Large-angle cosmic microwave background suppression and polarization predictions

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ABSTRACT

The anomalous lack of large-angle temperature correlations has been a surprising feature of the cosmic microwave background (CMB) since first observed by COBE-DMR and subsequently confirmed and strengthened by the Wilkinson Microwave Anisotropy Probe. This anomaly may point to the need for modifications of the standard model of cosmology or may indicate that our Universe is a rare statistical fluctuation within that model. Further observations of the temperature auto-correlation function will not elucidate the issue; sufficiently high-precision statistical observations already exist. Instead, alternative probes are required. In this work, we explore the expectations for forthcoming polarization observations. We define a prescription to test the hypothesis that the large-angle CMB temperature perturbations in our Universe represent a rare statistical fluctuation within the standard cosmological model. These tests are based on the temperature-\(Q\) Stokes parameter correlation. Unfortunately, these tests cannot be expected to be definitive. However, we do show that if this \(TQ\)-correlation is observed to be sufficiently large over an appropriately chosen angular range, then the hypothesis can be rejected at a high confidence level. We quantify these statements and optimize the statistics we have constructed to apply to the anticipated polarization data. We find that we can construct a statistic that has a 25 per cent chance of excluding the hypothesis that we live in a rare realization of \(\Lambda\) cold dark matter at the 99.9 per cent confidence level.

Key words: cosmic background radiation – large-scale structure of Universe.

1 INTRODUCTION

In the two decades since the Cosmic Background Explorer (COBE) first detected the primordial fluctuations in the cosmic microwave background (CMB) temperature (Wright et al. 1992), and perhaps even more so in the past decade over which the Wilkinson Microwave Anisotropy Probe (WMAP) has provided ever more accurate full-sky maps of those fluctuations (see Komatsu et al. 2011, for example), the CMB has become a keystone in the remarkable transition of cosmology from a qualitative to a precision science.

An important element of the role of the CMB in precision cosmology has been that the canonical theory of cosmology, inflationary \(\Lambda\) cold dark matter (\(\Lambda\)CDM), makes clear predictions for the statistical properties of the spherical harmonic coefficients of the temperature fluctuations,

\[
a_{\ell m} \equiv \int Y_{\ell m}^*(\theta, \phi) T(\theta, \phi) d(\cos \theta) d\phi,
\]

(1)

which are predicted to be statistically isotropic realizations of independent Gaussian random variables of zero mean and with variance \(C_\ell\) depending only on \(\ell\),

\[
C_\ell = \frac{1}{2\ell + 1} \sum_{m=-\ell}^{\ell} |a_{\ell m}|^2.
\]

(2)

In modern discussions of the CMB, the two-point angular power spectrum, embodied in these \(C_\ell\), plays a central role and is the source of the remarkable precision of the cosmological parameters (Komatsu et al. 2011).

Before the COBE era, it was the two-point angular correlation function of the fluctuations,

\[
C(\theta) \equiv \langle T(\hat{n}_1)T(\hat{n}_2) \rangle_{\hat{n}_1 \cdot \hat{n}_2 = \cos \theta},
\]

(3)

rather than the angular power spectrum that was of primary interest to astronomers. Statistically, the two-point angular correlation function is an ensemble average but, in practice, this must be replaced by an average over pairs of points separated by an angle \(\theta\), as denoted by the bar over the expression. In fact, the COBE differential microwave radiometer (COBE-DMR) did report \(C(\theta)\),

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though only in their final, four-year paper (Bennett et al. 1996). When extracted from a full-sky map, both $C(\theta)$ and the $C_\ell$ contain the same information, albeit in different forms. The same is true for a function and its Fourier transform; signals are typically most easily seen in one or the other forms but not both. In the case of the CMB, $C(\theta)$ and the $C_\ell$ are related by a Legendre series. The $C_\ell$ most easily show the small angular scale behaviour, microphysics at last scattering, whereas the $C(\theta)$ most easily shows the large angular scale behaviour.

