x_1, 0) = \frac{F_2(V \cos \theta)}{\mu} \int_{-\infty}^{\infty} f(s) H(s - x_1) ds. \quad (24)$$

In the range $x_1 > a$, this reduces to

$$\begin{aligned} u_z(x_1, 0) &= \frac{F_2(V \cos \theta)}{\mu} \int_{x_1}^{\infty} f(s) ds \\ &= -\frac{FF_2(V \cos \theta)}{\pi^2 a^2 \mu} [x_1 - \sqrt{x_1^2 - a^2}], \end{aligned} \quad (25)$$

after substituting for $f(s)$ from (11), (12) and performing the integrations.

Proceeding to the limit as $a \rightarrow 0$, we obtain

$$u_z(x_1, 0) = -\frac{FF_2(V \cos \theta)}{2\pi^2 \mu x_1}. \quad (26)$$

A similar procedure applied in the range $x_1 < -a$ yields the same result. Thus, the complete solution for the surface displacements in the two-dimensional problem is

$$u_z(x_1, 0) = -\frac{F}{2\mu} \left[F_1(V \cos \theta) \delta(x_1) + \frac{F_2(V \cos \theta)}{\pi^2 x_1} \right]. \quad (27)$$

2.5 Inversion of the Surface Displacement. The normal surface displacement due to the moving point load can now be found by substituting (27) into the integral superposition (3) or (4) and evaluating the resulting integrals.

The contribution from the first (delta function) term in Eq. (27) is easily evaluated as

$$\begin{aligned} u_z(r, \phi, 0) &= -\frac{F}{2\mu} \int_{-\pi/2}^{\pi/2} F_1(V \cos \theta) \delta(r \cos(\theta - \phi)) d\theta \\ &= -\frac{FF_1(V \sin \phi)}{2\mu r}. \end{aligned} \quad (28)$$

The inversion of the second term in Eq. (27) leads to a more complicated integral, which is evaluated in the Appendix.

3 Results

3.1 Subsonic Speeds, $0 < V < c_R$. Equation (28) defines the complete solution to the problem for $0 < V < c_2$, since in this range $F_2(V \cos \theta) = 0$ for all θ . It agrees with the results

of Churilov (1977), when recast in his notation. However, Eq. (28) draws out an interesting relation between the three-dimensional point force solution and the corresponding two-dimensional problem for the moving line load, summarized in Eq. (13)–(15). The same method could also be used to establish a similar relation between the point and line load problems for the elastostatic problem for a generally anisotropic material. This was proved by a different (and less straightforward) method by Barber and Sturla (1992). Notice that the surface displacement is proportional to r^{-1} , as we deduced from self-similarity considerations in the Introduction.

The function $F_1(V \sin \phi)$ increases with ϕ to a maximum at $\phi = \pi/2$, indicating that, for a given distance r from the load, the maximum normal surface displacement occurs on the line orthogonal to the direction of motion. As $V \rightarrow c_R$, the displacement along this line increases without limit, causing the half-space to mimic the elastostatic behavior of a very flexible material with stiffeners orthogonal to the direction of motion. We would therefore anticipate that the Hertzian contact area for all quadratic indenters would tend to an ellipse elongated in the direction of motion as the speed approaches the Rayleigh wave speed.

3.2 Speeds in the Range $c_R < V < c_2$. In this speed range, $F_2(V \cos \theta)$ remains zero for all θ , so the solution is still given by Eq. (28). However, there will now be two lines defined by

$$\phi = \pm \frac{\pi}{2} \pm \gamma; \quad \gamma = \cos^{-1}(c_R/V) \quad (29)$$

along which the normal surface displacement will be singular and two sectors $\pi/2 - \gamma < |\phi| < \pi/2 + \gamma$ in which it is negative—i.e., opposite in direction to the applied force.

For $V > c_R$, the solution is not unique, since the homogeneous problem of the unloaded half-space has nontrivial Rayleigh wave solutions that can be superposed to yield a fairly general standing wave disturbance.

