Characterization of Signals from Multiscale edges

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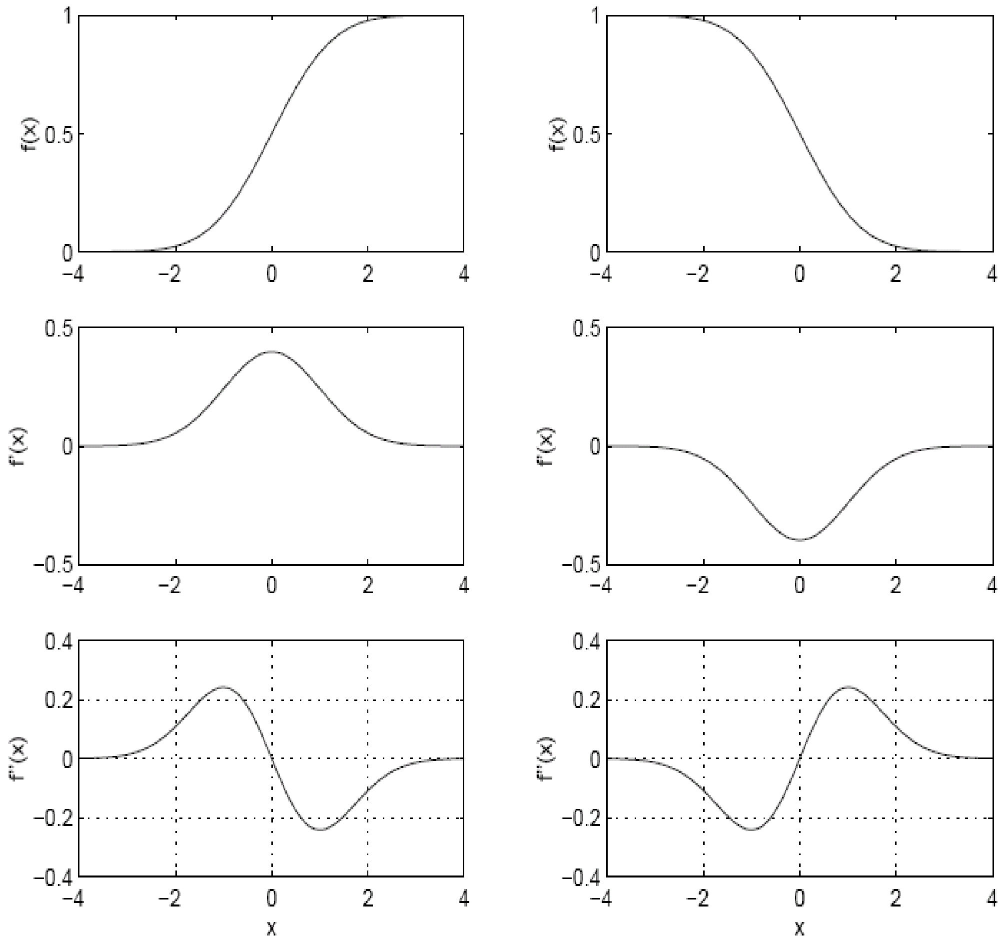
Edges

- Points of sharp variations

- Why do we need edge information?
  - to discriminate objects from their background
  - a very important precursor in many applications like region segmentation, image retrieval, data hiding or recognition and tracking of objects in image sequences.

- Reconstruct Images from Multiscale edges
  - Process Image information with edge based algorithms
  - Image compression
  - Image restoration