As observed by the COBE-DMR, $C(\theta)$ had an unexpected property – it was consistent with zero for angular separations between approximately $60^\circ$ and $160^\circ$. This was duly noted at the time but mainly remembered today as a low quadrupole. The WMAP team confirmed the COBE-DMR observation of a lack of large-angle correlation with significantly smaller error bars. In their initial, one-year release Spergel et al. (2003) phrased the anomaly in terms of a statistic

$$S_{1/2} \equiv \sqrt{\frac{1}{2} \int_{-1}^{1} [C(\theta)]^2 \, d(\cos \theta) \; .}$$

(4)

In the best-fitting $\Lambda$CDM model, the expected value of this statistic is approximately 50 000 $(\mu K)^4$, whereas the observed value is approximately 8500 $(\mu K)^4$ on the full sky, e.g. from the WMAP independent linear combination (ILC) map, with a $p$-value of approximately 0.05. Even more striking is that if one considers only the part of the sky outside a conservative Galaxy cut, then $S_{1/2} \approx 1000$–1150 $(\mu K)^4$ and is only $\sim 1300$ $(\mu K)^4$ in each of the $V$ and $W$ frequency bands, which are expected to be dominated by the CMB signal. The cut-sky $S_{1/2}$ has a $p$-value of about $2.5 \times 10^{-4}$, depending on the precise map (Copi et al. 2009).

We have argued that such absence of the two-point angular correlation is unlikely to result solely from a small quadrupole, or even a small quadrupole and octopole, and that it instead requires a ‘conspicacy’ among the first several multipoles (Copi et al. 2009). Such covariance among the $C_\ell$ is likely contrary to the fundamental prediction of the canonical cosmological model that the $a_{\ell m}$ underlying the $C_\ell$ are independent Gaussian random variables with variances depending only on $\ell$. This is one of a number of large-scale anomalies that suggest that modifications of the standard models are required on large angular scales (see Bennett et al. 2011; Copi et al. 2010, and references therein for further details).

One possible explanation of the absence of large-angle correlations among the CMB temperature fluctuations is that it is merely a statistical fluke. In this paper, we explore the consequences of this hypothesis. In particular, since the fluctuations in the CMB temperature and in its polarization arise largely from the same source – the gravitational potential – one might have hoped that a small temperature–temperature ($TT$) correlation function on large angular scales would predict a similarly small cross-correlation between CMB temperature and CMB Stokes parameter $Q$, or in the polarization–polarization ($QQ$) correlation.

Unfortunately, as we shall see, the connection between temperature and polarization fluctuations is too weak for a general definitive test of the origin of the vanishing correlation function. However, we do find that if the $S_{1/2}$ is small because of a statistical fluke within $\Lambda$CDM cosmology, then the cross-correlation between temperature and polarization is unlikely to be large on large angular scales. Therefore, were we to infer a large value of this cross-correlation from future data, that would be evidence against a statistical fluke as an explanation of the vanishing $TT$ correlation.

In this paper, we provide a prescription to follow in order to test this hypothesis. In Section 2, we describe the construction of an ensemble of realizations of $\Lambda$CDM that are constrained to resemble our observed Universe in the properties of their $TT$ angular power spectrum and full-sky and cut-sky two-point angular correlation functions. In Section 3, we construct statistics that, like $S_{1/2}$ for the $TT$ correlations, can be used to quantify the smallness of the correlations between temperature and polarization fluctuations. Section 4 contains a discussion of the results of applying the temperature–polarization cross-correlation statistics to the ensemble of constrained realizations and looks forward to what might be learned by applying them to future polarization data. Given the constructed realizations, the values of the statistics are fixed and the optimal application of them to polarization data can be determined as discussed in this section. This optimization is independent of polarization observations. Finally, Section 5 contains the conclusions.

2 CONSTRAINED REALIZATIONS

To study the signature of the lack of large-angle correlations in the WMAP temperature data on upcoming polarization measurements, such as from Planck, we require realizations of $\Lambda$CDM consistent with these large-angle results. For this purpose, we have generated 300 000 such realizations as follows.

