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Existence and uniqueness of attractors in frictional systems with uncoupled tangential displacements and normal tractions

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ABSTRACT

We consider the class of two or three-dimensional discrete contact problems in which a set of contact nodes can make frictional contact with a corresponding set of rigid obstacles. Such a system might result from a finite element discretization of an elastic contact problem after the application of standard static reduction operations. The Coulomb friction law requires that the tractions at any point on the contact boundary must lie within or on the surface of a 'friction cone', but the exact position of any 'stuck' node (*i.e.*, a node where the tractions are strictly within the cone) depends on the initial conditions and/or the previous history of loading. If the long-term loading is periodic in time, we anticipate that the system will eventually approach a steady periodic cycle.

Here we prove that if the elastic system is 'uncoupled', meaning that changes in slip displacements alone have no effect on the instantaneous normal contact reactions, the time-varying terms in this steady cycle are independent of initial conditions. In particular, we establish the existence of a unique 'permanent stick zone' $\mathcal{T}$ comprising the set of all nodes that do not slip after some finite number of cycles. We also prove that the tractions and slip velocities at all nodes not contained in $\mathcal{T}$ approach unique periodic functions of time, whereas the (time-invariant) slip displacements in $\mathcal{T}$ may depend on initial conditions.

Typical examples of uncoupled systems include those where the contact surface is a plane of symmetry, or where the contacting bodies can be approximated locally as half spaces and Dundurs' mismatch parameter $\beta = 0$. An important consequence of these results is that systems of this kind will exhibit damping characteristics that are independent of initial conditions. Also, the energy dissipated at each slipping node in the steady state is independent of initial conditions, so wear patterns and the incidence of fretting fatigue failure should also be so independent.

1. Introduction

Engineering systems are frequently assembled from components that are held in conjunction by contact and friction forces. Examples include bolted joints (Berczynski and Gutowski, 2006; Law et al., 2006), blade root contacts in jet engines (Murthy et al., 2004) and shrink fit assemblies (Booker et al., 2004). These assemblies are generally designed with a sufficient normal contact force to prevent slip, but because of the deformability of the components, regions of microslip can develop. This is particularly problematic in situations where the loading is periodic (vibratory), since the corresponding periodic reversed microslip can then lead to a local fretting fatigue failure (Nowell et al., 2006).

Since fretting fatigue is an irreversible process, it is reasonable to expect that its severity will correlate with the rate of frictional energy dissipation at the interface (Davies et al., 2012). However, the problem of determining this dissipation is complicated by the fact that frictional slip is a history-dependent process, so that the instantaneous state generally depends on the loading history and/or the initial conditions. For example, in a bolted joint, the energy dissipation (which also correlates with the effective hysteretic damping associated with the joint) can change significantly if the joint is disassembled and then reassembled, presumably because the bolt-tightening protocol was inadvertently changed (Segalman, 2013). However, in some circumstances, notably in quasi-Hertzian contact between similar materials, fretting fatigue experimental data is found to be extremely repeatable (Hills

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et al., in press), suggesting that in this system at least, the frictional energy dissipation is independent of initial conditions or assembly protocol.

In some cases, frictional systems subjected to periodic loading 'shake down' to a steady state that involves no slip, and for many years it was believed that a frictional 'Melan's theorem (Melan, 1936) applied, that might be stated as *"If a set of time-independent tangential displacements at the interface can be identified such that the corresponding residual stresses when superposed on the time-varying stresses due to the applied loads cause the interface tractions to satisfy the conditions for frictional stick throughout the contact area at all times, then the system will eventually shake down to a state involving no slip, though not necessarily to the state so identified."* However, recently Klarbring et al. (2007) it has been established that such a result is rigorously true if and only if the frictional system is 'uncoupled', meaning that slip at the interface has no effect on the normal tractions. It was later conjectured (Barber, 2011) that this result might be a special case of a more general theorem to the effect that in the steady state, the energy dissipation in friction, or more specifically, the tractions and slip velocity distributions in regions that are slipping, should be independent of the initial conditions for uncoupled systems. In this paper, we shall give a rigorous proof of such a theorem. The frictional Melan theorem (Klarbring et al., 2007) can then be seen as the special case where the slip velocity in the steady state is everywhere zero.

Central to the proof is the concept of a *permanent stick zone* comprising the set $\mathcal{T}$ of all nodes (or points in the contact area) that do not slip during the steady state. We shall prove that this set is independent of the initial conditions. As in Klarbring et al. (2007), we shall identify a norm representing a measure of the difference between two distinct solutions of the evolutionary equations for the same time-varying loads, but different initial conditions. We shall show that this norm decreases monotonically to zero if $\mathcal{T} = \emptyset$ and to a (generally) non-zero positive constant if $\mathcal{T} \neq \emptyset$. In the latter case, we shall show that the slip velocities and tangential tractions outside $\mathcal{T}$ are unique, though the tangential tractions within $\mathcal{T}$ may depend on the initial conditions.