As discussed in Section 2.3, a two-dimensional disturbance $f(x_1)$ moving at speed V in direction x is indistinguishable from the same disturbance moving at speed $V \cos \theta$ in direction x_1 , since the y_1 component of V involves no change in the field. Such a disturbance will be a solution of the homogeneous problem—i.e., it will correspond to a moving displacement field with no associated surface tractions—if we choose θ such that $V \cos \theta = c_R$. This in turn implies that $\theta = \pm \gamma$. It follows that the most general surface displacement that is invariant in the moving frame of reference in the speed range $c_R < V < c_2$ is

$$\begin{aligned} u_z(r, \phi, 0) &= -\frac{FF_1(V \sin \phi)}{2\mu r} \\ &+ h_1(r \cos(\phi - \gamma)) + h_2(r \cos(\phi + \gamma)), \end{aligned} \quad (30)$$

where h_1, h_2 are arbitrary functions.

For the steady-state problem, there is no reason for preferring any particular form (including zero) for the two arbitrary functions. However, if we conceive of the steady-state problem as the limit at large time of a transient problem in which an initially quiescent half-space is loaded by a point load that moves at constant speed V , we would expect to see Rayleigh wave disturbances in the sector behind the lines $\phi = \pm(\pi/2 + \gamma)$ but not ahead of the load. In order to achieve such a solution, the functions h_1, h_2 must be chosen to cancel the Rayleigh singularities along the forward facing lines $\phi = \pm(\pi/2 - \gamma)$, which in turn requires that $h_1(x) = h_2(x) \sim x^{-1}$ and hence that

$$\begin{aligned} h_1(r \cos(\phi - \gamma)) + h_2(r \cos(\phi + \gamma)) \\ = \frac{C \cos \phi}{r(m_1 - M_2^2 \sin^2 \phi)}, \end{aligned} \quad (31)$$

where C is a constant to be chosen to ensure boundedness at $\phi = \pm(\pi/2 - \gamma)$.

The singular term in $F_1(V \sin \phi)$ can be extracted using (A2) and the factorizations (A15), (A17) from the Appendix. Choosing C to cancel this singularity in (30), we obtain

$$C = \frac{FM_2\sqrt{1 - \Lambda m_1}[(2 - m_1)^2 + 4\sqrt{(1 - m_1)(1 - \Lambda m_1)}]}{2\pi\mu(m_3 - m_1)(m_1 - m_2)\sqrt{M_2^2 - m_1}} \quad (32)$$

leaving the solution

$$u_z(r, \phi, 0) = -\frac{F}{2\mu r} \left[F_1(V \sin \phi) - \frac{M_2 \cos \phi \sqrt{1 - \Lambda m_1}[(2 - m_1)^2 + 4\sqrt{(1 - m_1)(1 - \Lambda m_1)}]}{\pi(m_3 - m_1)(m_1 - m_2)(m_1 - M_2^2 \sin^2 \phi)\sqrt{M_2^2 - m_1}} \right] \quad (33)$$

for the moving load problem in the range $c_R < V < c_2$.

In this solution, the normal surface displacement is positive in the trailing Rayleigh sector, $(\pi/2) + \gamma < |\phi| < \pi$, but negative elsewhere including all points ahead of the moving load.

3.3 Transonic Speeds, $c_2 < V < c_1$. In this range, the surface displacement contains an additional contribution from F_2 , through the integral I_1 of the Appendix, since $M_2 > 1$ and hence (A14) defines a real angle α . We obtain

$$u_z(r, \phi, 0) = -\frac{F}{2\pi\mu r} \left\{ G_1(V \sin \phi)F_3(V \sin \phi) + G_2(V \sin \phi)F_4(V \sin \phi) + \left[\sum_{i=1}^3 \frac{A_i M_2 \cos \phi}{(M_2^2 \sin^2 \phi - m_i)} \sqrt{\frac{m_i - 1}{m_i - M_2^2}} \right] - G_1(V \sin \phi)F_3(V \sin \phi) \operatorname{sgn}(\cos \phi) \right\} + \frac{C \cos \phi}{r(m_1 - M_2^2 \sin^2 \phi)} \quad (34)$$

from (A1), (A12), (A22), where we have expanded the function F_1 using (A2) and also included the standing Rayleigh wave solution from (31). The identity $\operatorname{sgn}(x) \equiv 1 - 2H(-x)$ permits the two terms involving $G_1 F_3$ to be combined. As in the previous section, we choose C to cancel the singularity at $\phi = \pm(\pi/2 - \gamma)$, with the result

$$u_z(r, \phi, 0) = -\frac{F}{2\pi\mu r} \left\{ 2G_1(V \sin \phi)F_3(V \sin \phi)H(-\cos \phi) + G_2(V \sin \phi)F_4(V \sin \phi) + \left[\sum_{i=2}^3 \frac{A_i M_2 \cos \phi}{(M_2^2 \sin^2 \phi - m_i)} \sqrt{\frac{m_i - 1}{m_i - M_2^2}} \right] - \frac{M_2 \cos \phi \sqrt{1 - \Lambda m_1}(2 - m_1)^2}{(m_3 - m_1)(m_1 - m_2)(m_1 - M_2^2 \sin^2 \phi)\sqrt{M_2^2 - m_1}} \right\} \quad (35)$$