(i) In the standard, $\Lambda$CDM model, our Universe is a realization from an ensemble, the width of which (the cosmic variance) for low-$\ell$ information is quite large. However, once measured our realization can and has been precisely determined. In the work reported here this information is given in the WMAP reported $C_\ell^{TT}$. We are interested in producing realizations of our Universe as represented by the WMAP observations, not realizations of the full $\Lambda$CDM model. For this purpose, we treat the observational errors in the WMAP reported $C_\ell^{TT}$ as Gaussian distributed and generate realizations accordingly. Thus, we generate random $C_\ell^{TT}$ from Gaussian distributions centred on the WMAP-reported values. This produces a power spectrum consistent with that reported by WMAP. Again it is important to stress that this is a power spectrum consistent with the observation of our particular realization of the Universe as measured by WMAP, not a general realization of the best-fitting $\Lambda$CDM model. For this reason, cosmic variance is not relevant nor do we generate $C_\ell$ from a $\chi^2$ distribution. It is also true that on a partial sky the $C_\ell$ are slightly correlated. To correct for this we actually use the Fisher matrix from the WMAP likelihood code without the contribution from cosmic variance in drawing these $C_\ell^{TT}$. In practice, this is a small correction but has been included for completeness.

(ii) The $C_\ell^{TT}$ generated in the previous step contain the statistical information about the power in each mode consistent with the WMAP observations. For further analysis we need a map, not just the power spectrum. A map is a particular realization of this power spectrum. As in $\Lambda$CDM we assume that the modes in the map have random phases. In practice, this means we choose the $a_{\ell m}$ randomly on the $2\ell$-sphere such that

$$\frac{1}{2\ell + 1} \sum_{m = -\ell}^{\ell} |a_{\ell m}^T|^2 = C_\ell^{TT} \; ,$$

(5)

where the $C_\ell^{TT}$ in this expression are exactly the values generated from the previous step. By this construction the resulting sky realization is guaranteed to have a $S_{1/2}$ consistent with the small value in the full-sky WMAP ILC map. For each set of $C_\ell^{TT}$ from the

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1 The ILC map and all data from the WMAP mission used in this work are freely available from http://lambda.gsfc.nasa.gov.
previous step, we generate a single complete sky realization, i.e. a single map.

(iii) To further be consistent with WMAP observations the $S_{1/2}$ on the cut sky must also be small. For our realizations we require $S_{1/2}^2 \leq 1292.6 (\mu K)^4$, the value from the WMAP seven year, KQ75y7 masked ILC map. This $S_{1/2}$ value is calculated for a realization by first constructing a map at $N_{\text{SIDE}} = 64$ from the $a_{1m}^T$ generated above. The pseudo-$C_\ell$ are extracted from this map based on the region outside the KQ75y7 mask using SPICE (Chon et al. 2004). Finally, $S_{1/2}$ is calculated using these $C_\ell$ up to $\ell_{\text{max}} = 100$.

(iv) For temperature realizations that satisfy the cut sky constraint, we also generate realizations of the $a_{1m}^T$. See Appendix A for a review of the process.

The construction of a set of constrained realizations is the basis for the prescription we are describing. It will next provide predictions for the expectations from the observations of the CMB polarization.

3 STATISTICS

For the temperature auto-correlation, the $S_{1/2}$ statistic was defined posteri (Spergel et al. 2003) to be

$$ S_{1/2} \equiv \int_{-1}^{1/2} [C^{TT}(\theta)]^2 \, d(\cos \theta). \tag{6} $$

Inspired by this we define a comparable statistic for $C^{TQ}(\theta)$, the two-point angular correlation function between fluctuations in the temperature and the Stokes parameter $Q$.