2. Quasistatic evolution problems and rate problems

2.1. Notation and definitions

We consider a frictional discrete linearly elastic system with two or three spatial dimensions and with N contact nodes. The subsequent presentation covers the three-dimensional case, but the modifications needed for two dimensions are trivial. Some of the nodes are fixed so that the stiffness matrix of the system is positive definite. The system is subjected to time-dependent external forces and we assume that the inertial forces can be neglected. The contact nodes may be in contact with flat obstacles under Coulomb friction. The quasi-static problem then is to determine the time-dependent displacements and reaction forces. In a standard way one may reduce the problem to the contact nodes and write

$$\boldsymbol{K}\boldsymbol{u} = \boldsymbol{f} + \boldsymbol{r} \tag{2.1}$$

where $\boldsymbol{r}$ is a vector of nodal reactions and the reduced stiffness matrix $\boldsymbol{K}$ acts on the displacement contact vector $\boldsymbol{u}$. The vector $\boldsymbol{f}$ is the reduced external force field ($-\boldsymbol{f}$ is the force vector that would be needed to keep the contact node vector equal to zero when applying the external forces). Let us introduce the notation

$$\boldsymbol{f} = -\begin{pmatrix} \boldsymbol{q}^w \\ \boldsymbol{p}^w \end{pmatrix}, \quad \boldsymbol{r} = \begin{pmatrix} \boldsymbol{q} \\ \boldsymbol{p} \end{pmatrix}, \quad \boldsymbol{u} = \begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{w} \end{pmatrix}$$

and

$$\boldsymbol{K} = \begin{pmatrix} \boldsymbol{A} & \boldsymbol{B}^t \\ \boldsymbol{B} & \boldsymbol{C} \end{pmatrix}.$$

Notice that $\boldsymbol{K}$ is the reduced stiffness matrix for the system and hence must be symmetric and positive definite. It follows that $\boldsymbol{A}$, $\boldsymbol{C}$ must also be symmetric and positive definite. We also make the assumption that the normals of the obstacles are directed so that the reaction forces have non-negative normal components, *i.e.*, such that $p_i \geqslant 0$ and so that $w_i \geqslant 0$.

We next make the very special assumption that there is no coupling between tangential and normal forces and displacements, *i.e.*, that the matrix $\boldsymbol{B} = \boldsymbol{0}$.

Then the vectors of normal and tangential nodal contact forces, $\boldsymbol{p}$, $\boldsymbol{q}$ respectively, can be written

$$\begin{Bmatrix} \boldsymbol{q} \\ \boldsymbol{p} \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{q}^w \\ \boldsymbol{p}^w \end{Bmatrix} + \begin{bmatrix} \boldsymbol{A} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{C} \end{bmatrix} \begin{Bmatrix} \boldsymbol{v} \\ \boldsymbol{w} \end{Bmatrix}, \tag{2.2}$$

where $\boldsymbol{w}$ is a vector of nodal normal displacements (separations) and $\boldsymbol{q}^w$, $\boldsymbol{p}^w$ are the contact tractions that would be produced if the nodes were all welded in contact with $\boldsymbol{v}_i = \boldsymbol{0}$, $w_i = 0$. This may be rewritten

$$\boldsymbol{q} = \boldsymbol{q}^w + \boldsymbol{A}\boldsymbol{v} \tag{2.3}$$
$$\boldsymbol{p} = \boldsymbol{p}^w + \boldsymbol{C}w$$

Here $\boldsymbol{p}^w(t)$ and $\boldsymbol{q}^w(t)$ are given external forces, $\boldsymbol{p}(t)$, $w(t)$, and $\boldsymbol{q}(t)$, $\boldsymbol{v}(t)$ is the solution of $\boldsymbol{QE}$.

We now have the following formulations of the so called *evolution* and *rate* problems.

