Singularity of Cubic Bézier Curves

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joint work with

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Outline

Introduction

Singularity of a Parametric Curve

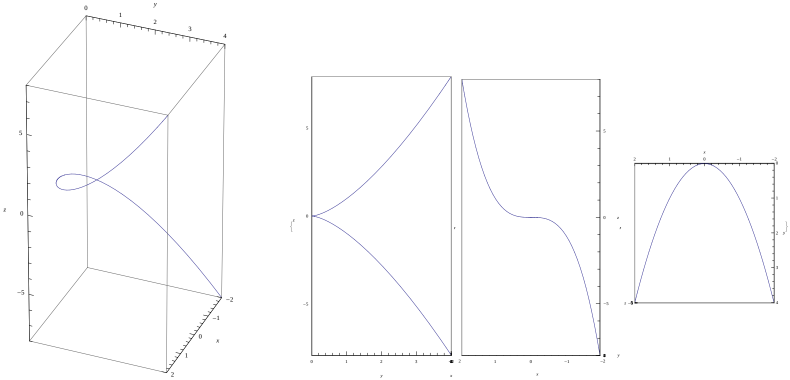
Bézier Curves

Singularity of Bézier Curves

Parametric Cubic Curve

$$\mathcal{C}(t) = \mathbf{a}_0 + \mathbf{a}_1 t + \mathbf{a}_2 t^2 + \mathbf{a}_3 t^3$$

Example (“twisted cubic”): $\mathcal{C}(t) = \left\langle\, t,\; t^2,\; t^3 \,\right\rangle$



Singularity of a Parametric Curve

Singularity of a curve $\mathcal{C}(t)$: t^* where $\mathcal{C}'(t^*) = \vec{0}$

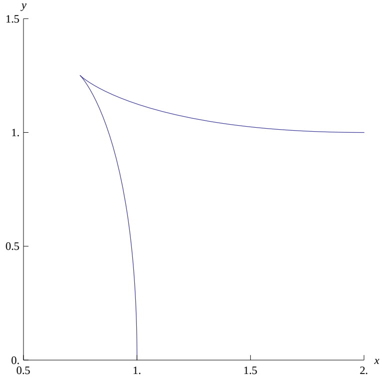
Geometrically, a *cusp*, except when also $\mathcal{C}''(t^*) = 0$,

which for cubic can happen only when curve is a *line*

Example: $\mathcal{C}(t) = \left\langle\, 4t^3 - 3t^2 + 1,\ 4t^3 - 9t^2 + 6t \,\right\rangle,\ t \in [0,1]$

$$\mathcal{C}'(t) = \left\langle\, 12t^2 - 6t,\ 12t^2 - 18t + 6 \,\right\rangle$$

$$t^* = \tfrac{1}{2},\ \mathcal{C}(t^*) = \left\langle\, \tfrac{3}{4},\ \tfrac{5}{4} \,\right\rangle$$

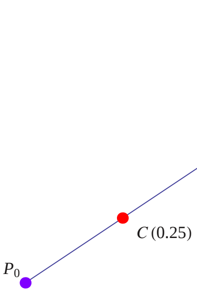


Bézier Curves

- A representation of parametric polynomial curves
- Geometric and intuitive, facilitating creative design process
- Computationally efficient and stable
- At the core of Computer Aided Geometric Design (CAGD)

Bézier Curves of degree 1

$$\mathcal{C}(t) = (1-t)P_0 + t\,P_1,\ t \in [0,1]$$



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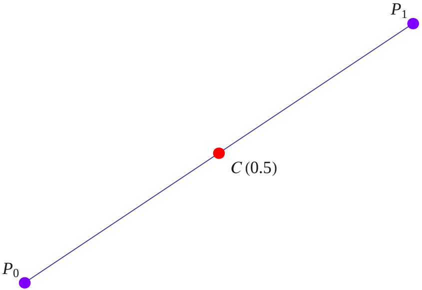

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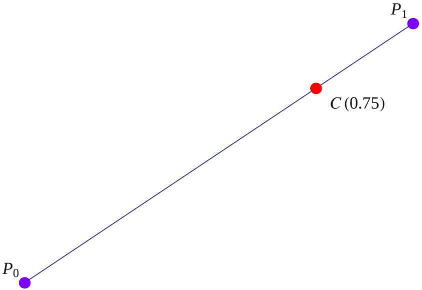

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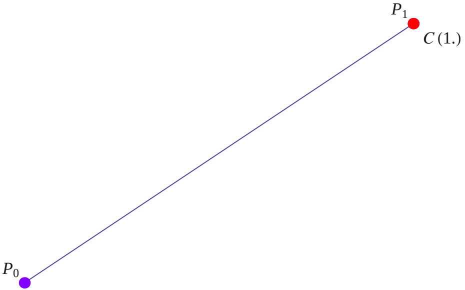

Bézier Curves of degree 2

$$P_{01} = (1-t)P_0 + t\,P_1\,; \qquad P_{12} = (1-t)P_1 + t\,P_2$$

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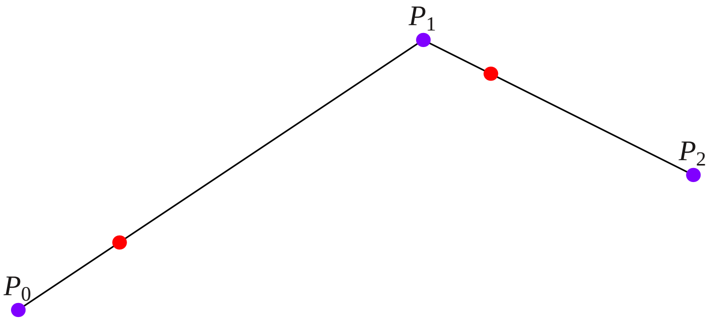

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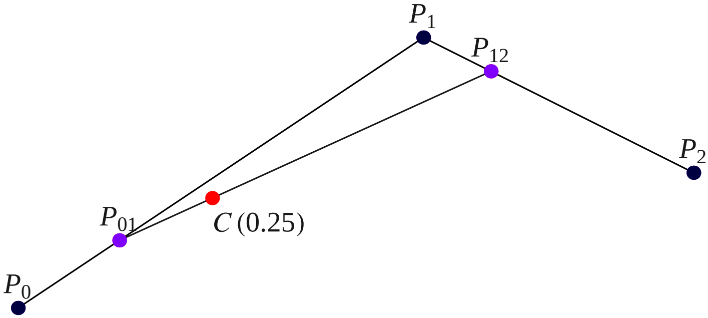

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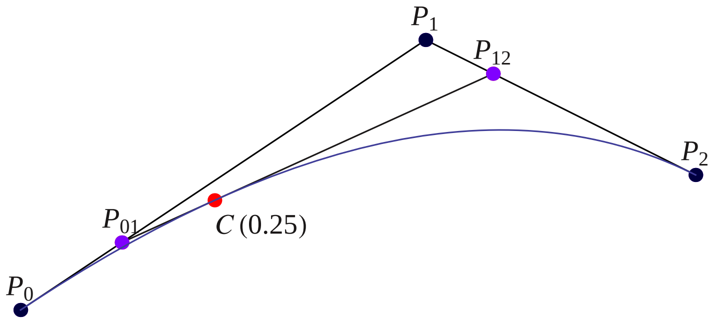

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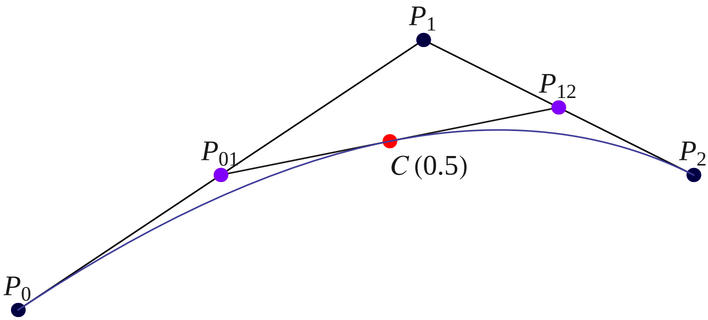

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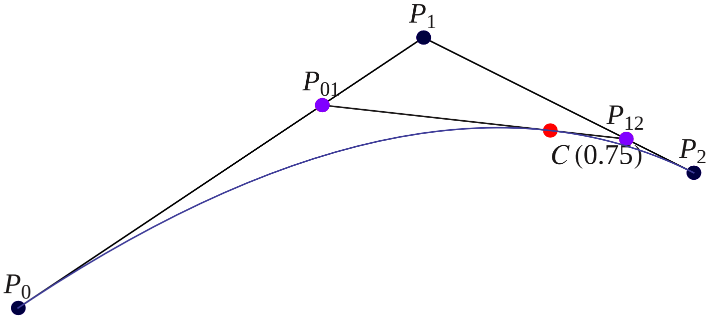

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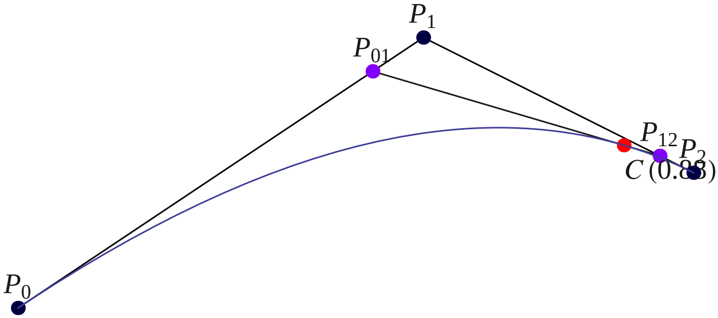

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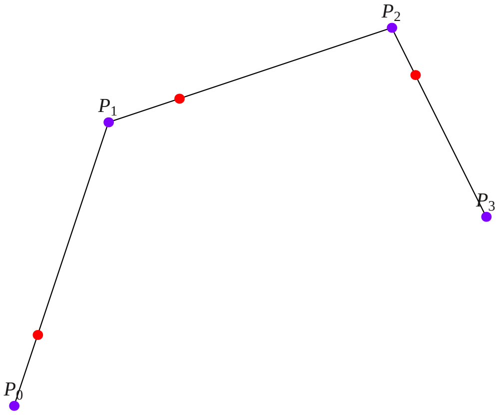

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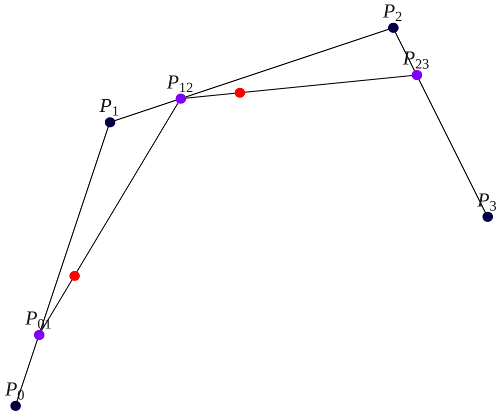