Observable properties of photons can be characterized by the Stokes parameters. For the CMB the relevant quantities are the intensity, conventionally represented by the temperature, $T$, and the linear polarization given by the $Q$ and $U$ parameters. For the CMB the circular polarization, represented by the $V$ Stokes parameter, is expected to be zero and not considered further. When working in real space the natural correlations to construct are amongst these observables, $T$, $Q$, and $U$. These correlations are constructed such that they only depend on the angular separation along the great circle connecting each pair of point on the sky and are thus rotationally invariant despite the fact that the definition of $Q$ and $U$ depend on the choice of coordinate axes (Kamionkowski, Kosowsky & Stebbins 1997). When working in harmonic space, it is natural to decompose the polarization into ‘gradient’ and ‘curl’ modes (Kamionkowski et al. 1997) alternatively called $E$ and $B$ modes (Zaldarriaga & Seljak 1997), which are similarly rotationally invariant quantities. These latter names will be used throughout. Thus, in real space we will work with the $TQ$ two-point angular correlation function, $C^{TQ}(\theta)$, which may be written in terms of the two-point angular power spectrum coefficients, $C^{TE}_\ell$.

3.1 $S^{TQ}$ statistic

In the case of polarization a priori the optimal range over which to integrate the correlation function is unknown and will be explored below so we define the general statistic

$$ S^{TQ}(\theta_1, \theta_2) \equiv \int_{\cos \theta_2}^{\cos \theta_1} [C^{TQ}(\theta)]^2 \, d(\cos \theta). \tag{7} $$

As with $S_{1/2}$ we may calculate this easily in terms of the power spectrum coefficients, $C^{TE}_\ell$. Using (Kamionkowski et al. 1997)

$$ C^{TQ}(\theta) = \sum_{\ell = 2}^\infty \frac{2\ell + 1}{4\pi} \sqrt{\frac{(\ell - 2)!}{(\ell + 2)!}} C^{TE}_\ell P_2^\pm(\cos \theta), \tag{8} $$

we may show that

$$ S^{TQ}(\theta_1, \theta_2) = \sum_{\ell, \ell'} C^{TE}_\ell I_{\ell \ell'}^{TQ}(\theta_1, \theta_2) C^{TE}_{\ell'} \tag{9} $$

where $I_{\ell \ell'}^{TQ}(\theta_1, \theta_2)$ are components of a known matrix calculated in Appendix B. A histogram of the $S^{TQ}$ statistic for a particular choice of $\theta_1$ and $\theta_2$ is shown in Fig. 1 comparing the constrained realizations to $\Lambda$CDM.

3.2 $s^{TQ}$ Statistic

Motivated solely by its simplicity and ease of computation, we also define a new statistic which is linear, rather than quadratic, in the $TQ$ correlation function

$$ s^{TQ}(\theta_1, \theta_2) \equiv \int_{\cos \theta_2}^{\cos \theta_1} C^{TQ}(\theta) \, d(\cos \theta). \tag{10} $$

As with $S^{TQ}(\theta_1, \theta_2)$ we may calculate this easily in terms of the $C^{TE}_\ell$,

$$ s^{TQ}(\theta_1, \theta_2) = \sum_{\ell = 2}^\infty C^{TE}_\ell i_{\ell}^{TQ}(\theta_1, \theta_2), \tag{11} $$

where $i_{\ell}^{TQ}(\theta_1, \theta_2)$ are the components of a known vector calculated in Appendix B. A histogram of the $s^{TQ}$ statistic for a particular choice of $\theta_1$ and $\theta_2$ is shown in Fig. 2 comparing the constrained realizations to $\Lambda$CDM.