The quasi-static evolution problem $\boldsymbol{QE}$:

Assume that we are given an initial state $\boldsymbol{u}(0)$, $\boldsymbol{f}(0)$, $\boldsymbol{r}(0)$ satisfying $\boldsymbol{K}\boldsymbol{u}(0) = \boldsymbol{f}(0) + \boldsymbol{r}(0)$ and

$$w_i(0) \geqslant 0, \quad p_i(0) \geqslant 0, \quad w_i(0)p_i(0) = 0 \tag{2.4}$$

$$\text{if} \quad w_i(0) = 0 \quad \text{then} \quad 0 \leqslant |\, \boldsymbol{q}_i(0) \,| \leqslant \mu_i p_i(0) \qquad \text{(contact)} \tag{2.5}$$

$$\text{if} \quad w_i(0) > 0 \quad \text{then} \quad \boldsymbol{q}_i(0) = \boldsymbol{0}, \quad \text{and}$$
$$p_i(0) = 0 \quad \text{(non-contact)} \tag{2.6}$$

Then, for a given external force field $\boldsymbol{f}(t)$ for $t \geqslant 0$, find $\boldsymbol{u}(t)$ and $\boldsymbol{r}(t)$ such that

$$w_i(t) \geqslant 0, \quad p_i(t) \geqslant 0, \quad w_i(t)p_i(t) = 0 \tag{2.7}$$

$$\text{if} \quad w_i(t) = 0 \quad \text{then} \quad 0 \leqslant |\, \boldsymbol{q}_i(t) \,| \leqslant \mu_i p_i(t) \qquad \text{(contact)} \tag{2.8}$$

$$\text{if} \quad w_i(t) > 0 \quad \text{then} \quad \boldsymbol{q}_i(t) = \boldsymbol{0}, \quad \text{and}$$
$$p_i(t) = 0 \quad \text{(non-contact)} \tag{2.9}$$

and so that

$$\text{if } w_i(t) = 0 \quad \text{and} \quad 0 < |\, \boldsymbol{q}(t) \,| = \mu_i p_i(t), \quad \text{then} \quad \dot{\boldsymbol{v}}_i(t) = -\lambda_i \boldsymbol{q}_i(t)$$
$$\text{where} \tag{2.10}$$

$$\lambda_i \geqslant 0, \tag{2.11}$$

$$\frac{(\dot{\boldsymbol{q}}_i(t), \boldsymbol{q}_i(t))}{|\, \boldsymbol{q}_i(t) \,|} - \mu_i \dot{p}_i(t) \geqslant 0 \tag{2.12}$$

with equality in at least one of (2.11) and (2.12) and

$$\text{if } w_i(t) = 0 \quad \text{and} \quad 0 \leqslant |\, \boldsymbol{q}_i(t) \,| < \mu_i p_i(t), \quad \text{then} \quad \dot{\boldsymbol{v}}_i(t) = \boldsymbol{0} \tag{2.13}$$

The quasi-static rate problem, $\boldsymbol{QR}$:

With fixed t and for given $\boldsymbol{f}(t)$, $\boldsymbol{u}(t)$ and $\boldsymbol{r}(t)$ satisfying (2.1), (2.7)–(2.9) and for a given derivative $\dot{\boldsymbol{f}}(t)$, find $\dot{\boldsymbol{u}}(t)$ and $\dot{\boldsymbol{r}}(t)$ such that $\boldsymbol{K}\dot{\boldsymbol{u}}(t) = \dot{\boldsymbol{f}}(t) + \dot{\boldsymbol{r}}(t)$, and (2.10)–(2.13) are satisfied.

2.2. Decoupling of the evolution problem

Consider the following subproblem: For a given function $\boldsymbol{p}^w(t)$, find $\boldsymbol{p}(t),\ \boldsymbol{w}(t)$ such that $\boldsymbol{p}(t) = \boldsymbol{p}^w(t) + \boldsymbol{C}\boldsymbol{w}(t)$ and so that (2.7) is satisfied.

For each t this problem has the unique solution

$$\boldsymbol{w}(t) = \operatorname{argmin}_{\boldsymbol{w}\leqslant \boldsymbol{0}}((\boldsymbol{C}\boldsymbol{w}, \boldsymbol{w})/2 + (\boldsymbol{p}^w(t), \boldsymbol{w})), \quad \boldsymbol{p}(t) = \boldsymbol{p}^w(t) + \boldsymbol{C}\boldsymbol{w}(t).$$

The solution exists and is unique since $\boldsymbol{C}$ is positive definite and the domain for $\boldsymbol{w}$ is closed and convex. Consequently, the functions $\boldsymbol{p}(t)$ and $\boldsymbol{w}(t)$ are known a priori in the evolution and rate problems. Therefore the evolution problem reduces to the following. For some initial state $\boldsymbol{v}(0),\ \boldsymbol{q}(0)$ satisfying (2.4)–(2.6) and for given functions $\boldsymbol{p}(t)$ and $\boldsymbol{w}(t)$, find $\boldsymbol{q}(t),\ \boldsymbol{v}(t)$ such that the conditions (2.8)–(2.13) are satisfied.