Detection of edges



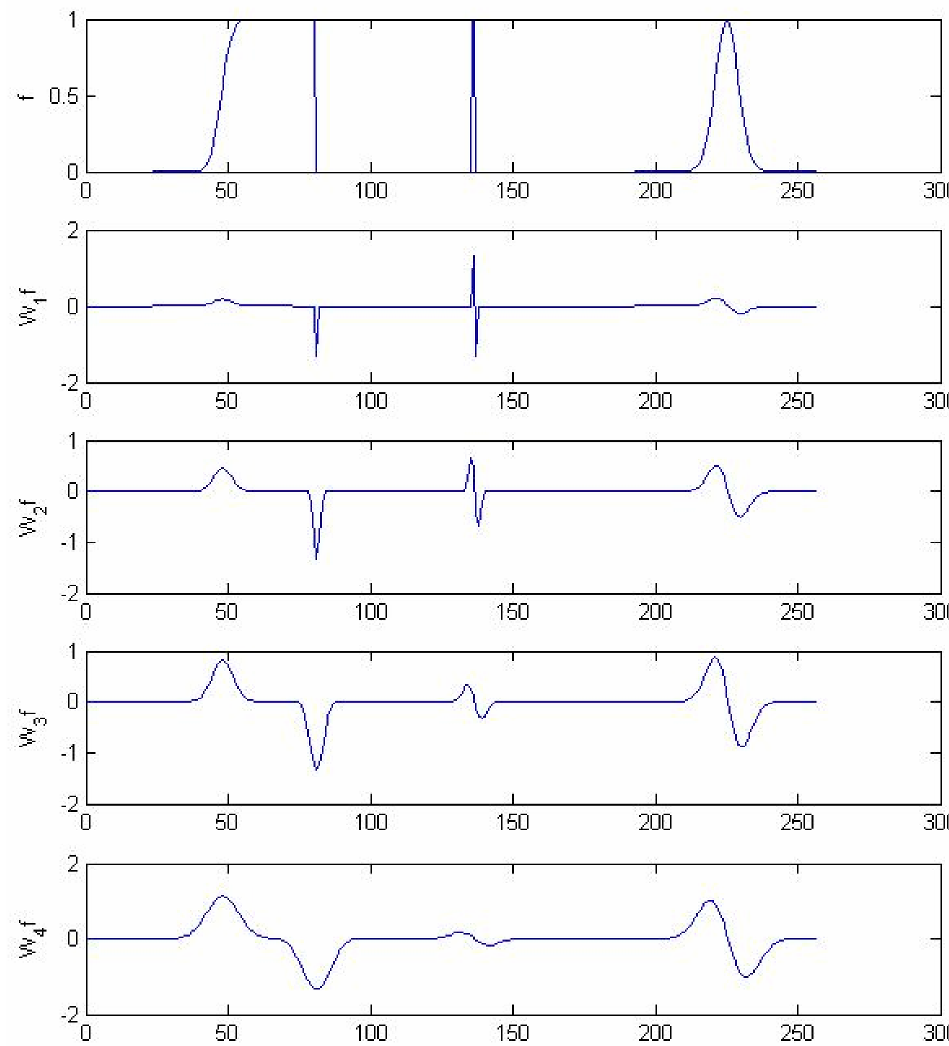
Edge Detection via Wavelet transform

- How is edge detection related to wavelet transform?
- The difference coefficients of a wavelet transform are nothing but the differentiation of the signal smoothed at different scales.

- Consider the daub 1 wavelet filter

g = [-1 1]

$$W_j f(x) = f * \Psi_j(x)$$
$$= f_{j-1} * g$$
$$= f_{j-1}(x-1) - f_{j-1}(x)$$



- How to combine these different values to characterize the signal variation?

- The wavelet theory gives an answer to this question by showing that the evolution across scales of the wavelet transform depends on the local Lipschitz regularity of the signal.

Definition : Let $0 \leq \alpha \leq 1$. A function $f(x)$ is uniformly Lipschitz α over an interval (a,b) if and only if there exists a constant K such that for any $(x_0,x_1) \in (a,b)^2$

$$|f(x_0) - f(x_1)| \leq K\,|x_0 - x_1|^{\alpha}$$

Theorem[1] : Let $0 < \alpha < 1$. A function $f(x)$ is uniformly Lipschitz α over (a,b) if and only if there exists a constant $K > 0$ such that for all $x \in (a,b)^2$ the wavelet transform satisfies