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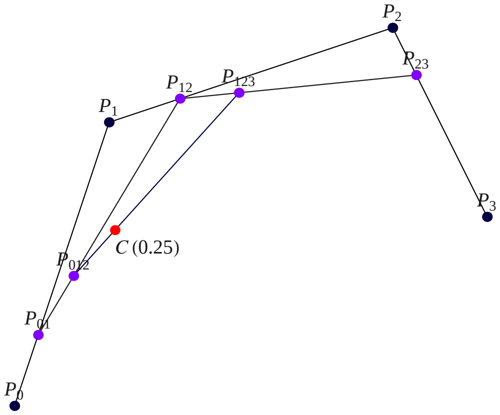

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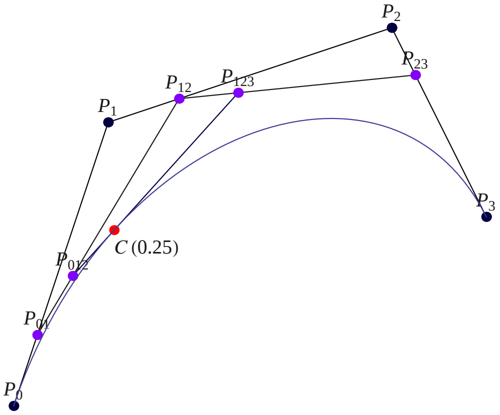

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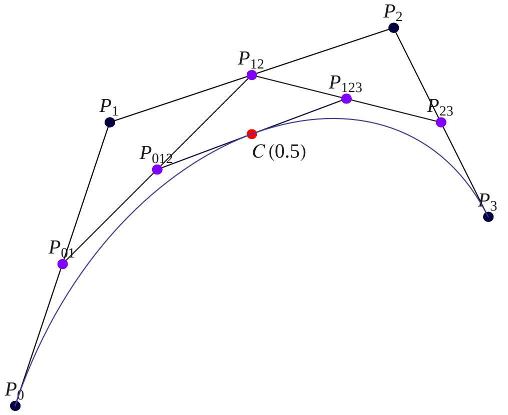

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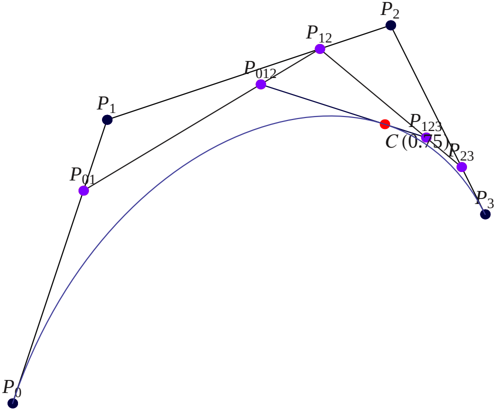

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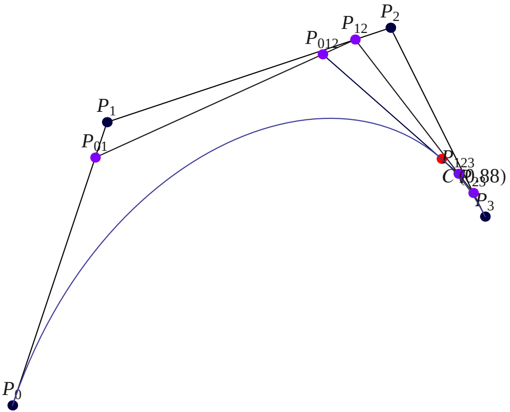

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&\qquad + t\,\Big[(1-t)[(1-t)P_1 + t\,P_2] + t\,[(1-t)P_2 + t\,P_3]\Big] \\
&= (1-t)^3 t^0 P_0 + 3(1-t)^2 t^1 P_1 + 3(1-t)^1 t^2 P_2 + (1-t)^0 t^3 P_3 \\
&= \sum_{i=0}^{3} \binom{3}{i} (1-t)^{3-i} t^i\, P_i
\end{aligned}$$

Bézier Curve – Definition

Degree 3:

$$\mathcal{C}(t) = \sum_{i=0}^{3} \binom{3}{i} (1-t)^{3-i} t^{i} P_i$$
$$= \sum_{i=0}^{3} B_i^3(t) P_i$$

where

$B_i^3(t) = \binom{3}{i} (1-t)^{3-i} t^{i}$ is
the i^{th} *Bernstein* (basis) *polynomial* of degree 3, and

P_i are known as (Bézier) *control points*.

Bézier Curve – Definition

Degree n:

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Bernstein Basis Polynomials

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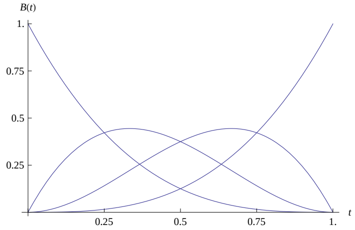
$$\{B_i^3(t)\}_{i=0}^3 = \{(1-t)^3,\ 3(1-t)^2 t,\ 3(1-t)t^2,\ t^3\}$$



Partition of unity:

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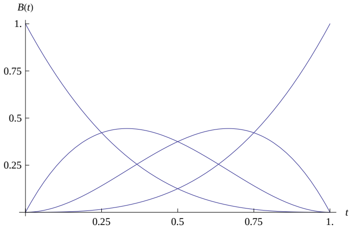

Partition of unity:

$$\begin{aligned}\sum_{i=0}^{n} B_i^n(t) &= \sum_{i=0}^{n} \binom{n}{i} (1-t)^{n-i} t^i \\ &= \left(1 - t\ + t\right)^n \\ &= 1\end{aligned}$$

Historical Notes

- *Bernstein polynomials* were used by Sergei Bernstein in 1910 in his elegant proof of the *Weierstrass Approximation Theorem* (1885): a continuous function on a closed interval can be uniformly approximated by polynomials.
- Bézier curves were first developed in the 1950s in the French automobile industry.

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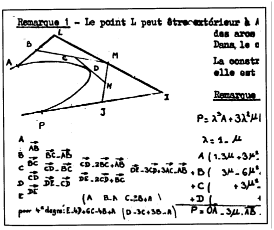
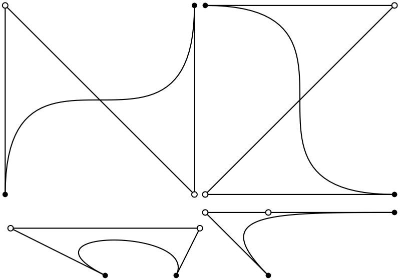
Paul de Casteljau (Citroën) developed in 1959 the geometric algorithm presented – bearing his name – for evaluating points on a Bézier curve. It is the most robust and numerically stable method for evaluating polynomials, and one of the most important algorithms in CAGD.



from de Casteljau's writings

Pierre Bézier (Rénault) also worked on Bézier curves and surfaces, which are now used in most computer-aided design and computer graphics systems.

Examples of Cubic Bézier Curves



from Farin & Hansford 2000

Properties of Bézier Curves

- Endpoint interpolation: $\mathcal{C}(0) = P_0$ and $\mathcal{C}(1) = P_n$
- Endpoint tangency to control polygon: $\mathcal{C}'(0) \| (P_1 - P_0)$ and $\mathcal{C}'(1) \| (P_n - P_{n-1})$
- *Convex Hull Property*: $\mathcal{C}[\{P\}] \subset \mathrm{ConvexHull}(\{P\})$ implies $\left(\{P\} \text{ planar} \Longrightarrow \mathcal{C}[\{P\}] \text{ planar}\right)$
- Convexity preservation for planar curves: $\{P\} \text{ convex} \Longrightarrow \mathcal{C}[\{P\}] \text{ convex}$
- Affine invariance: $\mathcal{C}[\Phi\{P\}] = \Phi\mathcal{C}[\{P\}]$
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Derivative of Bézier Curve

$$\begin{aligned}\mathcal{C}'(t) &= \sum_{i=0}^{n} B_i^{n\,\prime}(t)\, P_i \\ &= \sum_{i=0}^{n} \binom{n}{i} \left((1-t)^{n-i} t^i\right)' P_i \\ &= n \sum_{i=0}^{n-1} B_i^{n-1\,\prime}(t)\, (P_{i+1} - P_i)\end{aligned}$$

That is, the Bézier control points of $\mathcal{C}'$ are simply

$$\{n\,(P_{i+1} - P_i)\}_{i=0}^{n-1}$$

Differentiate a Bézier Curve by *differencing* its control points!

The curve $\mathcal{C}'$ is known as the *hodograph* of the curve $\mathcal{C}$.

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That is, the Bézier control points of $\mathcal{C}'$ are simply

$$\{n\,(P_{i+1} - P_i)\}_{i=0}^{n-1}$$

Differentiate a Bézier Curve by *differencing* its control points!

The curve $\mathcal{C}'$ is known as the *hodograph* of the curve $\mathcal{C}$.

Derivative of Bézier Curve

$$\begin{aligned}
\mathcal{C}'(t) &= \sum_{i=0}^{n} B_i^{n\,\prime}(t)\, P_i \\
&= \sum_{i=0}^{n} \binom{n}{i} \left((1-t)^{n-i} t^{\,i}\right)' P_i \\
&= n \sum_{i=0}^{n-1} B_i^{n-1\,\prime}(t)\, (P_{i+1} - P_i)
\end{aligned}$$

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Singularity of Bézier Curve of degree 1

Recall definition of *singularity* of a curve $\mathcal{C}(t)$:

$$t^* \text{ where } \mathcal{C}'(t^*) = \vec{0}$$

Apply this to Bézier Curve of degree 1:

$$\begin{aligned} \mathcal{C}(t) &= (1-t)P_0 + t\,P_1 \\ \mathcal{C}'(t) &= P_1 - P_0 \\ &= \vec{0} \quad \forall t \quad \text{iff } P_0 = P_1 \end{aligned}$$

That is, the only case of singularity of a polynomial curve of degree 1 is the trivial case when its two endpoints agree!