4 RESULTS

The $S^{TQ}$ and $s^{TQ}$ statistics defined above have been calculated for the constrained realizations discussed in Section 2 and for a comparable number of realizations of $\Lambda$CDM. In both cases, these have been calculated from maps produced at $N_{\text{SIDE}} = 64$. Data from the temperature map outside the KQ75y7 mask and from the seven-year polarization analysis mask, both provided by WMAP, have been used. As shown in Figs 1 and 2, the constrained and $\Lambda$CDM
CMB polarization predictions

Figure 2. Example histogram of the $s^{TQ}$ statistic, defined in equation (10), for constrained (solid, black line) and $\Lambda$CDM (dashed, red line) realizations. We note that the constrained realizations are more sharply peaked near zero than $\Lambda$CDM which, though also peaked near zero, has a long tail particularly to large, positive values. The dashed, vertical lines represent the values of $s^{TQ}$ for which 99 and 99.9 per cent, respectively, of the constrained realizations have smaller values.

realizations have most likely values for the $S^{TQ}$ and $s^{TQ}$ statistics at nearly the same value. However, as we also see $\Lambda$CDM predicts much broader distributions for the two statistics. In particular, there is a significant probability of producing values larger than the constrained realizations. This provides a means of testing the hypothesis that our Universe is just a rare realization of $\Lambda$CDM. This results in a simple but not definitive test.

Consider the case of the $S^{TQ}$ statistic as represented in Fig. 1. For the constrained realizations 99 per cent of them have $S^{TQ}(48^{\circ}, 120^{\circ}) \leq 1.403 (\mu K)^2$ and 99.9 per cent have $S^{TQ}(48^{\circ}, 120^{\circ}) \leq 2.195 (\mu K)^2$. Unconstrained $\Lambda$CDM (with the best-fitting values of cosmological parameters) randomly generates realizations with values larger than these 38.6 and 25.6 per cent of the time, respectively. If observations of the polarization show our Universe to have a $S^{TQ}(48^{\circ}, 120^{\circ})$ value larger than these values, then we can reject the random $\Lambda$CDM realization hypothesis at the appropriate confidence level. Alternatively, if the $S^{TQ}(48^{\circ}, 120^{\circ})$ value is smaller, then no definitive statement can be made; the polarization fluctuations would be consistent with the hypothesis that we live in a rare $\Lambda$CDM realization but do nothing to advance that hypothesis. This is the main point of the paper.

Similar statements may be made about the $s^{TQ}$ statistic shown in Fig. 2. It provides similar information as $S^{TQ}$.

We still have freedom to choose the optimal range of angles over which to evaluate the statistics. We define optimal to mean the maximum discriminatory power between the distribution of the statistic in the constrained versus $\Lambda$CDM realizations. For a given percent cut-off from the constrained realizations, we wish to find the angle range $[\theta_1, \theta_2]$ that has the maximum fraction of $\Lambda$CDM realizations above this value.

The results of such a study are shown in Figs 3 and 4 for the $S^{TQ}(\theta_1, \theta_2)$ statistic and in Figs 5 and 6 for the $s^{TQ}(\theta_1, \theta_2)$ statistic. The statistics are non-zero only up to the diagonal $\theta_1 = \theta_2$, shown as the dashed, black line in the figures, but not along it. For this reason, the contours are truncated at the diagonal. They have been made symmetric in $\theta_1$ and $\theta_2$ (by taking $|S^{TQ}|$ and $|s^{TQ}|$) so the results are shown as identical when reflected through the diagonal. The optimal ranges and fractions of $\Lambda$CDM realizations are listed in Table 1. We note that the optimal surfaces represented by the contours seen in the figures are relatively broad, at least in one direction. Due to this the values of $\theta_1$ and $\theta_2$ in a neighbourhood of those listed in Table 1 can be employed with nearly the same efficacy.

5 CONCLUSIONS

The absence of two-point angular correlations on large angular scales in the CMB temperature data is, by now, well established. It was first measured by COBE-DMR, but became more significant
functions, such as $TQ$. These would be the basis for an observational test of this statistical fluke hypothesis.

In this paper, we have discussed a prescription to follow in order to test this hypothesis of our Universe being a statistical fluke. The prescription may be simply stated as follows.

(i) Construct realizations of our Universe consistent with the observed temperature fluctuations. This means construct sets of $a_{Tm}$ and $a_{Em}$ consistent with the observed $C_{lT}$ and full and cut sky $S_{1/2}$ as discussed in Section 2.