Notice that the term involving A_1 is of the same form as the standing Rayleigh wave and is therefore cancelled by the assignment of C . The resulting expression is continuous in the sector $0 < |\phi| < (\pi/2) + \alpha$. There is a slope discontinuity along the trailing shear wave fronts $\phi = \pm((\pi/2) + \alpha)$ and a Cauchy singularity, with a change of sign in the displacements at the Rayleigh wave fronts $\phi = \pm((\pi/2) + \gamma)$. The displacement is positive behind the Rayleigh wave front and usually negative in front. However, for certain tran-

sonic values of M_2 there is also a small range of positive displacements ahead of the Rayleigh wave front.

3.4 Supersonic Speeds, $V > c_1$. In this case, contributions are obtained from both I_1, I_2 of (A12), giving

$$u_z(r, \phi, 0) = -\frac{F}{2\pi\mu r} \left\{ G_1(V \sin \phi)F_3(V \sin \phi) + G_2(V \sin \phi)F_4(V \sin \phi) \right.$$

$$\left. + \left[\sum_{i=1}^3 \frac{A_i M_2 \cos \phi}{(M_2^2 \sin^2 \phi - m_i)} \sqrt{\frac{m_i - 1}{m_i - M_2^2}} \right] - G_1(V \sin \phi)F_3(V \sin \phi) \operatorname{sgn}(\cos \phi) + \left[\sum_{i=1}^3 \frac{B_i M_2 \cos \phi}{(M_2^2 \sin^2 \phi - m_i)} \sqrt{\frac{\Lambda m_i - 1}{m_i - M_2^2}} \right] - G_2(V \sin \phi)F_4(V \sin \phi) \operatorname{sgn}(\cos \phi) \right\} + \frac{C \cos \phi}{r(m_1 - M_2^2 \sin^2 \phi)} \quad (36)$$

The two series terms can be added to give

$$\sum_{i=1}^3 \frac{M_2 \cos \phi}{(M_2^2 \sin^2 \phi - m_i)} \left[A_i \sqrt{\frac{m_i - 1}{m_i - M_2^2}} + B_i \sqrt{\frac{\Lambda m_i - 1}{m_i - M_2^2}} \right]$$

For $i = 2, m_2 > 1$ and we can substitute for A_2, B_2 from (A16), (A18) to obtain

$$A_2 \sqrt{\frac{m_2 - 1}{m_2 - M_2^2}} + B_2 \sqrt{\frac{\Lambda m_2 - 1}{m_2 - M_2^2}} = \frac{\sqrt{\Lambda m_2 - 1}[(2 - m_2)^2 - 4\sqrt{(\Lambda m_2 - 1)(m_2 - 1)}]}{(m_1 - m_2)(m_2 - m_3)\sqrt{m_2 - M_2^2}} \quad (37)$$

By definition, we have that $D = 0$ when $M_2^2 = m_2$ and it follows on comparing (A9) and (37) that the factor $[(2 - m_2)^2 - 4\sqrt{(\Lambda m_2 - 1)(m_2 - 1)}]$ in (37) must be zero. Thus, there is no contribution to (36) from the terms with $i = 2$. A similar argument can be made for the terms with $i = 3$.

For $i = 1$, a nonzero result is obtained because the arguments of the square roots change sign, but the resulting term has the

Fig. 1 Displacement as a function of polar angle ϕ for $\nu = 0.2$, ($c_R = 0.911c_2$, $c_1 = 1.633c_2$), and various speeds. The dotted lines are asymptotes to the unbounded displacements at the Rayleigh wave front $\phi = \gamma + 90$ deg, which occurs at 106.5 deg, 130.6 deg, 145.3 deg, 158.6 deg for $M_2 = 0.95, 1.2, 1.6, 2.5$, respectively.