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Singularity of Bézier Curve of degree 2 – part 1

$\mathcal{C}'(t^*) = \vec{0}$ for $n = 2$:

$$\begin{aligned} \mathcal{C}(t) &= B_0^2 P_0 + B_1^2 P_1 + B_2^2 P_2 \\ \mathcal{C}'(t) &= B_0^1(P_1 - P_0) + B_1^1(P_2 - P_1) \quad \text{by derivative formula} \\ &= (1-t)(P_1 - P_0) + t\,(P_2 - P_1) \end{aligned}$$

Hence, the only cases of singularity of a polynomial curve of degree 2 occur when its Bézier control points satisfy

$$(P_1 - P_0) \| (P_2 - P_1)$$

i.e., they are *collinear*

Hence, by the *Convex Hull Property*, the curve actually lies on a *line*.

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Singularity of Bézier Curve of degree 2 – part 2

Equation for singularity:

$$\mathcal{C}'(t^*) = (1 - t^*)(P_1 - P_0) + t^*\,(P_2 - P_1) = \vec{0},\ t^* \in [0, 1]$$

For singularity, in addition to being collinear, must have P_0, P_1, P_2 “out of order”, i.e.,
P_0 between P_1 and P_2: $t^* \in [0, \frac{1}{2}]$
OR
P_2 between P_0 and P_1: $t^* \in [\frac{1}{2}, 1]$

In all cases, the singularity is at P_1; curve reverses direction there.

Special cases of coincident adjacent end control points:

- If $P_0 = P_1$, singularity there at $t = 0$
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Singularity of Bézier Curve of degree 3 – part 1

Basics

$\mathcal{C}'(t^*) = \vec{0}$ for $n = 3$:

$$\begin{aligned} \mathcal{C}(t) &= B_0^3 P_0 + B_1^3 P_1 + B_2^3 P_2 + B_3^3 P_3 \\ \tfrac{1}{3}\mathcal{C}'(t) &= B_0^2(P_1 - P_0) + B_1^2(P_2 - P_1) + B_2^2(P_3 - P_2) \\ &= (1-t)^2(P_1 - P_0) + 2(1-t)t\,(P_2 - P_1) + t^2(P_3 - P_2) \end{aligned}$$

Hence, the only cases of singularity of a polynomial curve of degree 3 occur when a linear combination of $\{(P_1 - P_0), (P_2 - P_1), (P_3 - P_2)\}$ equals $\vec{0}$

Hence, for singularity, these three vectors, and hence, $\{P_0, P_1, P_2, P_3\}$ themselves, must be *coplanar*

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Singularity of Bézier Curve of degree 3 – part 2

Construct singular curve, given some *translate* of its hodograph

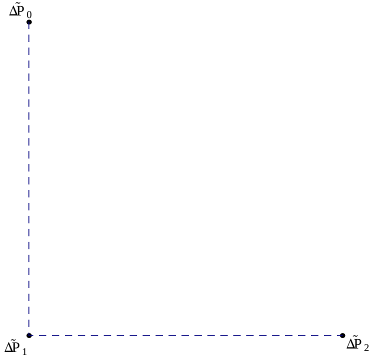
$\Delta P_i = P_{i+1} - P_i$

Hodograph

$\mathcal{C}[\{P\}]' = \mathcal{C}[\{3\Delta P\}] \Longrightarrow$

$$\boxed{\text{singularity} : \mathcal{C}[\{\Delta P\}](t^*) = \vec{0}}$$

$\mathcal{C}[\{\widetilde{\Delta P}\}](t)$, $\widetilde{\Delta P_i} = \Delta P_i + \vec{C}$



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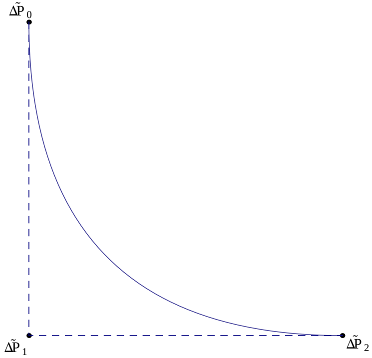

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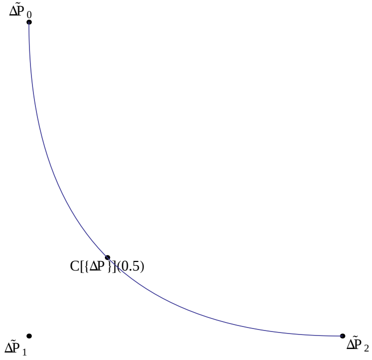

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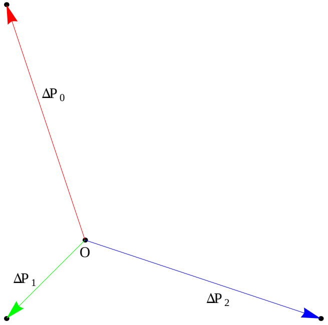

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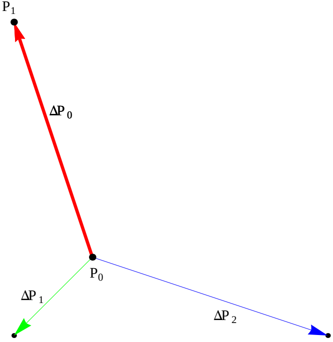
$P_0 = \vec{0}$ (arbitrary)
$P_1 = P_0 + \Delta P_0$



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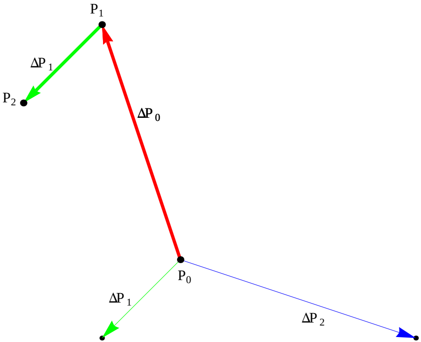

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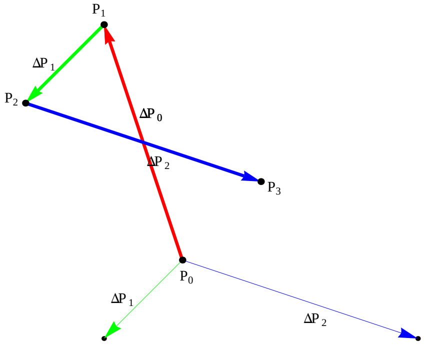

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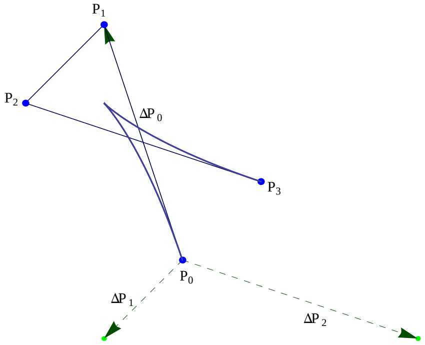
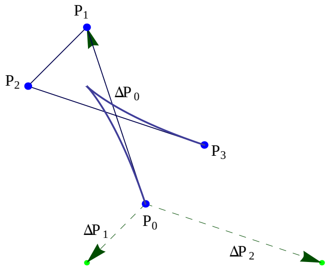
$\mathcal{C}[\{P\}]$ *singular*, with
$\mathcal{C}[\{P\}]'(t^*) = \vec{0}$



Singularity of Bézier Curve of degree 3 – part 3

Examples of singular cubics with various values of t^*, using the construction:



t_sing=0.5

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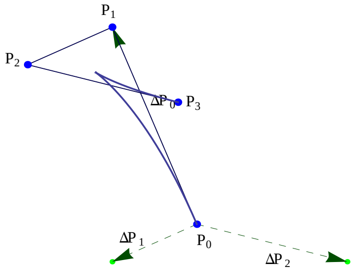

t_sing=0.6

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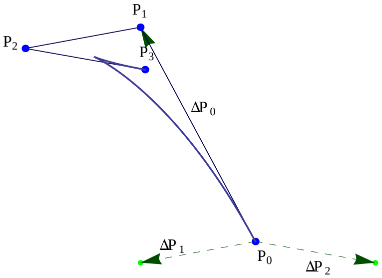

t_sing=0.7

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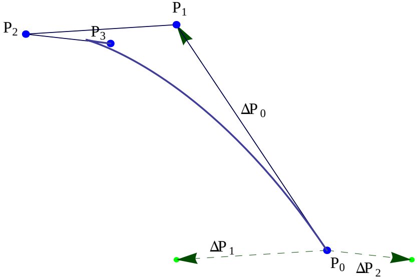

t_sing=0.8

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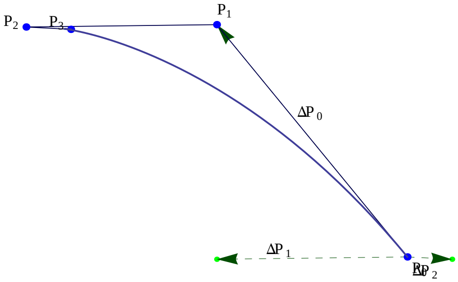


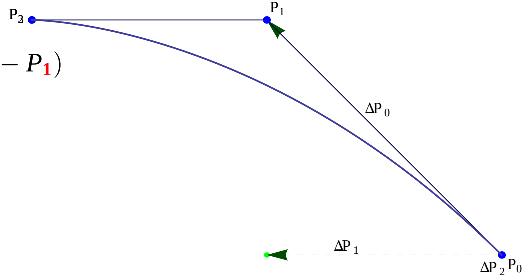
t_sing=0.9

Singularity of Bézier Curve of degree 3 – part 3

Examples of singular cubics with various values of t^*, using the construction:

$P_2 = P_3$

tangent@sing. $\| (P_3 - P_{\mathbf{1}})$



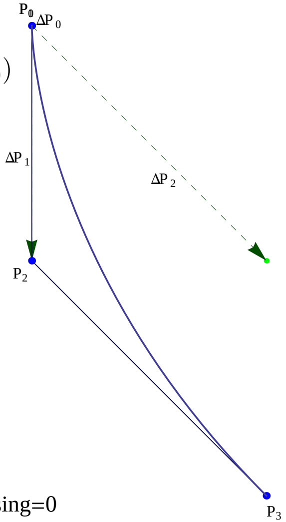
t_sing=1.