(ii) Apply the $S^Q$ and $s^Q$ statistics as defined in Section 3 to these constrained realizations.

(iii) Also apply the $S^Q$ and $s^Q$ statistics to a comparable number of best-fitting $\Lambda$CDM realizations and use these to find the optimal ranges $[\theta_1, \theta_2]$ for each statistic. Optimal here means that the maximum fraction of $\Lambda$CDM realizations fall above the value at some confidence level in the constrained realization, e.g. the 99 or 99.9 per cent level.

(iv) Given the optimal ranges from the previous step now apply these particular cases to the observed polarization signal. If the observations produce values for these statistics larger than that expected from the constrained realizations, then the statistical fluke hypothesis can be rejected at the appropriate confidence level. Alternatively, if the values are smaller, then the hypothesis remains consistent but unproven.

We further note that the optimization in this prescription is independent of the polarization observations, or, in fact, whether the polarization has been observed or not.

Our work is, in spirit, related to Dvorkin, Peiris & Hu (2008). While they consider observables in the polarization signal for ‘models’ of three dimensional primordial power modulation that might explain the breaking of statistical isotropy in the temperature field, we predict the polarization statistics starting directly from realizations of $\Lambda$CDM models that are constrained to show the suppressed $TT$ correlation at large angular scales.

In the work reported here we have generated realizations and performed the optimization based on the WMAP seven-year data release. The prescription could be applied to the WMAP nine-year data release and the results are not expected to differ significantly. We have also said nothing about applying the statistics to the WMAP reported polarization observations. Unfortunately, the signal-to-noise ratio in the polarization observations is not yet sufficient to make meaningful statements. To see this, using the WMAP nine-year reported $C_{lE}$ (Hinshaw et al. 2012), we find for the optimal ranges $S^Q_{\Lambda C D M}(48', 120') = (1.0 \pm 0.8) (\mu K)^4$ (12)

Table 1. Optimal angle ranges for the $S^Q$ statistic (7) and $s^Q$ statistic (10). The optimal ranges are determined by finding the maximum fraction of $\Lambda$CDM realizations with the appropriate statistic above the 99 or 99.9 per cent level of the constrained realizations. The histograms for the optimal 99.9 per cent ranges are shown in Figs 1 and 2. The full contours are shown in Figs 3–6.

<table>
<thead>
<tr>
<th>Statistic</th>
<th>C.L. (per cent)</th>
<th>$\theta_1$ (deg)</th>
<th>$\theta_2$ (deg)</th>
<th>Fraction (per cent)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S^Q$</td>
<td>99</td>
<td>46</td>
<td>122</td>
<td>38.8</td>
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<tr>
<td></td>
<td>99.9</td>
<td>48</td>
<td>120</td>
<td>25.6</td>
</tr>
<tr>
<td>$s^Q$</td>
<td>99</td>
<td>20</td>
<td>177</td>
<td>38.5</td>
</tr>
<tr>
<td></td>
<td>99.9</td>
<td>23</td>
<td>174</td>
<td>26.4</td>
</tr>
</tbody>
</table>

in the WMAP temperature maps. This absence of correlation is difficult to accommodate within the standard cosmological model, especially since it seems to imply covariance among low-$\ell$ multipoles of the CMB. A simple explanation that has been proffered is that we just live in a rare realization of $\Lambda$CDM that happens to have a lack of large-angle $TT$ correlations. If so, one might hope that constraining $\Lambda$CDM realizations to have low $TT$ correlations at large angles would have observable consequences for other correlation