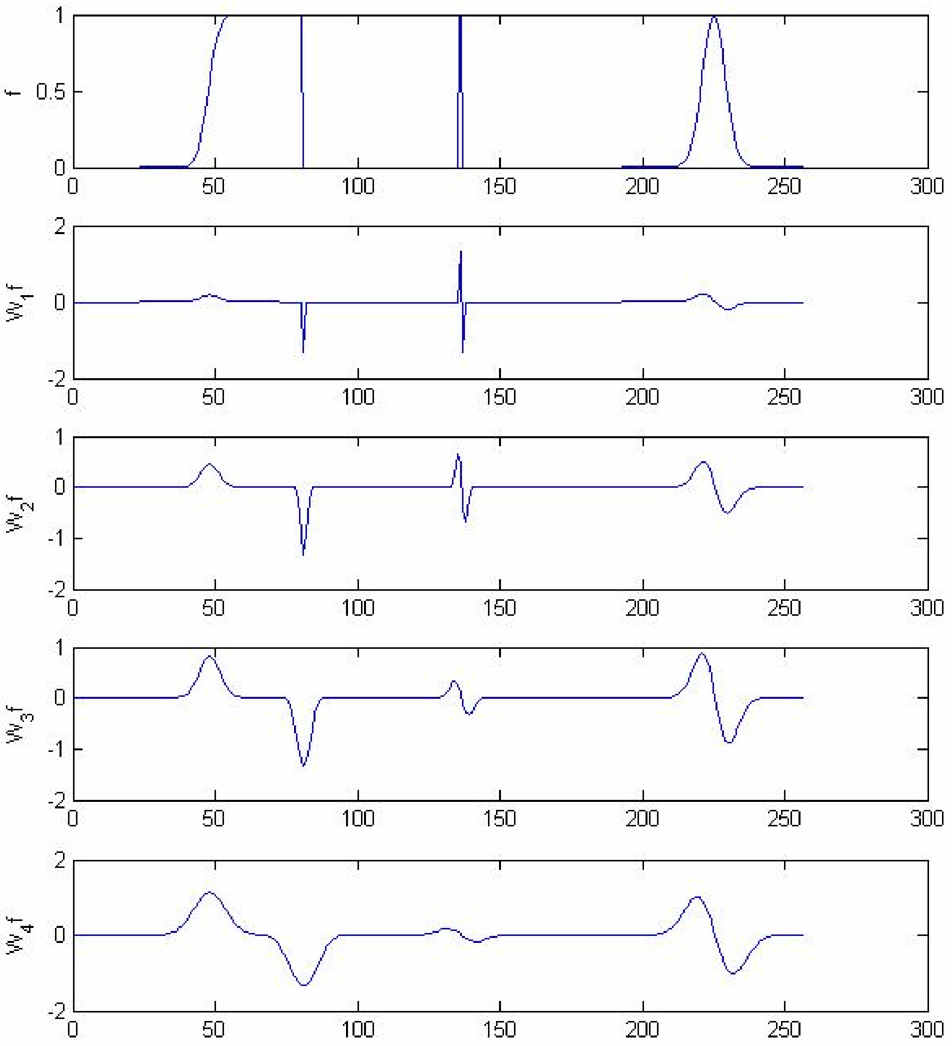
$$|W_{2^j} f(x)| \leq K (2^j)^{\alpha}$$

1 Meyer, '*Ondelettes et Operatuers',* 1990

$$|W_{2^j} f(x)| \le K(2^j)^\alpha$$

- If the uniform Lipschitz regularity is positive, the above condition implies that the amplitude of the wavelet transform modulus maxima should decrease when the scale decreases.