Singularity of Bézier Curve of degree 3 – part 3

Examples of singular cubics with various values of t^*, using the construction:

$$P_1 = P_0$$

tangent@sing. $\parallel (P_{\mathbf{2}} - P_0)$



t_sing=0

Singularity of Bézier Curve of degree 3 – part 3

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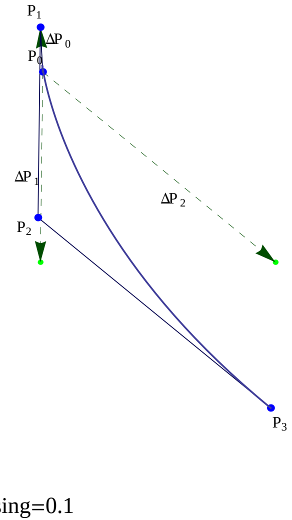

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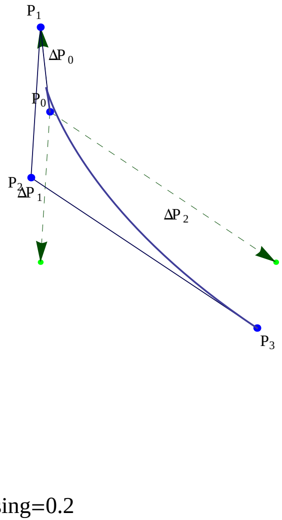

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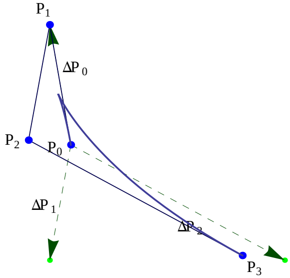

t_sing=0.3

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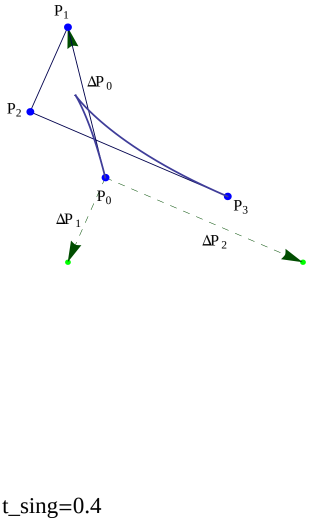
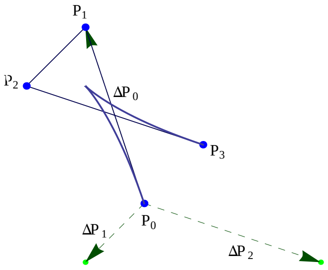


Singularity of Bézier Curve of degree 3 – part 3

Examples of singular cubics with various values of t^*, using the construction:



t_sing=0.5

Singularity of Bézier Curve of degree 3 – part 4a

Solution for singularity using Bézier singularity condition

Define points
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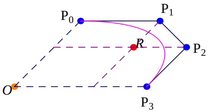

From the geometry above, $R - O = \Delta P_0 - \Delta P_2$,

$$\Delta P_0 \parallel (P_3 - O) \text{ and } \Delta P_2 \parallel (P_0 - O)$$

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with (x, y) capturing the essential shape of the control polygon.
Under the Bézier singularity condition $\mathcal{C}[\{\Delta P\}](t^*) = \vec{0}$, $(1), (2) \longrightarrow$

$$(x, y) = \left(\frac{2t^*}{3t^*-1}, \frac{2(1-t^*)}{2-3t^*}\right), \quad \text{which satisfies} \quad \boxed{\left(x - \frac{4}{3}\right)\left(y - \frac{4}{3}\right) = \frac{4}{9} \quad (*)}$$

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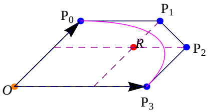

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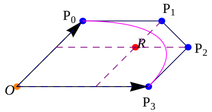

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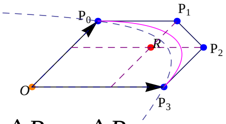

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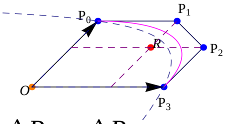

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Condition $(*)$ for singularity was found by [Su & Liu 1990] using other methods that did not make essential use of the Bézier form.

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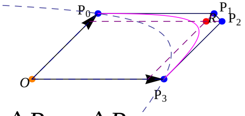

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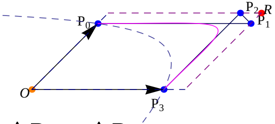

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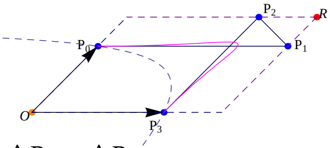

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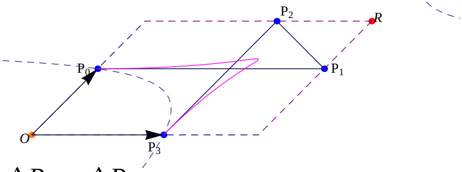

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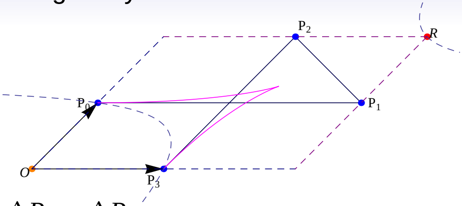

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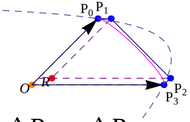

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Additional case: special *doubly* degenerate case of $(x, y) = (0, 0) \implies$
$P_1 = P_0$ & $P_2 = P_3 \implies$ singular at $t = 0$ & $t = 1$

Singularity of Bézier Curve of degree 3 – part 4a

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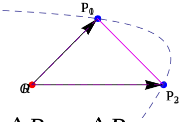

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Singularity of Bézier Curve of degree 3 – part 4b

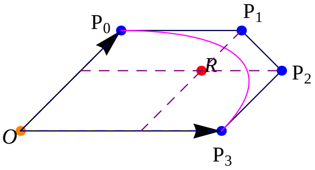
Summary of main result

Define affine coordinates (x, y) of the control polygon of a cubic Bézier curve by $R - O = (P_3 - O)x + (P_0 - O)y$; see graph at bottom left.

The curve has a singularity at $t = t^*$ iff

$$(x, y) = \left(\frac{2t^*}{3t^* - 1}, \frac{2(1 - t^*)}{2 - 3t^*} \right), \text{ which satisfies } \boxed{\left(x - \frac{4}{3}\right)\left(y - \frac{4}{3}\right) = \frac{4}{9}}$$

or *two* singularities at $t^* \in \{0, 1\}$, for the case $(x, y) = (0, 0)$.



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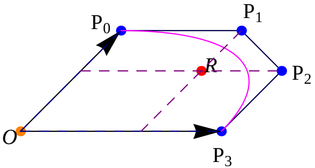

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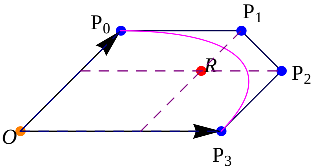
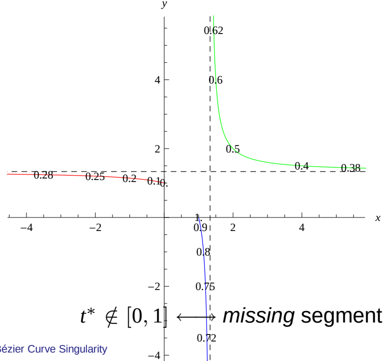
The x-y hyperbola,with some values of t^*:



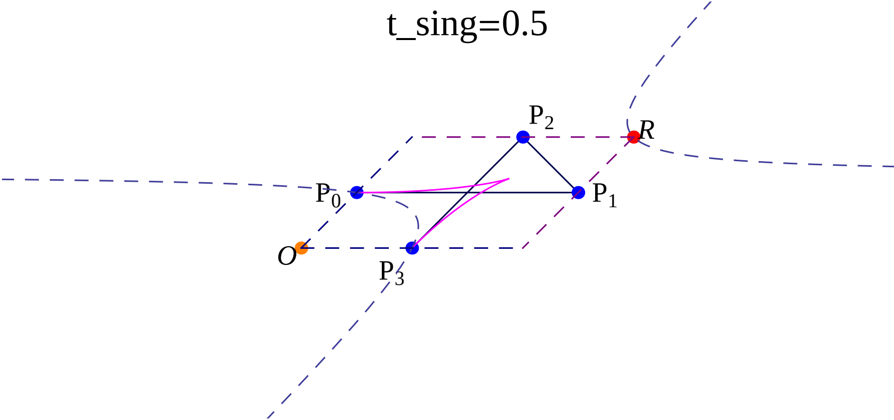


$t^* \notin [0, 1] \longleftrightarrow$ *missing* segment

Singularity of Bézier Curve of degree 3 – part 5a

Examples of singular solution

Singular cubic curves with various values of $t^* \in [0,1]$, with P_0,P_3, and the *directions* of $P_1 - P_0$ and $P_3 - P_2$ fixed.



Singularity of Bézier Curve of degree 3 – part 5a

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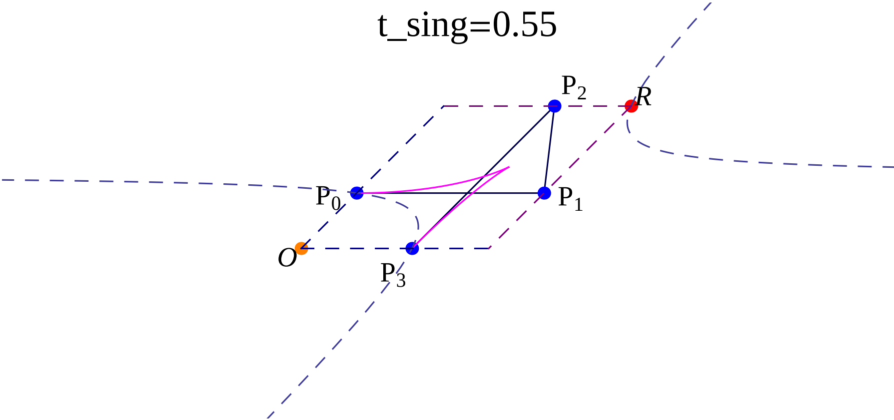

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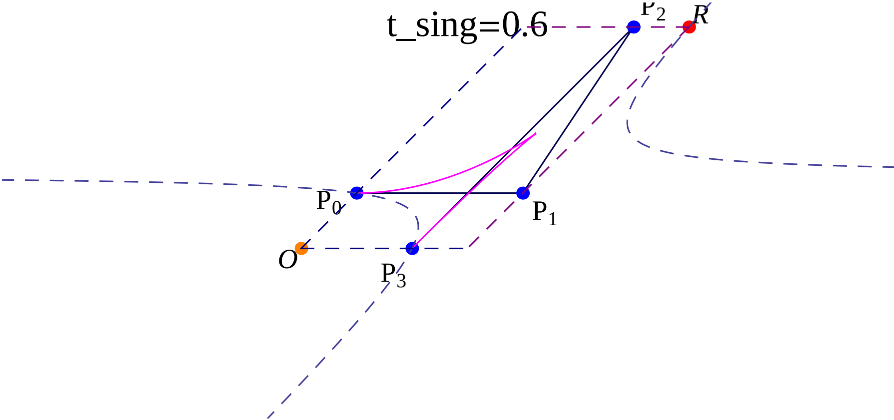