...
and

\[ a_{\ell}^{E} = \frac{C_{\ell}^{EE}}{\sqrt{C_{\ell}^{TT}}} \xi_1 + \sqrt{C_{\ell}^{EE} - \left(\frac{C_{\ell}^{EE}}{C_{\ell}^{TT}}\right)^2} \xi_2, \quad \text{(A2)} \]

where \( \xi_1 \) and \( \xi_2 \) are Gaussian random variables drawn from a distribution with zero mean and unit variance and the index \( j \) refers to the pair of indices \((\ell, m)\). To be precise, \( j \) takes the values 0 to \( 2\ell \) and the complex coefficients are constructed as

\[ a_{\ell m}^{\ell,0} = \left\{ \begin{array}{ll} a_{\ell 0}^0, & m = 0 \\ \frac{1}{\sqrt{2}} (a_{\ell m-1}^0 + i a_{\ell m}^0), & m > 0 \end{array} \right. \quad \text{(A3)} \]

For our purposes we need to generate constrained realizations of \( \Lambda \text{CDM} \) so the above procedure must be modified. The steps discussed in Section 2 lead to the generation of \( a_{\ell m}^{\ell,0} \). In other words, we have determined \( a_{\ell m}^{\ell,0} \) by which equation (A1) means we have also determined \( \xi_1 \). That is, instead of choosing \( \xi_1 \) as a Gaussian random variable we have used observational constraints to determine its value and find it by inverting that equation. Since the temperature and \( E \)-type polarization are correlated, this constrained temperature realization affects \( a_{\ell m}^{\ell,0} \). The real and imaginary components of \( a_{\ell m}^{\ell,0} \) may now be generated from (A2) as

\[ a_{\ell m}^{\ell,0} = \frac{C_{\ell}^{EE}}{C_{\ell}^{TT}} a_{\ell m}^{\ell,0} + \sqrt{C_{\ell}^{EE} - \left(\frac{C_{\ell}^{EE}}{C_{\ell}^{TT}}\right)^2} \xi_2, \quad \text{(A4)} \]

where \( \xi_2 \) is still to be chosen as a Gaussian random variable.

**APPENDIX B: DERIVATION OF STATISTICS FORMULAS**

**B1 \( S^{TQ}(\theta_1, \theta_2) \)**

We wish to evaluate \( S^{TQ}(\theta_1, \theta_2) \) as discussed in the text (7). Consider the simpler case

\[ S^{TQ}(x) = \int_0^x \left[ C^{TQ}(\theta) \right]^2 d(\cos \theta) \]

\[ = \sum_{\ell, \ell'} (2\ell + 1)(2\ell' + 1) \frac{\pi^2}{(4\pi)^2} \sqrt{\frac{(\ell - 2)!}{(\ell + 2)!}} C_{\ell}^{EE} C_{\ell'}^{EE} \times \int_0^x P_{\ell}^2(\cos \theta) P_{\ell'}^2(\cos \theta) d(\cos \theta). \quad \text{(B1)} \]

To evaluate this expression we need to perform the integral of two associated Legendre functions, \( P_{\ell}^m \), of order \( m = 2 \),

\[ \bar{T}_{\ell,\ell}^{\ell}(x) = \int_0^x P_{\ell}^2(x) P_{\ell}^2(x) dx. \quad \text{(B2)} \]

Note that this integral is only defined for \( \ell, \ell' \geq 2 \).

For \( \ell \neq \ell' \), we may proceed by directly integrating the associated Legendre differential equation to find

\[ \bar{T}_{\ell,\ell}^{\ell}(x) = \frac{1 - x^2}{\ell(\ell + 1) - \ell'(\ell' + 1)} \times \left[ P_{\ell}^2(x) \frac{dP_{\ell}^2(x)}{dx} - P_{\ell}^2(x) \frac{dP_{\ell}^2(x)}{dx} \right]. \quad \text{(B3)} \]