We may note that this reduced rate problem is for every t a Linear complementarity problem, with a system matrix independent of the coefficients of frictions. Further this matrix is positive definite and consequently a so called P-matrix, We conclude that the rate problem is unique solvable for every t and that we have an a priori estimate $|\dot{\boldsymbol{q}}(t)| \leqslant C\,|\dot{\boldsymbol{f}}(t)|$, see Cottle et al. (1992) and Klarbring (1999). Using this one may also prove that the evolution problem has at least one solution, see Andersson (1999). More precisely, for a given external force field $\boldsymbol{f}(t)$, with $t > 0$ and $\dot{\boldsymbol{f}} \in L^1_{loc}(0,\infty)$ there is a unique solution $\dot{\boldsymbol{q}}$ of the reduced quasi-static evolution problem, valid for almost all t and with $\dot{\boldsymbol{q}} \in L^1_{loc}(0,\infty)$ and $|\dot{\boldsymbol{q}}(t)| \leqslant C\,|\dot{\boldsymbol{f}}(t)|$. In the next section we will show that the solution of the evolution problem is indeed unique.

3. Attractors for the system

Suppose that $\boldsymbol{q}^1(t),\ \boldsymbol{v}^1(t)$ and $\boldsymbol{q}^2(t),\ \boldsymbol{v}^2(t)$ are two solutions corresponding to initial conditions $\boldsymbol{v}^1(0),\ \boldsymbol{v}^2(0)$ respectively. We define the difference between these solutions such that $\boldsymbol{q}(t) = \boldsymbol{q}^2(t) - \boldsymbol{q}^1(t)$ and $\boldsymbol{v}(t) = \boldsymbol{v}^2(t) - \boldsymbol{v}^1(t)$ and we also define the norm of this difference as

$$\|\boldsymbol{v}\|^2 \equiv (\boldsymbol{q}(t), \boldsymbol{v}(t)) = (\boldsymbol{A}\boldsymbol{v}(t), \boldsymbol{v}(t)) \tag{3.1}$$

which is related to the norm used in the Melan theorem proof (Klarbring et al., 2007). We then have the following key lemma.

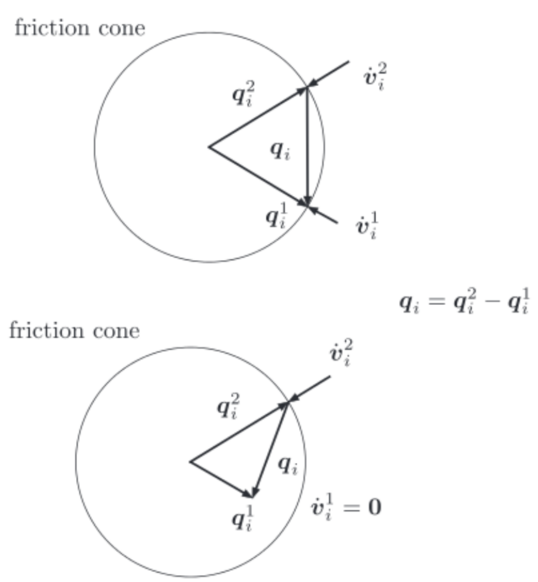

Fig. 1. The product $(\boldsymbol{q}_i^2(t) - \boldsymbol{q}_i^1(t),\ \dot{\boldsymbol{v}}_i^2(t) - \dot{\boldsymbol{v}}_i^1(t))$.

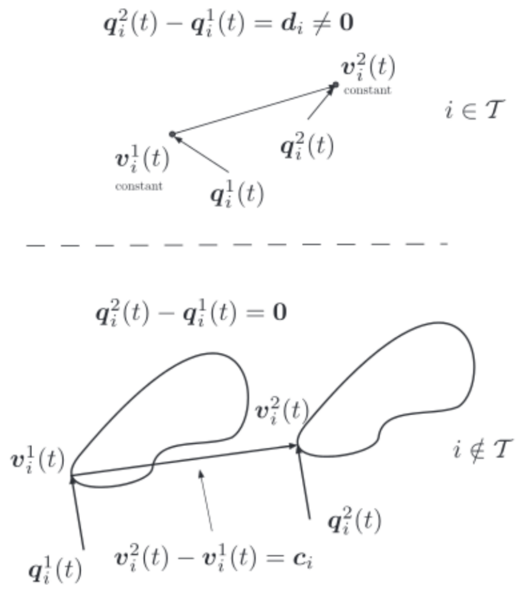

Fig. 2. Orbits $\boldsymbol{q}_i^{1,2},\ \boldsymbol{v}_i^{1,2}$.