Singularity of Bézier Curve of degree 3 – part 5a

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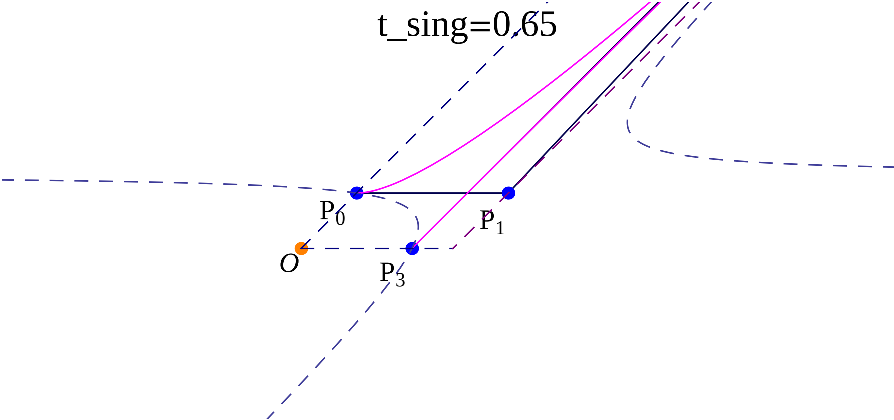

Singularity of Bézier Curve of degree 3 – part 5a

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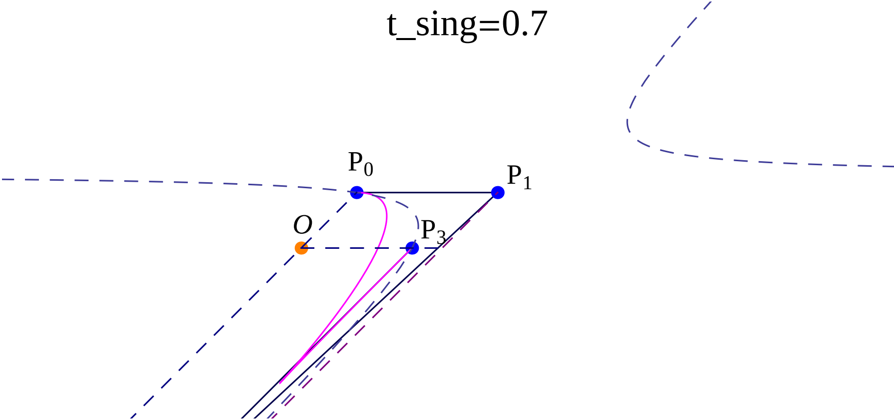

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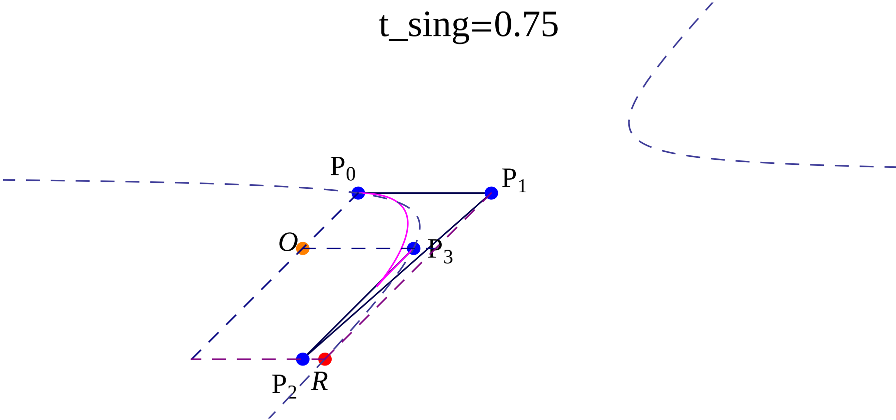

Singularity of Bézier Curve of degree 3 – part 5a

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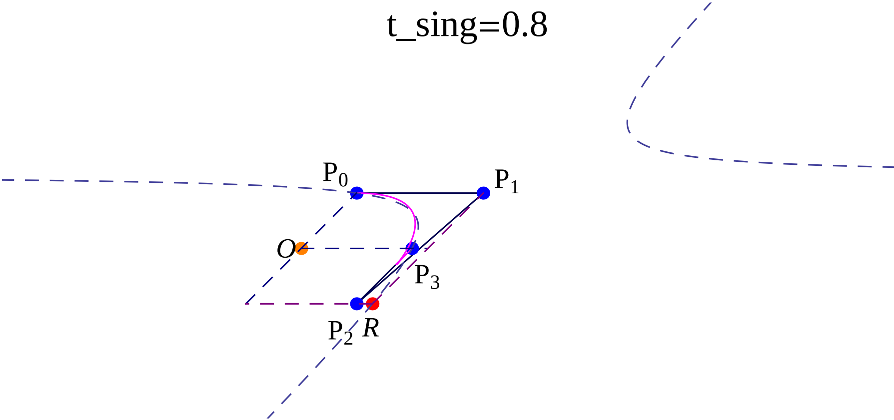

Singularity of Bézier Curve of degree 3 – part 5a

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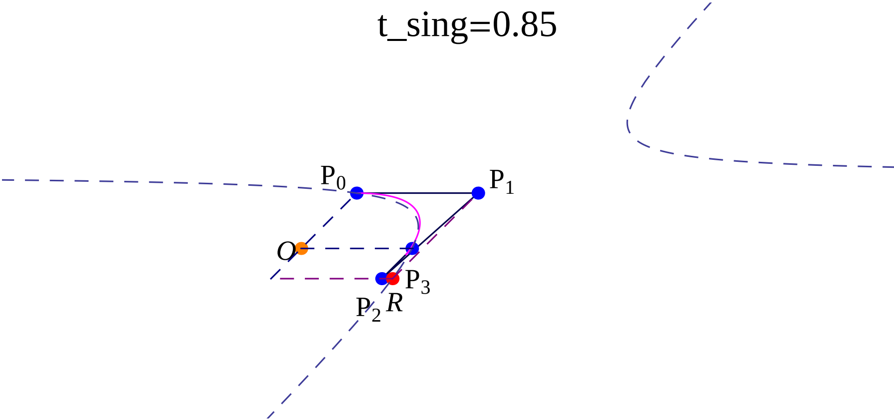

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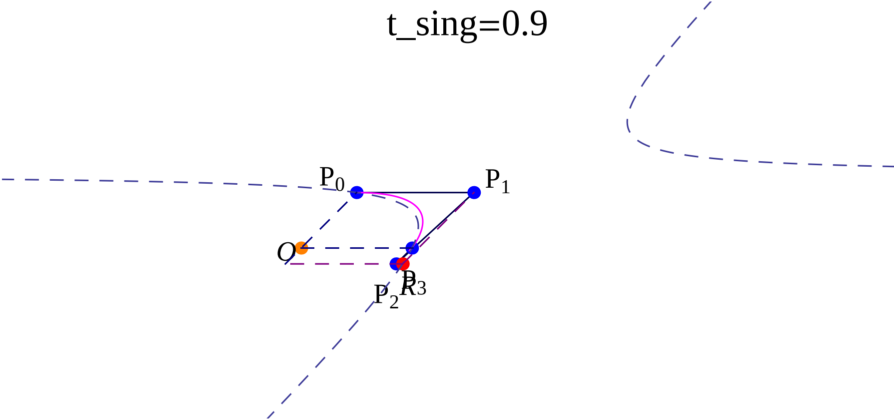

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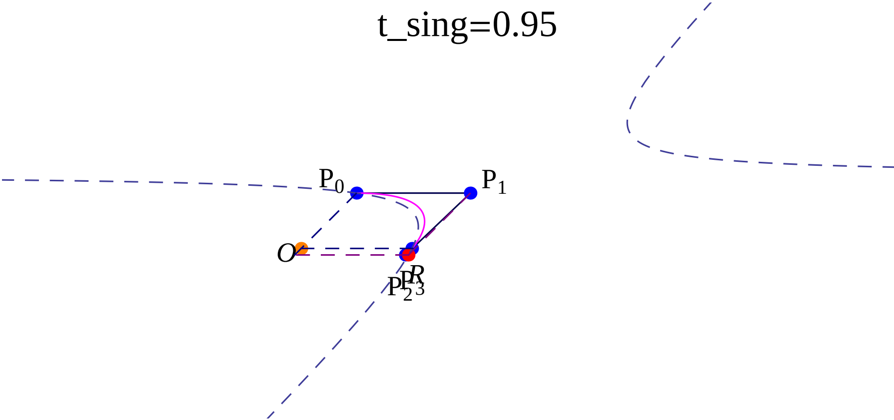

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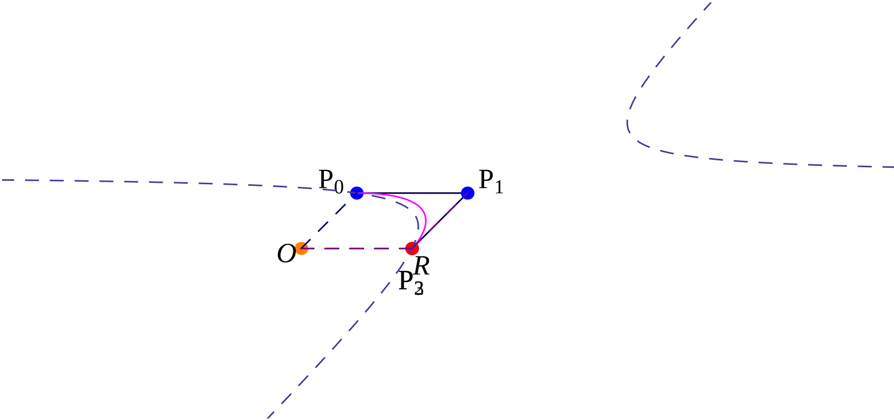

Singularity of Bézier Curve of degree 3 – part 5a

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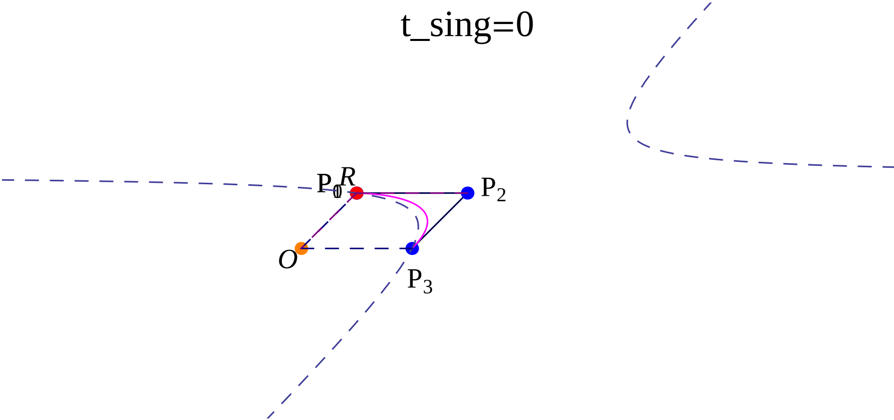