- The singularity at abscissa 3 produces wavelet transform maxima that increase when the scale decreases. These can be decribed by a negative Lipschitz exponent.



$$f(x) = h * g_\sigma(x)$$

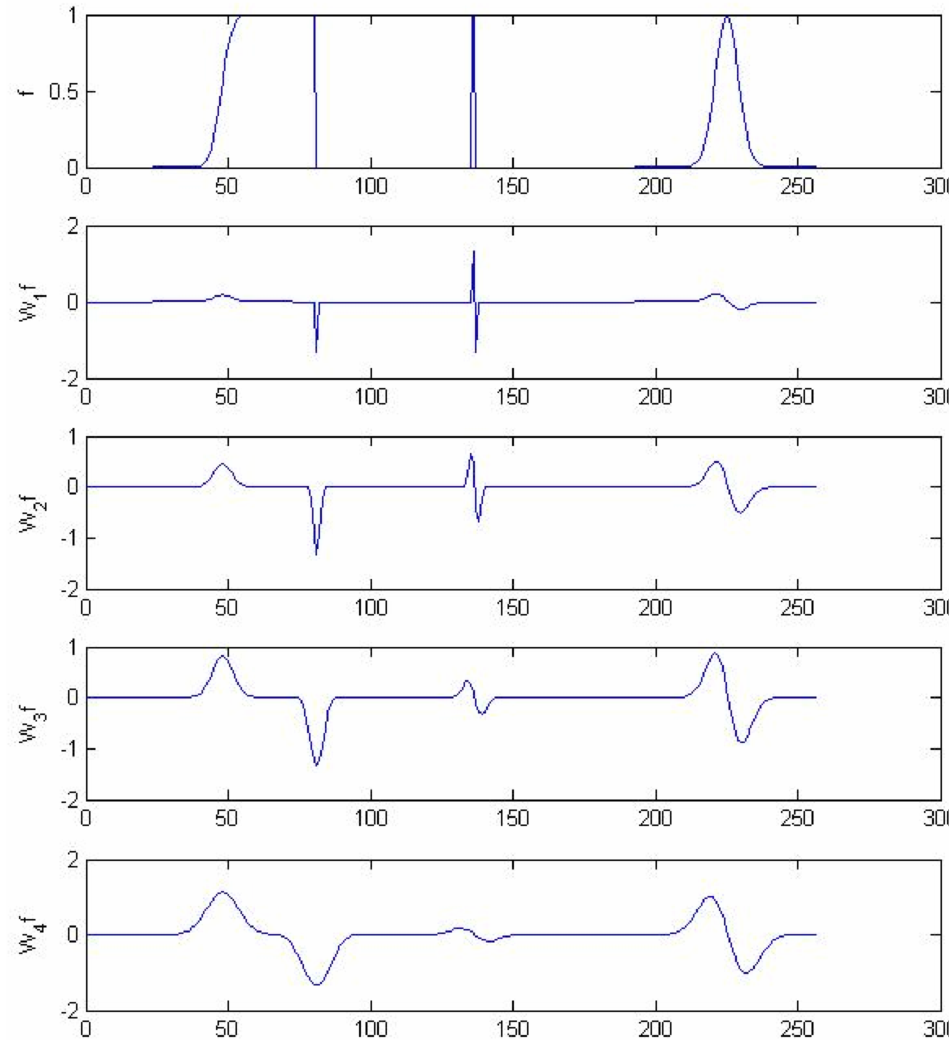
$$g_\sigma(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-\frac{x^2}{2\sigma^2})$$

$$W_{2^j} f(x) = 2^j \frac{d}{dx}(f * \theta_{2^j})(x) = 2^j \frac{d}{dx}(h * g_\sigma * \theta_{2^j})(x)$$

$$\approx 2^j \frac{d}{dx}(h * \theta_{s_o})(x) = \frac{2^j}{s_0} W_{s_o} h(x)$$

where $s_0 = \sqrt{2^{2j} + \sigma^2}$

$$|W_s h(x)| \le K(s)^\alpha \Rightarrow |W_{2^j} f(x)| \le K 2^j (s_o)^{\alpha - 1}$$



- Concluding remarks:

  - Complete information about the discontinuities in the signal is embedded in its wavelet transform across scales.

  - the lipschitz exponent and the smoothness of the discontinuity can be completely retrieved from the wavelet transform modulus maxima values at different scales.

- Task : To reconstruct the signal from this information!