Lemma 3.1. *With the above notation, we have for all i,*

$$(\boldsymbol{q}_i(t), \dot{\boldsymbol{v}}_i(t)) \leqslant 0 \tag{3.2}$$

for almost all t, with equality only if $\dot{\boldsymbol{v}}_i^1(t) = \dot{\boldsymbol{v}}_i^2(t) = 0$ *or* $\boldsymbol{q}_i(t) = 0$ *and* $|\boldsymbol{q}_i^1(t)| = |\boldsymbol{q}_i^2(t)| = -\mu_i p_i^w(t)$. *In particular, since* $\boldsymbol{q}(t) = \boldsymbol{A}\boldsymbol{v}(t)$, *and* $(\boldsymbol{A}\boldsymbol{v}(t), \dot{\boldsymbol{v}}(t)) = \sum_i(\boldsymbol{q}_i(t), \dot{\boldsymbol{v}}_i(t))$ *and* $\frac{\partial}{\partial t}\|\boldsymbol{v}(t)\|^2 = 2(\boldsymbol{A}\boldsymbol{v}(t), \dot{\boldsymbol{v}}(t))$, *we conclude that the energy norm* $\|\boldsymbol{v}(t)\|$ *is a decreasing function of t.*

The proof follows by simple geometric considerations, see Fig. 1. In particular, since $\dot{\boldsymbol{v}}_i^1$ must be directed into the friction cone at $\boldsymbol{q}_i^1$ and $\dot{\boldsymbol{v}}_i^2$ must be directed into the friction cone at $\boldsymbol{q}_i^2$, we have $(\boldsymbol{q}_i, \dot{\boldsymbol{v}}_i^1) \geqslant 0$ and $(\boldsymbol{q}_i, \dot{\boldsymbol{v}}_i^2) \leqslant 0$ and hence $(\boldsymbol{q}_i(t), \dot{\boldsymbol{v}}_i(t)) \leqslant 0$.

We next have the following result, see also Fig. 2.

Lemma 3.2. *Assume that we are given some external loading, and that we have two orbits* $\boldsymbol{v}^1(t),\ \boldsymbol{q}^1(t)$ *and* $\boldsymbol{v}^2(t),\ \boldsymbol{q}^2(t)$, *which are such that for the differences* $\boldsymbol{v}(t) = \boldsymbol{v}^2(t) - \boldsymbol{v}^1(t)$, $\boldsymbol{q}(t) = \boldsymbol{q}^2(t) - \boldsymbol{q}^1(t)$ *the energy norm* $\|\boldsymbol{v}(t)\| = (\boldsymbol{q}(t), \boldsymbol{v}(t))^{1/2}$ *is constant. Then* $\boldsymbol{q}^2(t), \boldsymbol{q}^1(t)$ *and* $\boldsymbol{v}^2(t) - \boldsymbol{v}^1(t)$ *are constant functions of t. Further, for some set* $i \in \mathcal{T}^{1,2}$ $\boldsymbol{v}_i^1(t)$ *and* $\boldsymbol{v}_i^2(t)$ *are constant if and only if* $i \in \mathcal{T}^{1,2}$.

Proof. By the existence theorem the derivatives $\dot{\boldsymbol{v}}(t)$ and $\dot{\boldsymbol{q}}(t) = \boldsymbol{A}\dot{\boldsymbol{v}}(t)$ exist for all $t \in H$ where $H \subset (0,\infty)$ and $m(((p,\infty)\setminus H) = 0$, m denoting the Lebesgue measure. Further for all $t \in H$ we have $\frac{d}{dt}(\boldsymbol{A}\boldsymbol{v}(t), \boldsymbol{v}(t)) = 2(\boldsymbol{A}\boldsymbol{v}(t), \dot{\boldsymbol{v}}(t)) = 2\sum_i(\boldsymbol{q}_i(t), \dot{\boldsymbol{v}}(t)) = 0$, and by Lemma 3.1 we conclude that for all $t \in H$ and all $i, \boldsymbol{q}_i(t) = 0$ or $\dot{\boldsymbol{v}}_i^2(t) = \dot{\boldsymbol{v}}_i^1(t) = \boldsymbol{0}$.