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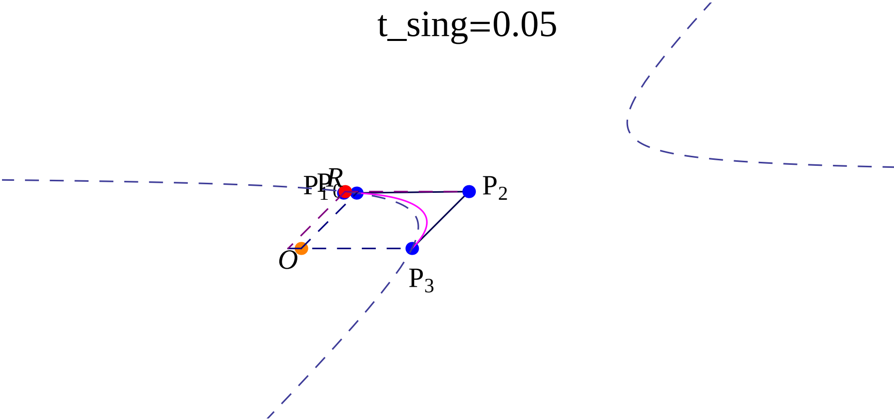

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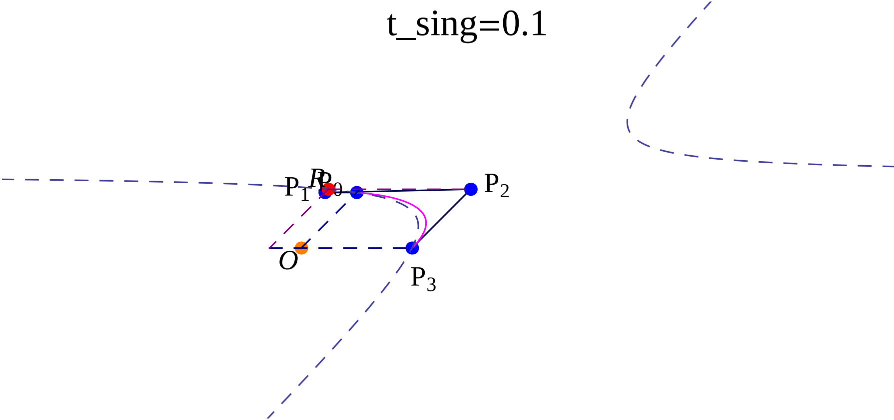

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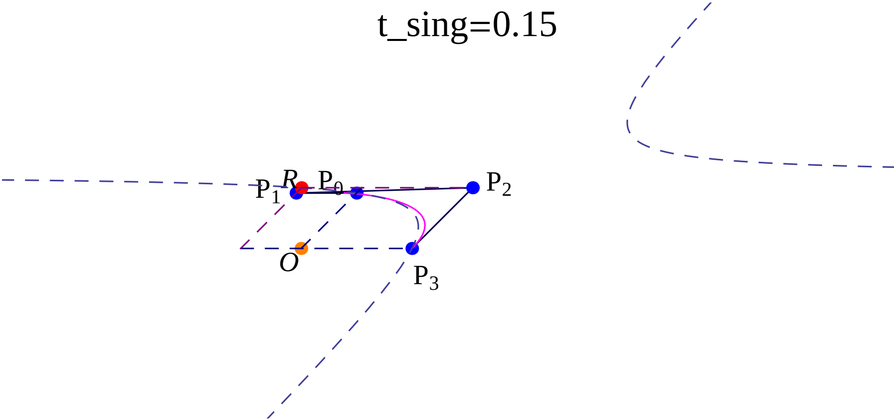

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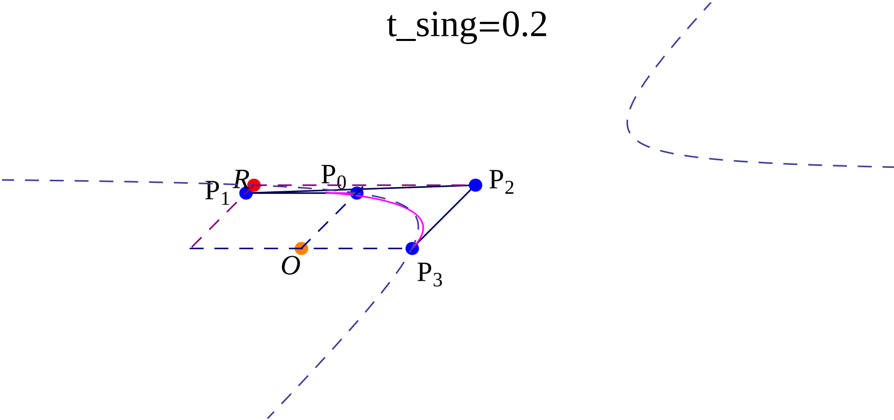

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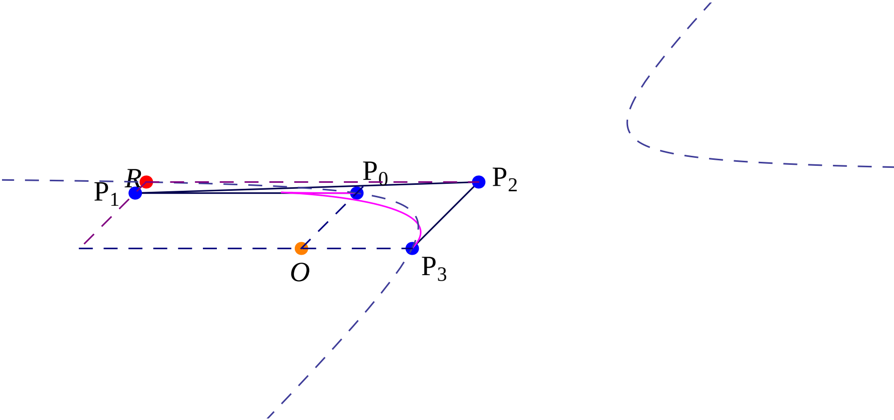

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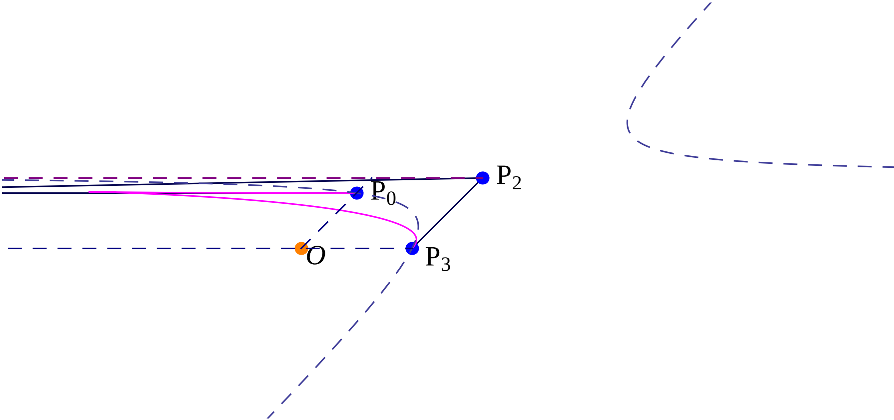

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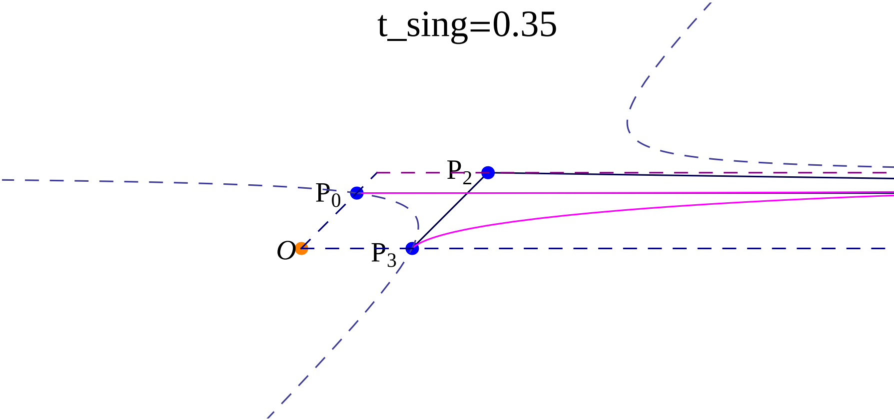

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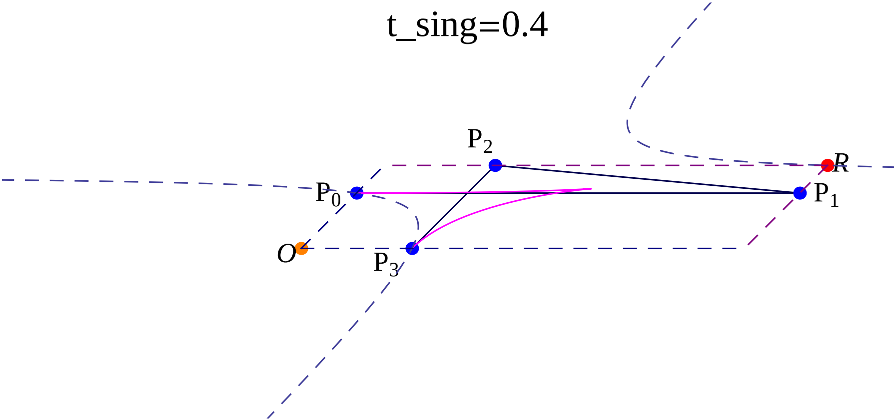

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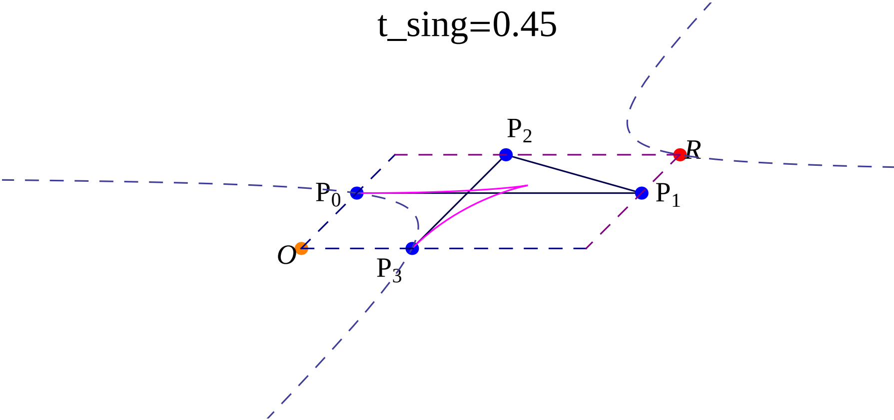