Fig. 2 Displacement as a function of polar angle ϕ for $\nu = 0.3$, ($c_R = 0.927c_2$, $c_1 = 1.871c_2$), and various speeds. The dotted lines are asymptotes to the unbounded displacements at the Rayleigh wave front $\phi = \gamma + 90$ deg, which occurs at 102.6 deg, 129.4 deg, 149.0 deg, 158.2 deg for $M_2 = 0.95, 1.2, 1.8, 2.5$, respectively.

same form as the standing Rayleigh wave and is eliminated when C is chosen so as to cancel the leading Rayleigh singularity.

The final solution therefore contains only the two step-function terms—i.e.,

$$u_z(r, \phi, 0) = -\frac{F}{\pi\mu r} [G_1(V \sin \phi)F_3(V \sin \phi)H(-\cos \phi) + G_2(V \sin \phi)F_4(V \sin \phi)H(-\cos \phi)] \\ = -\frac{FF_1(V \sin \phi)H(-\cos \phi)}{\mu r} \quad (38)$$

using Eq. (A2).

As we should expect, when the force moves at supersonic speed, the surface displacement is zero in front of the trailing dilatational wave fronts $\phi = \pm((\pi/2) + \beta)$, where $\cos \beta = M_1^{-1}$. A slope discontinuity occurs at the shear wave fronts and a Cauchy singularity with a sign change at the Rayleigh wave fronts.

It is interesting to note that this result can be predicted without performing the integrations on F_2 explicitly. Since we know that the total expression must be zero for $|\phi| < (\pi/2)$, it follows that the contribution in this sector from F_2 must be equal and opposite to that from F_1 and given by (28). However, the integral from F_1 is symmetrical about the line $\phi = \pm(\pi/2)$, whereas that from F_2 is antisymmetrical, so it follows that the contributions must add in the trailing sector, giving just twice that due to F_1 alone, as is confirmed by (38).

4 Conclusions

The above results define in closed form the normal surface displacements due to the steadily moving normal point force throughout the range of speeds. Collecting results for convenience we have

$$u_z(r, \phi, 0) = -\frac{FF_1(V \sin \phi)}{2\mu r}; \quad 0 < V < c_R \quad (39)$$

$$= -\frac{F}{2\mu r} \left[F_1(V \sin \phi) - \frac{M_2 \cos \phi \sqrt{1 - \Lambda m_1} [(2 - m_1)^2 + 4\sqrt{(1 - m_1)(1 - \Lambda m_1)}]}{\pi(m_3 - m_1)(m_1 - m_2)(m_1 - M_2^2 \sin^2 \phi) \sqrt{M_2^2 - m_1}} \right]; \quad c_R < V < c_2 \quad (40)$$

$$= -\frac{F}{2\pi\mu r} \left\{ 2G_1(V \sin \phi)F_3(V \sin \phi)H(-\cos \phi) + G_2(V \sin \phi)F_4(V \sin \phi) \right. \quad (41)$$

$$\left. + \left[\sum_{i=2}^3 \frac{A_i M_2 \cos \phi}{(M_2^2 \sin^2 \phi - m_i) \sqrt{m_i - M_2^2}} - \frac{M_2 \cos \phi \sqrt{1 - \Lambda m_1} (2 - m_1)^2}{(m_3 - m_1)(m_1 - m_2)(m_1 - M_2^2 \sin^2 \phi) \sqrt{M_2^2 - m_1}} \right] \right\}; \quad c_2 < V < c_1 \\ = -\frac{FF_1(V \sin \phi)H(-\cos \phi)}{\mu r}; \quad V > c_1. \quad (42)$$

Fig. 3 Displacement as a function of polar angle ϕ for $\nu = 0.45$, ($c_R = 0.949c_2$, $c_1 = 3.317c_2$), and various speeds. The dotted lines are asymptotes to the unbounded displacements at the Rayleigh wave front $\phi = \gamma + 90$ deg, which occurs at 101.9 deg, 127.7 deg, 151.7 deg, 161.6 deg, 167.8 deg for $M_2 = 0.97, 1.2, 2.0, 3.0, 4.5$, respectively.