Next, let $E_i = \{t \in H : \boldsymbol{q}_i(t) = \boldsymbol{0}\}$ and $F_i = \{t \in H : \boldsymbol{q}_i(t) \neq \boldsymbol{0}\}$. We claim that $\dot{\boldsymbol{q}}_i(t) = \boldsymbol{0}$ almost everywhere in E_i. In fact, the set of isolated points in E_i is countable, and for the subset $E_i \subset E_i$ of non-isolated points we have $m(E_i \setminus E_i') = 0$. Since the points in E_i' are non-isolated with respect to E_i it follows easily that $\dot{\boldsymbol{q}}(t) = 0$ for all $t \in E_i'$. Letting $H_i' = F_i \cup E_i'$ and $H' = \bigcap_i H_i'$ we also have $m((0,\infty)\setminus H') = 0$.

Now, for $t \in H'$, let $I(t) = \{i : \dot{\boldsymbol{q}}_i(t) = 0\}$ and $J(t) = \{i : \dot{\boldsymbol{q}}_i(t) \neq 0\}$. Then we have,

$$\dot{\boldsymbol{q}}_i(t) = \boldsymbol{0}, \quad \text{if} \quad i \in I(t)$$
$$\sum_{k\in J(t)} L_{jk}\dot{\boldsymbol{q}}_k(t) = \dot{\boldsymbol{v}}_j(t) = \boldsymbol{0} \quad \text{for} \quad j \in J(t)$$

where we have introduced the matrix $\boldsymbol{L} = \boldsymbol{A}^{-1}$. This is a homogeneous linear system of equations for the unknowns

$\dot{\boldsymbol{q}}_i(t),\ 1 \leqslant i \leqslant N$, and since the system matrix is positive definite we conclude that $\dot{\boldsymbol{q}}(t) = 0$, for all $t \in H'$. Since $\dot{\boldsymbol{v}}(t) = \boldsymbol{L}\dot{\boldsymbol{q}}(t)$ we also conclude that $\dot{\boldsymbol{v}}(t) = 0$ for all $t \in H'$. Consequently $\boldsymbol{q}(t)$ and $\boldsymbol{v}(t)$ are constant. Further, by Lemma 3.1, if $\boldsymbol{q}_i = \boldsymbol{q}_i^2 - \boldsymbol{q}_i^1 \neq \boldsymbol{0}$ then $\boldsymbol{v}_i^2(t)$ and $\boldsymbol{v}_i^1(t)$ are constant.

We have proved that if $\boldsymbol{v}^2(t)$ and $\boldsymbol{v}^1(t)$ are two orbits such that we have equality in (3.2) for almost all t and all i, then there is a set of indices $\mathcal{T}^{1,2}$ with the property that $\boldsymbol{v}_k^2(t)$ and $\boldsymbol{v}_k^1(t)$ are constant if and only if $k \in \mathcal{T}^{1,2}$. h

If, in particular, we have a T-periodic loading, Lemma 3.2 may be applied to periodic orbits. The following corollary follows immediately.

Corollary 3.1. *If, in Lemma 3.2, one of the orbits, say* $\boldsymbol{v}^1(t)$ *is T-periodic, then so is* $\boldsymbol{v}^2(t)$.

Proof. We have $\boldsymbol{v}^1(t) - \boldsymbol{v}^2(t) = \boldsymbol{c}$ where c is a constant vector. This proves the statement. h

Further we have the following theorem.

Theorem 3.1. *If the external loading is T-periodic, then there exists a set of indices* $\mathcal{T} \subset \{1, 2, \ldots, N\}$ *with the following property. For every T-periodic orbit* $\boldsymbol{v}^p(t)$ *we have* $\boldsymbol{v}_i^p(t)$ *constant if and only if* $i \in \mathcal{T}$. *Moreover, for the case that* $\mathcal{T} = \emptyset$ *there exists at most one such orbit* $\boldsymbol{v}^p(t)$.

Proof. By the proof of Lemma 3.2 it first follows that if $\boldsymbol{v}^2(t)$ and $\boldsymbol{v}^1(t)$ are two T-periodic orbits then there is a set of indices $\mathcal{T}^{1,2}$ with the property that $\boldsymbol{v}_k^2(t)$ and $\boldsymbol{v}_k^1(t)$ are non-constant if and only if $k \in \mathcal{T}^{1,2}$.

Repeating the argument with for example $\boldsymbol{v}_k^1(t)$ exchanged for a periodic orbit $\boldsymbol{v}_k^3(t)$, we conclude that $\mathcal{T}^{1,2} = \mathcal{T}^{3,2}$, *i.e.*, that there exists a set $\mathcal{T}$ so that for every periodic orbit $\boldsymbol{v}^p(t)$ we have $\boldsymbol{v}_k^p(t)$ constant if and only if $k \in \mathcal{T}$.