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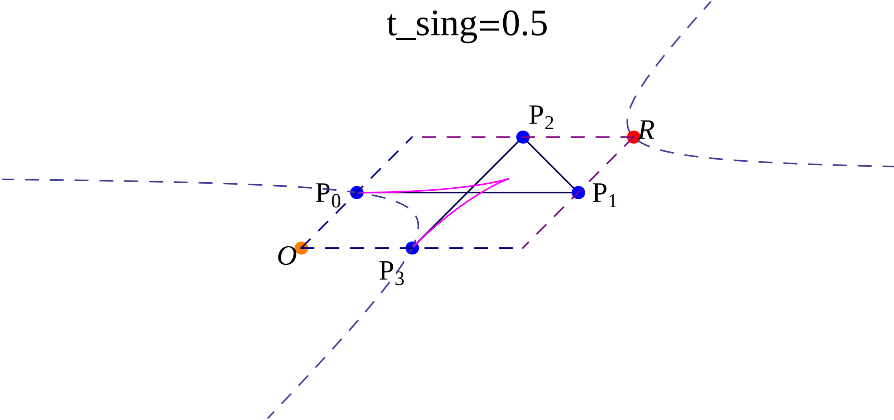
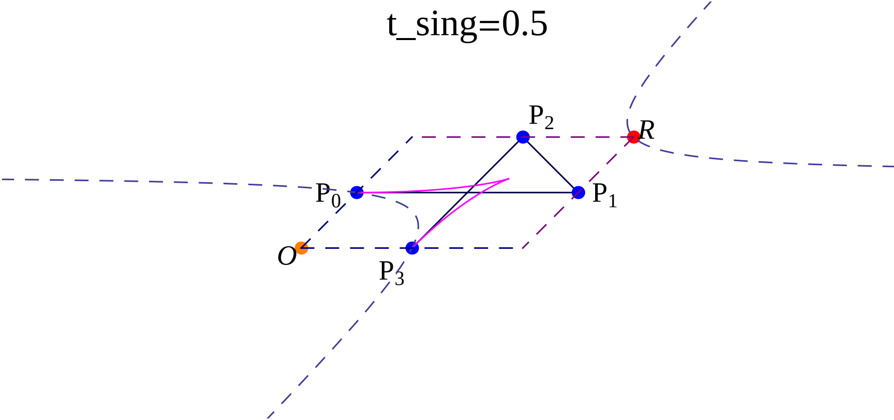


Singularity of Bézier Curve of degree 3 – part 5b

Examples of cubic curves in x-y space

A "tour" of cubic curves in other regions of x-y space, again, with P_0,P_3, and the *directions* of $P_1 - P_0$ and $P_3 - P_2$ fixed.



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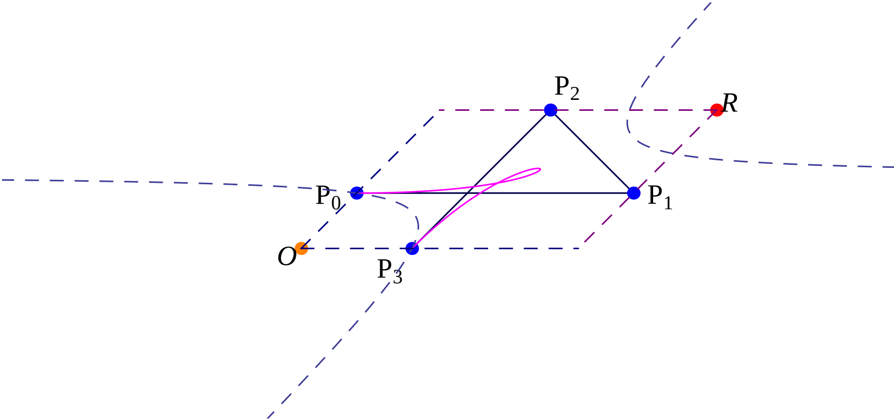

Singularity of Bézier Curve of degree 3 – part 5b

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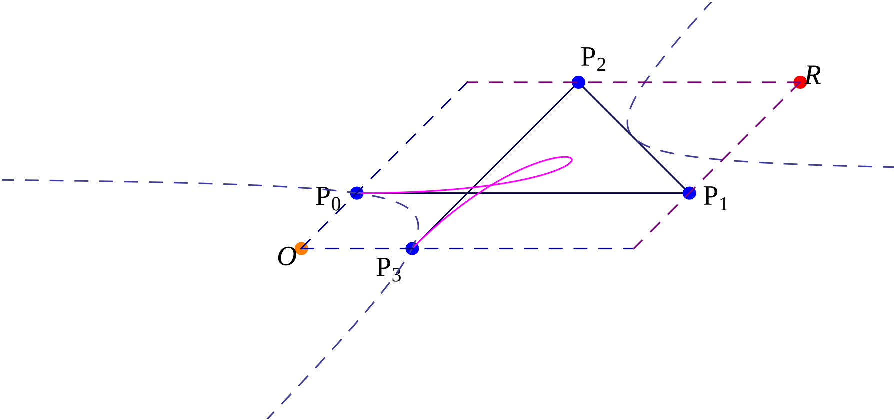

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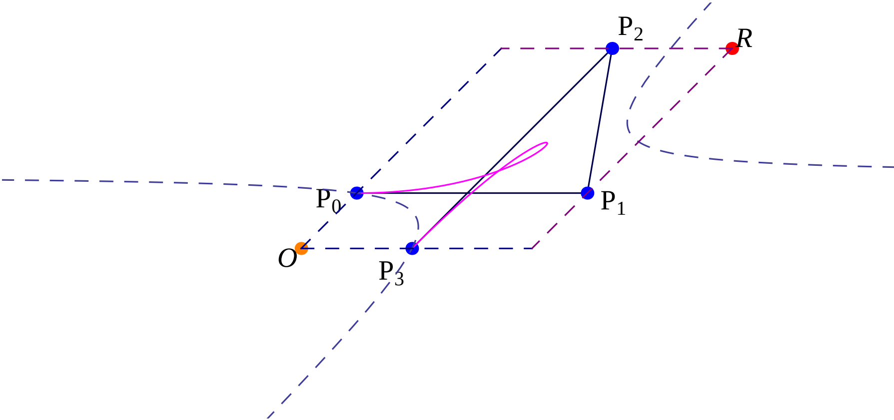

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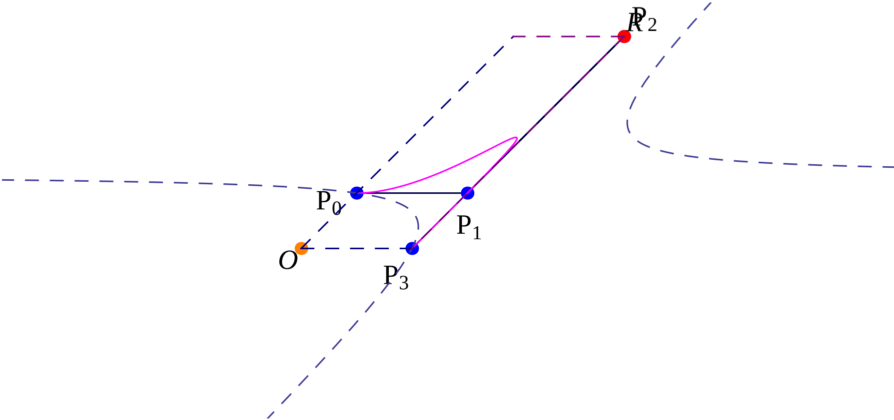

Singularity of Bézier Curve of degree 3 – part 5b

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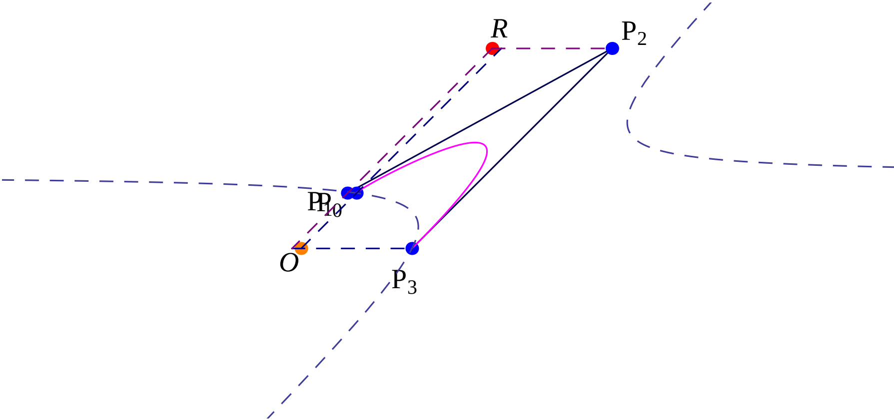

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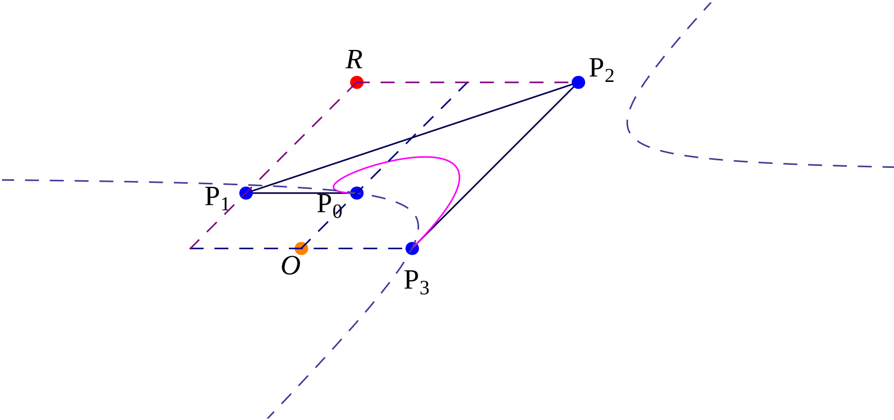