These expressions are easily evaluated, making use of the results of Rahman and Barber (1995) for m_1, m_2, m_3 as required. Sample results for various speeds are given in Figs. 1 to 3, for $\nu = 0.2, 0.3, 0.45$, respectively. In each case, the displacement is positive (i.e., in the same direction as the load) in any region accessible to Rayleigh waves emanating from the moving load. Displacements in regions not so accessible are usually negative, except for a small region in certain transonic examples. Of course, the displacement is zero in the supersonic case for all points that are not accessible to dilatational waves emanating from the source.

The displacements are continuous functions of speed through the values $V = c_2, c_1$ —e.g., Eqs. (41) with $V = c_1 - \epsilon$ and (42) with $V = c_1 + \epsilon$ converge on the same limit as $\epsilon \rightarrow 0$. However, a more dramatic change occurs as the speed passes through the Rayleigh wave speed c_R . The displacement at $\phi = \pm(\pi/2)$ increases without limit as $V \rightarrow 0$ from below, but above $V = c_R$ a Cauchy singularity occurs at the trailing Rayleigh lines $\phi = \pm((\pi/2) + \gamma)$.

The results can be used as Green's functions to formulate three-dimensional elastodynamic contact problems for a frictionless indenter moving at constant speed over the half-space.

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APPENDIX

Substituting the second term in (27) into (4), we obtain the expression

$$u_z(x, y, 0) = -\frac{F}{2\pi^2\mu r} \int_{-\pi/2}^{\pi/2} \frac{F_2(V \cos \theta) d\theta}{\cos(\theta - \phi)}. \quad (A1)$$

In this Appendix, we shall evaluate this integral for the two cases $c_2 < V < c_1$ and $V > c_1$.

It is convenient to rationalize the expressions $F_1(V)$, $F_2(V)$ of Eqs. (14), (15). We find

$$F_1 = (G_1 F_3 + G_2 F_4)/\pi; \quad F_2 = (G_3 F_3 + G_4 F_4), \quad (A2)$$

where

$$G_1(V) = H(c_2 - V)\sqrt{1 - M_2^2} \quad (A3)$$

$$G_2(V) = H(c_1 - V)\sqrt{1 - M_1^2} \quad (A4)$$

$$G_3(V) = H(V - c_2)\sqrt{M_2^2 - 1} \quad (A5)$$

$$G_4(V) = H(V - c_1)\sqrt{M_1^2 - 1}. \quad (A6)$$

$H(\cdot)$ is the Heaviside step function,

$$F_3(V) = \frac{4M_2^2(1 - M_1^2)}{D} \quad (A7)$$

$$F_4(V) = \frac{M_2^2(2 - M_2^2)^2}{D} \quad (A8)$$

and

$$D = (2 - M_2^2)^4 - 16(1 - M_1^2)(1 - M_2^2). \quad (A9)$$

Note that D can be expanded as a polynomial in M_2^2 , giving

$$D = M_2^2[M_2^6 - 8M_2^4 + 8(3 - 2\Lambda)M_2^2 - 16(1 - \Lambda)] \\ = M_2^2(M_2^2 - m_1)(M_2^2 - m_2)(M_2^2 - m_3), \quad (A10)$$

where

$$\Lambda = \frac{M_1^2}{M_2^2} = \frac{c_2^2}{c_1^2} = \frac{(1 - 2\nu)}{2(1 - \nu)} \quad (A11)$$

and m_1, m_2, m_3 are the roots of the Rayleigh polynomial—i.e., the values of M_2^2 at which $D(M_2) = 0$ —and are given explicitly in closed form by Rahman and Barber (1995). Note that one of these roots (taken here to be m_1) is always real and equal to $(c_R/c_2)^2$, but the other two may be real or a pair of complex conjugates depending on the value of ν . We shall argue below for the case where the roots are all real, but the complex conjugate case can be treated by the same method and yields the same final result.

With this notation, the evaluation of Eq. (A1) reduces to that of the two integrals

$$I_1 = \int_{-\pi/2}^{\pi/2} \frac{G_3(V \cos \theta) F_3(V \cos \theta) d\theta}{\cos(\theta - \phi)}; \\ I_2 = \int_{-\pi/2}^{\pi/2} \frac{G_4(V \cos \theta) F_4(V \cos \theta) d\theta}{\cos(\theta - \phi)}. \quad (A12)$$

The derivations are tedious but routine and will only be briefly outlined here.