For \( \ell = \ell' \) more care is required. Starting from the Rodriguez formula

\[ P_{\ell}^2(x) = (1 - x^2)^2 P_{\ell,0}^0(x) \quad \text{(B4)} \]
and the recursion relation
\[ \frac{dP_{\ell}(x)}{dx} = \frac{dP_{\ell-1}(x)}{dx} + \ell P_{\ell}(x), \]  
we can show that
\[ I_{\ell}^{(2)}(x) = 4J_{\ell}^{(2)}(x) - 4\ell(\ell - 1)J_{\ell}^{(1)}(x) + \ell^2(\ell - 1)^2I_{\ell}(x). \]  
Here, \( I_{\ell}(x) \) is the equivalent integral over Legendre polynomials encountered in the definition of \( S_{\ell}\); see appendix A of Copi et al. (2009) for details. The remaining quantities, \( J_{\ell}^{(1)}(x) \) and \( J_{\ell}^{(2)}(x) \), are calculated through integration by parts and use of the recursion relation to find
\[ J_{\ell}^{(1)}(x) = P_{\ell-1}(x)P_{\ell}(x) + \frac{1}{2} \left\{ 1 - x \left[ P_{\ell-1}(x) \right]^2 - (2\ell - 1)I_{\ell-1,\ell-1}(x) \right\}, \]  
and
\[ J_{\ell}^{(2)}(x) = J_{\ell-1}^{(2)}(x) + \ell \left[ P_{\ell-1}(x)P_{\ell}(x) + 1 \right]. \]  
Note that \( J_{0}^{(2)}(x) \) is defined recursively. We can directly show that \( J_{0}^{(1)}(x) = 0 \).

With these expressions for the integrals we may write
\[ S^{TE}(\theta_1, \theta_2) = \sum_{\ell,\ell'} C^{TE}_\ell \hat{I}^{TE}_{\ell,\ell}(\theta_1, \theta_2) C^{TE}_{\ell'}, \]  
where
\[ \hat{I}^{TE}_{\ell,\ell}(\theta_1, \theta_2) = \frac{(2\ell + 1)(2\ell' + 1)}{(4\pi)^2} \sqrt{\frac{(\ell - 2)!(\ell' - 2)!}{(\ell + 2)!(\ell' + 2)!}} \times \left[ \hat{I}^{TE}_{\ell}(\cos \theta_1) - \hat{I}^{TE}_{\ell}(\cos \theta_2) \right]. \]

This matrix may be precomputed for rapid evaluation of \( S^{TE}(\theta_1, \theta_2) \).

**B2 \( s^{TE}(\theta_1, \theta_2) \)**

We wish to evaluate \( s^{TE}(\theta_1, \theta_2) \) as discussed in the text (10). We proceed as above and consider the simpler case
\[ s^{TE}(x) = \int_{-1}^{1} C^{TE}(\theta) \cos \theta \, d\theta \]
\[ = \sum_{\ell=2}^{\infty} \frac{(2\ell + 1)}{4\pi} \sqrt{\frac{(\ell - 2)!}{(\ell + 2)!}} \cdot C^{TE}_\ell \times \int_{-1}^{1} P^{2}_\ell(\cos \theta) \, d\cos \theta. \]  
To evaluate this expression we need to perform the integral
\[ i^{TE}_{\ell}(x) \equiv \int_{-1}^{1} P^{2}_\ell(\cos \theta) \, d\cos \theta. \]

This integral is straightforward to evaluate starting from the Rodriguez formula (B4) and integrating by parts to find
\[ i^{TE}_{\ell}(x) = \ell P_{\ell-1}(x) - (\ell - 2)x P_{\ell}(x) + 2(-1)^\ell \]
\[ \frac{2}{2\ell + 1} [P_{\ell+1}(x) - P_{\ell-1}(x)]. \]

With this we may write
\[ s^{TE}(\theta_1, \theta_2) = \sum_{\ell=2}^{\infty} C^{TE}_\ell \cdot i^{TE}_{\ell}(\theta_1, \theta_2), \]
where
\[ i^{TE}_{\ell}(\theta_1, \theta_2) = \frac{2\ell + 1}{4\pi} \sqrt{\frac{(\ell - 2)!}{(\ell + 2)!}} \left[ i^{TE}_{\ell}(\cos \theta_1) - i^{TE}_{\ell}(\cos \theta_2) \right]. \]

This vector may be precomputed for rapid evaluation of \( s^{TE}(\theta_1, \theta_2) \).