Reconstruction

- The reconstruction of signals from multi scale edges has mainly been studied in the zero crossing framework[1].
- Issues : There are known counter examples that prove that the positions of zero crossings of $W^b_s f(x)$ do not characterize uniquely the function $f(x)$.

  Example : Wavelet transform of '$sin(x)$' and '$sin(x) + 0.2sin(2x)$' have the same zero crossings at all scales.
- *Mallat's Conjecture*[2] *:* To obtain a complete and stable zero crossing representation, it is sufficient to record the positions where $W^a_s f(x)$ has local extrema and its value at the corresponding locations.
- A reconstruction algorithm has been proposed by Mallat based on this conjecture.

1 B.Logan,"Information in the zero crossings of band pass signals", *Bell Syst. Tech. J.*, vol. 56, 1977.

2 Mallat,"Zero crossings of a wavelet transform," *IEEE Trans. Inform. Theory*, vol. 37, July 1991.

Reconstruction Algorithm

- Goal : To reconstruct an approximation of $(W_{2^j} f(x))_{j\in Z}$ given the positions of the local maxima of $|W_{2^j} f(x)|$ and the values of $W_{2^j} f(x)$ at these locations.
- Assume that the wavelet *Ψ(x)* is differentiable in the sense of Sobolev, hence the wavelet transform of *f(x)* is also differentiable in the sense of Sobolev, and it has, at most, a countable number of modulus maxima.
- The maxima constraints on $W_{2^j} h(x)$ can be decomposed in two conditions :
  - At each scale 2^j, for each local maximum located at x_n^j, $W_{2^j} h(x_n^j) = W_{2^j} f(x_n^j)$.
  - At each scale 2^j, the local maxima of $|W_{2^j} h(x)|$ are located at the abscissa $(x_n^j)_{n\in Z}$
- Condition 1 is equivalent to: $\left\langle f(k), \psi_{2^j}(x_n^j - k)\right\rangle = \left\langle h(k), \psi_{2^j}(x_n^j - k)\right\rangle$

  Hence the solution to this would be $h(x) = f(x) + g(x)$ with $g(x) \in O$ where O is the orthogonal complement to the space spanned by $\left\{\psi_{2^j}(x_n^j - x)\right\}_{(j,n)\in Z^2}$
- Condition 2 is more difficult to analyze because it is not convex. It can be replaced by an equivalent convex constraint

$$Min \sum_{j=-\infty}^{+\infty} \left(\left\| W_{2^j} h \right\|^2 + 2^{2j} \left\| \frac{dW_{2^j} h}{dx} \right\|^2 \right)$$

Reconstruction Algorithm

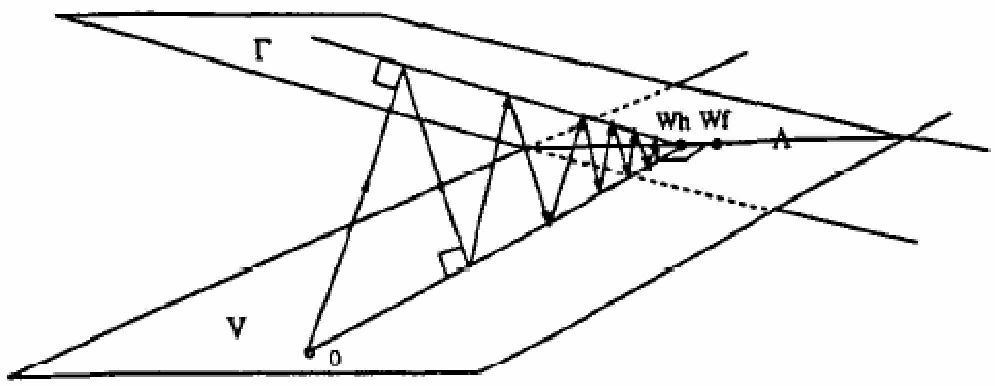
$$Min \sum_{j=-\infty}^{+\infty} \left(\left\| W_{2^j} h \right\|^2 + 2^{2j} \left\| \frac{dW_{2^j} h}{dx} \right\|^2 \right)$$