Finally, if $\boldsymbol{v}^2(t)$ and $\boldsymbol{v}^1(t)$ are two periodic orbits such that $\boldsymbol{v}_k^2(t)$ and $\boldsymbol{v}_k^1(t)$ are non-constant for all k then $\boldsymbol{q}_k = \boldsymbol{q}_k^2 - \boldsymbol{q}_k^1 = \boldsymbol{0}$ for all k, and therefore $\boldsymbol{v}_k(t) = \boldsymbol{v}_k^2(t) - \boldsymbol{v}_k^1(t) = \boldsymbol{0}$ for all k, *i.e.*, the solutions are identical. h

The node set $\mathcal{T}$ in Theorem 3.1 defines the permanent stick zone, as defined in the Introduction. The theorem establishes that this zone is the same for all initial conditions. It also establishes that the steady state is unique if the set $\mathcal{T}$ is empty, *i.e.*, if there is no permanent stick zone.

We also have the following theorem.

Theorem 3.2. *For a T-periodic loading the class of periodic orbits is non-empty. Further it is convex, in the sense that if* $\boldsymbol{v}^1(t)$ *and* $\boldsymbol{v}^2(t)$ *are T-periodic orbits, then so is* $(1-s)\boldsymbol{v}^1(t) + s\boldsymbol{v}^2(t)$, *for* $0 \leqslant s \leqslant 1$. *It is also closed in the sense that if* $\{\boldsymbol{v}^n(t\}_{n=1}^{\infty}$ *is a sequence of T-periodic orbits such that their initial states converge,* $\boldsymbol{v}^n(0) \to \boldsymbol{w}$, *then* $\boldsymbol{v}^n(t) \to \boldsymbol{v}^{\infty}(t)$ *uniformly in t, where* $\boldsymbol{v}^{\infty}(t)$ *is a periodic orbit, with* $\boldsymbol{v}^{\infty}(0) = \boldsymbol{w}$.

Proof. For a given external T-periodic force field $\boldsymbol{f}(t)$ we consider the set $D(0)$ of all possible initial states, satisfying $\boldsymbol{Ku}(0) = \boldsymbol{f}(0) + \boldsymbol{r}(0)$ and the equilibrium and friction conditions (2.4)–(2.6). It is easy to verify that the set $D(0)$ is a closed and convex subset of R^{3N}. Now the solution of the quasi static evolution problem defines a mapping

$$D(0) \ni (\boldsymbol{u}(0), \boldsymbol{r}(0)) \#\ (\boldsymbol{u}(T), \boldsymbol{r}(T)) \in D(0)$$

By Lemma 3.1 this mapping is continuous and therefore Brouwer's fixed point theorem implies that there exists at least one fixed point. Every such fixed points defines a periodic solution of the evolution problem and we have proved the first statement of the theorem. The other statements follow from Theorem 3.1 and Corollary 3.1. The details are omitted. h

We finally give the following theorem stating that for a periodic loading, every orbit is attracted by a periodic one.

Theorem 3.3. *If the external loading is T-periodic, then every orbit is attracted by some periodic orbit. If, in particular, there exist a periodic orbit* $\boldsymbol{v}^p(t)$ *with* $\boldsymbol{v}_k^p(t)$ *non-constant for all nodes k* ($\mathcal{T} = \emptyset$) *then all orbits are attracted by this periodic orbit* $\boldsymbol{v}^p(t)$.

Proof. Assume that $\boldsymbol{v}^1(t),\ \boldsymbol{q}^1(t),\ \boldsymbol{p}(t)$ is some periodic orbit (by Theorem 3.2 there exist at least one) and that $\boldsymbol{v}(t),\ \boldsymbol{q}(t),\ \boldsymbol{p}(t)$ is an arbitrary given orbit (for the same loading). Forming $v(t) = \boldsymbol{v}(t) - \boldsymbol{v}^1(t),\ \|v(t)\|$ is, by Lemma 3.1 a decreasing function of t. Then either $\|v(t)\| \searrow 0$ or $\|v(t)\| \searrow c > 0$.

In the first case we conclude that $v(t) = \boldsymbol{v}(t) - \boldsymbol{v}^1(t) \to \boldsymbol{0}$ as $t \to \infty$, *i.e.*, that $\boldsymbol{v}^1(t)$ attracts $\boldsymbol{v}(t)$.