We first note that $G_3(V)$ is nonzero only in the range $V > c_2$ and hence I_1 can be rewritten

$$I_1 = \int_{-\alpha}^{\alpha} \frac{G_3(V \cos \theta) F_3(V \cos \theta) d\theta}{\cos(\theta - \phi)}, \quad (A13)$$

where

$$\cos \alpha = M_2^{-1}. \quad (A14)$$

Substituting (A9) into (A7), we can write $F_3(V)$ in the form

$$F_3(V) = \frac{4(1 - M_1^2)}{(M_2^2 - m_1)(M_2^2 - m_2)(M_2^2 - m_3)} \\ = \sum_{i=1}^3 \frac{A_i}{(M_2^2 - m_i)}, \quad (A15)$$

where the constants A_i are given by

$$A_1 = -\frac{4(1 - \Lambda m_1)}{(m_3 - m_1)(m_1 - m_2)}; \\ A_2 = -\frac{4(1 - \Lambda m_2)}{(m_1 - m_2)(m_2 - m_3)}; \\ A_3 = -\frac{4(1 - \Lambda m_3)}{(m_2 - m_3)(m_3 - m_1)}. \quad (A16)$$

The corresponding expansion of $F_4(V)$ is

$$F_4(V) = \frac{(2 - M_2^2)^2}{(M_2^2 - m_1)(M_2^2 - m_2)(M_2^2 - m_3)} \\ = \sum_{i=1}^3 \frac{B_i}{(M_2^2 - m_i)}, \quad (A17)$$

where

$$B_1 = -\frac{(2 - m_1)^2}{(m_3 - m_1)(m_1 - m_2)}; \\ B_2 = -\frac{(2 - m_2)^2}{(m_1 - m_2)(m_2 - m_3)}; \\ B_3 = -\frac{(2 - m_3)^2}{(m_2 - m_3)(m_3 - m_1)}. \quad (A18)$$

Using (A15), (A5) in (A13), we have

$$I_1 = \sum_{i=1}^3 A_i \int_{-\alpha}^{\alpha} \frac{\sqrt{M_2^2 \cos^2 \theta - 1} d\theta}{(M_2^2 \cos^2 \theta - m_i) \cos(\theta - \phi)}. \quad (\text{A19})$$

We now make the monotonic change of variable $\sin \theta = x$ and rationalize the resulting integrals, with the result

$$I_1 = \sum_{i=1}^3 A_i \int_{-\sqrt{1-1/M_2^2}}^{\sqrt{1-1/M_2^2}} \frac{\sqrt{M_2^2 - 1 - M_2^2 x^2} \left[\cos \phi - \frac{x}{\sqrt{1-x^2}} \sin \phi \right] dx}{(M_2^2 - m_i - M_2^2 x^2)(\cos^2 \phi - x^2)}. \quad (\text{A20})$$

The term multiplying $\sin \phi$ in the integrand is odd in x and hence makes no contribution to the integral. The remaining $\cos \phi$ term can be decomposed further into two partial fractions and evaluated using 3.613 of Gradshteyn and Ryzhik (1980).

We obtain

$$I_1 = \sum_{i=1}^3 \frac{A_i \pi}{(M_2^2 \sin^2 \phi - m_i)} \times \left[M_2 \cos \phi \sqrt{\frac{m_i - 1}{m_i - M_2^2}} - \operatorname{sgn}(\cos \phi) \sqrt{1 - M_2^2 \sin^2 \phi} \right], \quad (\text{A21})$$

where the square roots are to be taken as zero when their arguments are negative.

Some simplification in this expression can be made by noting that the second term does not contain m_i . The summation on this term can therefore be performed, using (A15), with the result

$$I_1 = \left[\sum_{i=1}^3 \frac{A_i \pi M_2 \cos \phi}{(M_2^2 \sin^2 \phi - m_i)} \sqrt{\frac{m_i - 1}{m_i - M_2^2}} \right] - \pi G_1(V \sin \phi) F_3(V \sin \phi) \operatorname{sgn}(\cos \phi). \quad (\text{A22})$$

An essentially similar procedure for the integral I_2 yields the result

$$I_2 = \left[\sum_{i=1}^3 \frac{B_i \pi M_2 \cos \phi}{(M_2^2 \sin^2 \phi - m_i)} \sqrt{\frac{\Lambda m_i - 1}{m_i - M_2^2}} \right] - \pi G_2(V \sin \phi) F_4(V \sin \phi) \operatorname{sgn}(\cos \phi). \quad (\text{A23})$$