$$h(x) = f(x) + g(x) \text{ with } g(x) \in O$$

- Instead of computing the solution itself, we reconstruct its wavelet transform with an algorithm based on alternate projections.
- The solutions to condition 1 belong to the space $\Lambda = V \cap \Gamma$ where V is the space of all dyadic wavelet transforms of functions in $L^2(R)$ and Γ is the affine space of sequences of functions $(g_j(x))_{j \in Z}$ such that for any index j and all maxima positions x_n^j, $g_j(x_n^j) = W_{2^j} f(x_n^j)$
- The sequence that satisfies both the conditions is the element of Λ whose norm $|\quad|$ is minimum. This is done by alternately projecting onto V and Γ.

Reconstruction Algorithm

- The projection operator on V is $P_V = WoW^{-1}$ since any dyadic wavelet transform will be invariant under this operator.
- The projection operator P_Γ is implemented by adding piecewise exponentials curves to each function of the sequence that we project on Γ.



Reconstruction Algorithm

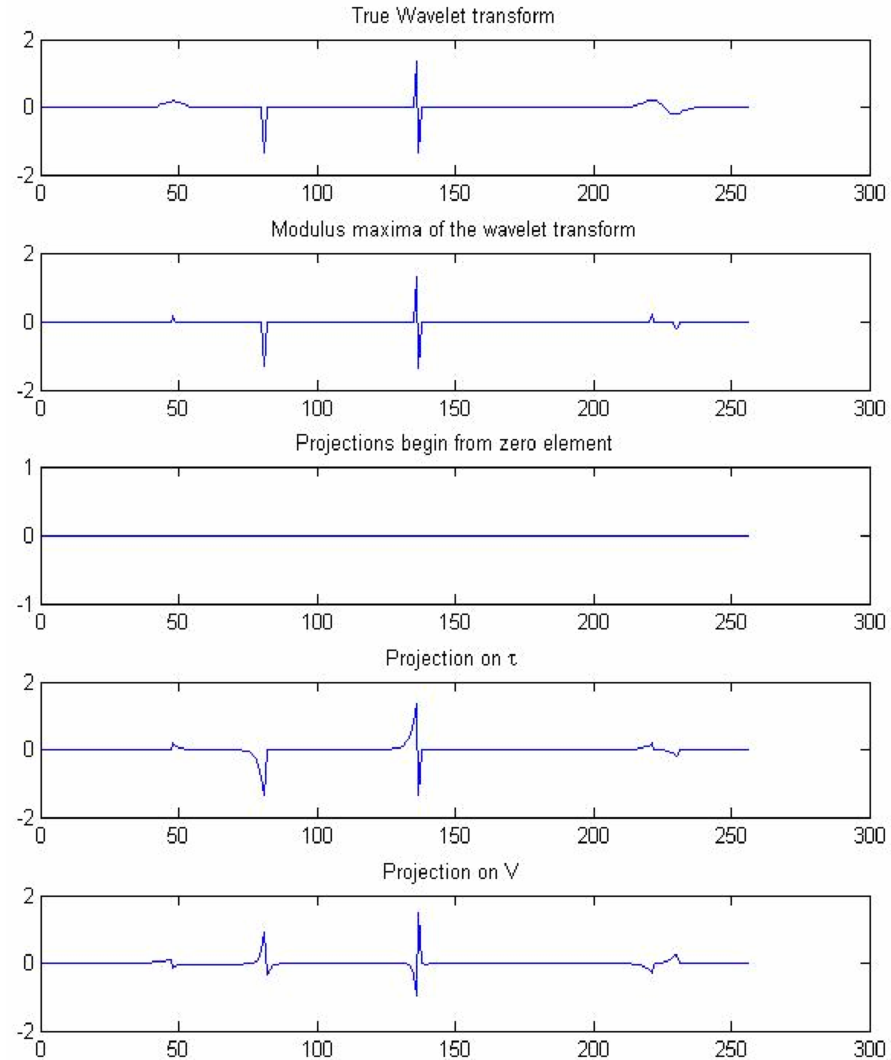
$$\varepsilon_j(x) = \alpha e^{2^{-j}x} + \beta e^{-2^{-j}x}$$

$$\text{where } \varepsilon_j(x_o) = W_{2^j} f(x_o) - g_j(x_o)$$

$$\varepsilon_j(x_1) = W_{2^j} f(x_1) - g_j(x_1)$$

- Any spurious oscillations that may result can be suppressed by imposing sign constraints.