In the second case we argue as follows. Since $\boldsymbol{v}(t)$ is bounded, there exists an increasing sequence of integers $\{n_\nu\}_{\nu=1}^{\infty}$ such that $\boldsymbol{v}(n_\nu T),\ \boldsymbol{p}(n_\nu T)$ and $\boldsymbol{q}(n_\nu T)$ converge, $\boldsymbol{v}(n_\nu T) \to \boldsymbol{v}^{\infty},\ \boldsymbol{p}(n_\nu T) \to \boldsymbol{p}^{\infty}$ and $\boldsymbol{q}(n_\nu T) \to \boldsymbol{q}^{\infty}$ as $n_\nu \to \infty$. Since $|\ \boldsymbol{q}_i(t)\ | \leqslant \mu_i p_i(t)$ for all t it follows that $|\ \boldsymbol{q}_i^{\infty}\ | \leqslant \mu_i p_i^{\infty}$. Therefore there exists an orbit $\boldsymbol{v}^2(t),\ \boldsymbol{p}(t),\ \boldsymbol{q}^2(t)$ with initial values given by $\boldsymbol{v}^2(0) = \boldsymbol{v}^{\infty}, \boldsymbol{q}^2(0) = \boldsymbol{q}^{\infty}$.

Now let $\|\boldsymbol{v}(n_\nu T) - \boldsymbol{v}^{\infty}\| = \varepsilon_\nu \to 0$ as $n_\nu \to \infty$. We also have $\|\boldsymbol{v}(n_\nu T + t) - \boldsymbol{v}^2(t)\| \leqslant \varepsilon_\nu$. Further

$$\begin{aligned}\|\boldsymbol{v}^1(t) - \boldsymbol{v}^2(t)\| &= \|(\boldsymbol{v}^1(t) - \boldsymbol{v}(n_\nu T+t)) + (\boldsymbol{v}(n_\nu T+t) - \boldsymbol{v}^2(t))\| \\ &= \|(\boldsymbol{v}^1(n_\nu T+t) - \boldsymbol{v}(n_\nu T+t)) + (\boldsymbol{v}(n_\nu T+t) - \boldsymbol{v}^2(t))\| \\ &\geqslant \|\boldsymbol{v}^1(n_\nu T+t) - \boldsymbol{v}(n_\nu T+t)\| - \|\boldsymbol{v}^2(t) - \boldsymbol{v}(n_\nu T+t)\| \\ &\geqslant c - \varepsilon_\nu\end{aligned}$$

and this is valid for all ν, *i.e.*, $\|\boldsymbol{v}^1(t) - \boldsymbol{v}^2(t)\| \geqslant c$. We also have that $\|\boldsymbol{v}^1(t) - \boldsymbol{v}^2(t)\| \leqslant \|\boldsymbol{v}^1(0) - \boldsymbol{v}^{\infty}\| = c$. We conclude that $\|\boldsymbol{v}^1(t) - \boldsymbol{v}^2(t)\| = c$ for all t and, by Corollary 3.1, that the orbit $\boldsymbol{v}^2(t)$ is T-periodic.

We will finally show that $\boldsymbol{v}(t) - \boldsymbol{v}^2(t) \to \boldsymbol{0}$ as $t \to \infty$. Since $\|\boldsymbol{v}(t) - \boldsymbol{v}^2(t)\|$ is a decreasing function of t it suffices to show that $\boldsymbol{v}(n_\nu T) - \boldsymbol{v}^2(n_\nu T) = \boldsymbol{v}(n_\nu T) - \boldsymbol{v}^2(0) = \boldsymbol{v}(n_\nu T) - \boldsymbol{v}^{\infty} \to \boldsymbol{0}$. This is true by the definition of $\boldsymbol{v}^{\infty}$. h

4. Conclusions

We have established that for an uncoupled discrete frictional elastic system, the response to a given periodic loading will always tend to a periodic displacement state and this state is unique if the permanent stick zone $\mathcal{T}$ is null *i.e.*, if all nodes slip at least once during each cycle.

If the loading is such that $\mathcal{T} \neq \emptyset$, then there exists a continuous range of steady states in which the locked-in slip displacements in $\mathcal{T}$ take different values, the state reached depending on the initial conditions. However, the difference between any two of these states is independent of time, showing that the slip velocities $\dot{\boldsymbol{v}}(t)$ are unique in the steady state. Since the tangential tractions during slip are given by $\boldsymbol{q}_i(t) = -\mu_i p_i(t)\dot{\boldsymbol{v}}_i(t)/|\dot{\boldsymbol{v}}_i(t)|$, these are also unique, and they must remain so during any period of stick. We conclude that the steady-state loading history at any nodes i R $\mathcal{T}$ is independent of initial conditions and hence that the work done against friction at each node is a unique function of t. This suggests that fretting damage in uncoupled systems should be independent of initial conditions.

Finally, the tangential tractions $\boldsymbol{q}_i(t)$ in the permanent stick zone ($i \in \mathcal{T}$) will depend on initial conditions, but their time-derivatives $\dot{\boldsymbol{q}}_i(t)$ will not.

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