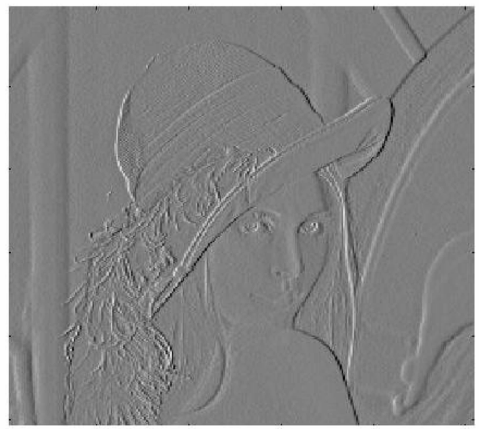
$$\begin{cases} sign(g_j(x) = sign(x_n^j) & \text{if } sign(x_n^j) = sign(x_{n+1}^j) \\ sign(\frac{dg_j(x)}{dx}) = sign(x_{n+1}^j - x_n^j) & \text{if } sign(x_n^j) \neq sign(x_{n+1}^j) \end{cases}$$



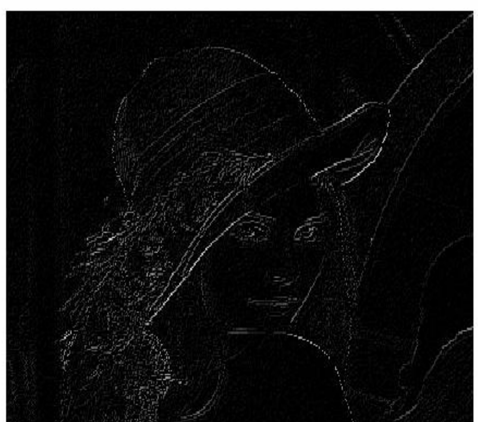
Reconstructed Lena

True Image

Wavelet transform, Wx



Wavelet transform, Wy



Edge Map

Reconstructed Image

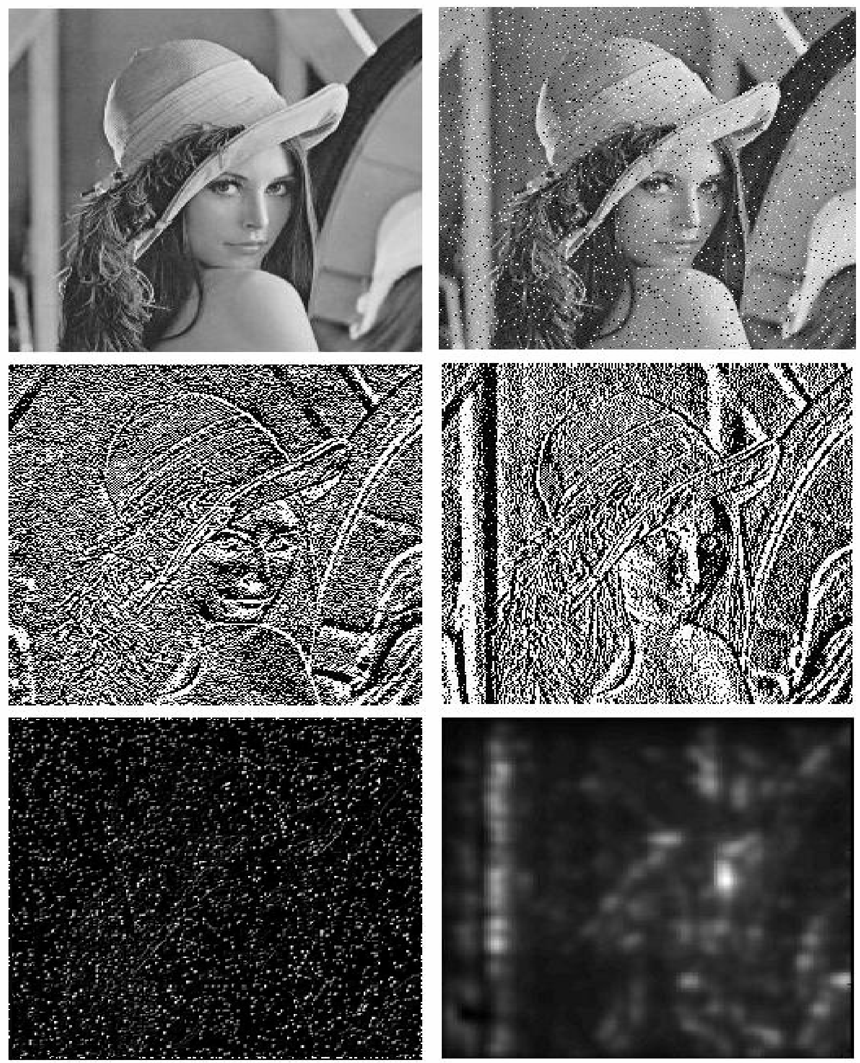
Application

Image Restoration

True Image

Noisy Image

Reconstructed Image



True Image

Image in salt & pepper noise

Wavelet transform, Wy

Wavelet transform, Wx

Edge map!

Reconstructed Image

Can you see Lena???

Drawbacks!!

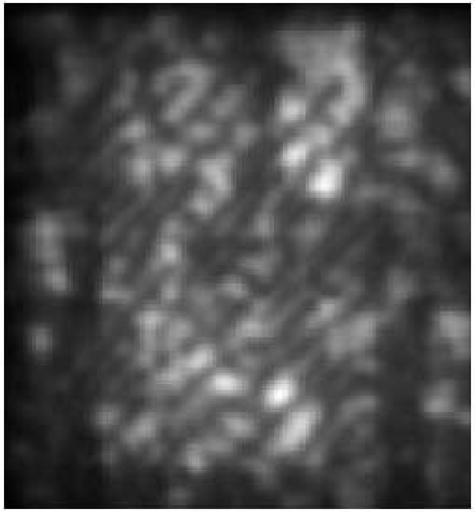
- The completeness of the representation used in the algorithm depends on the choice of the smoothing function *θ(x)* and the conjecture is not valid in general[1].

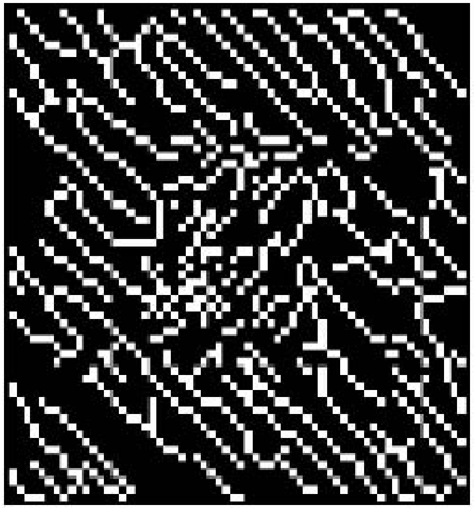
- A discrete analysis of the completeness conjecture was done independently by Berman[2], who found numerical examples that contradict the completeness conjecture.

- Convergence Issues : The computation of the solution might be unstable, in which case, the alternate projections converge very slowly.

1 Meyer, “*Un contre-example a la conjecture de Marr et a celle de S.Mallat*,” 1991
2 Z.Berman, “The uniqueness question of discrete wavelet maxima representation,” Tech. Rep, Univ of Maryland, Apr 1991.

Failure!!





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Thank U