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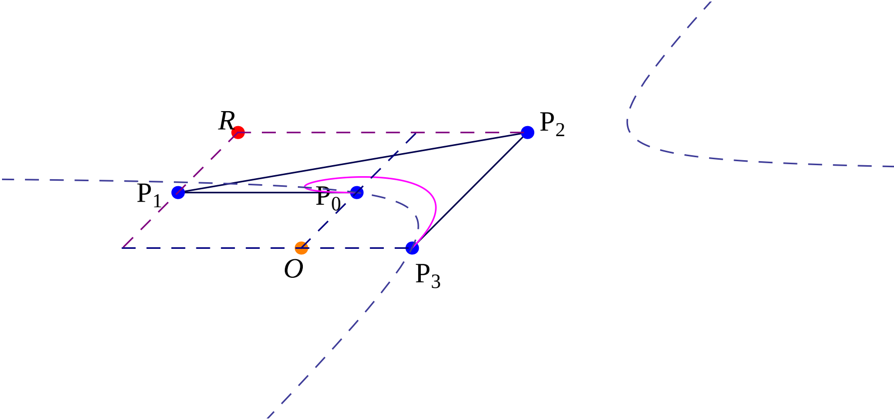

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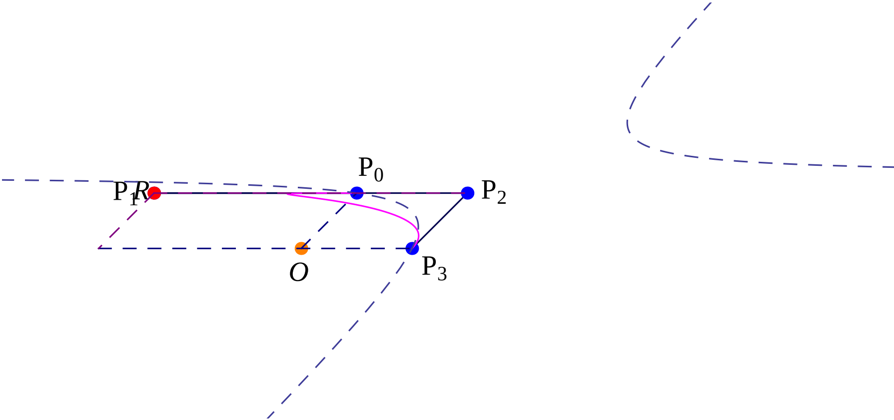

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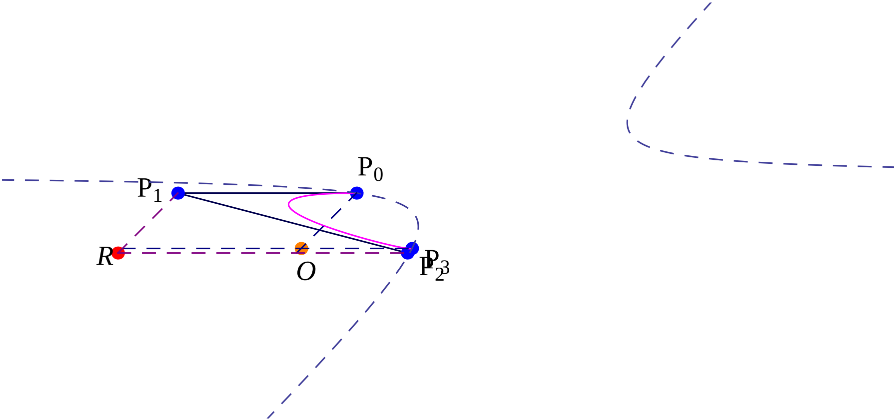

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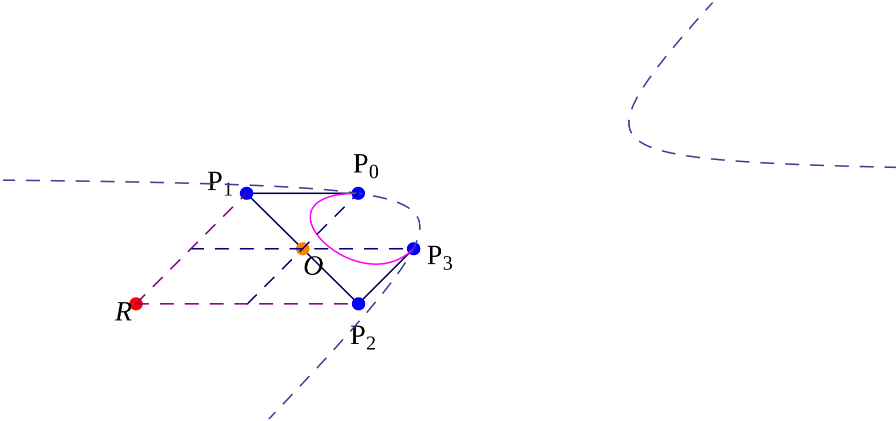

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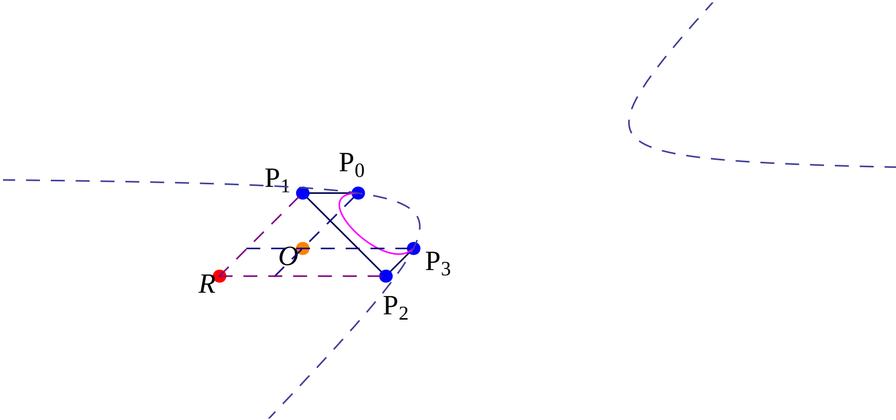

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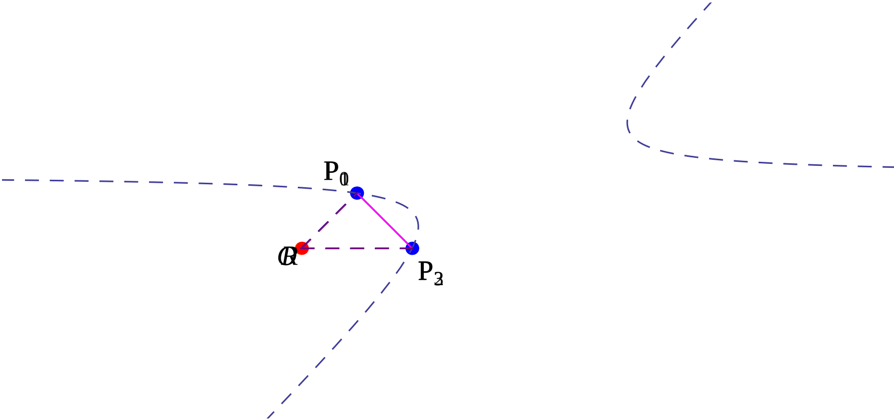

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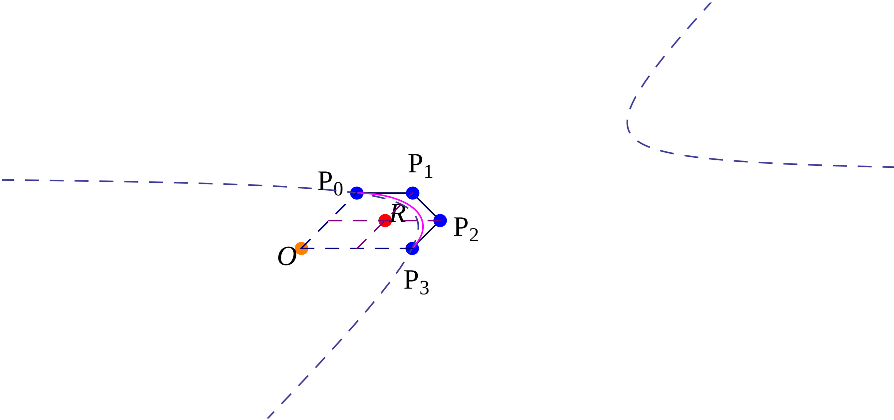

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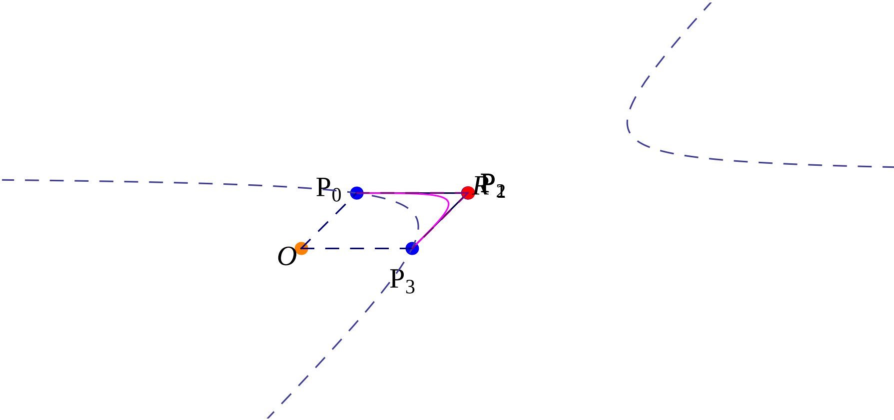

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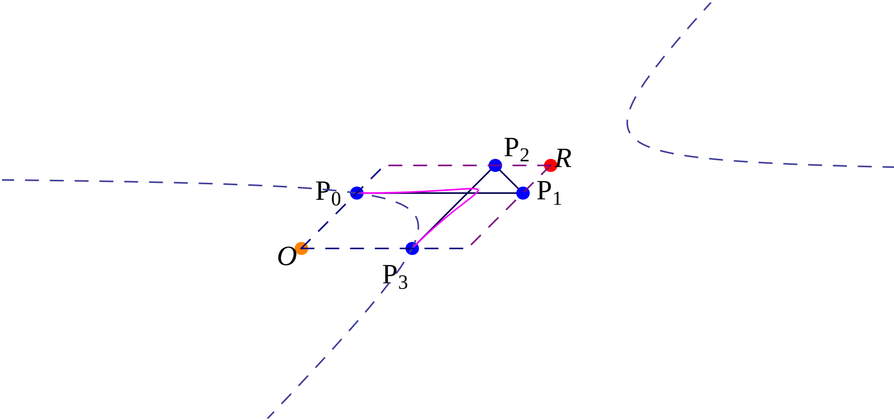

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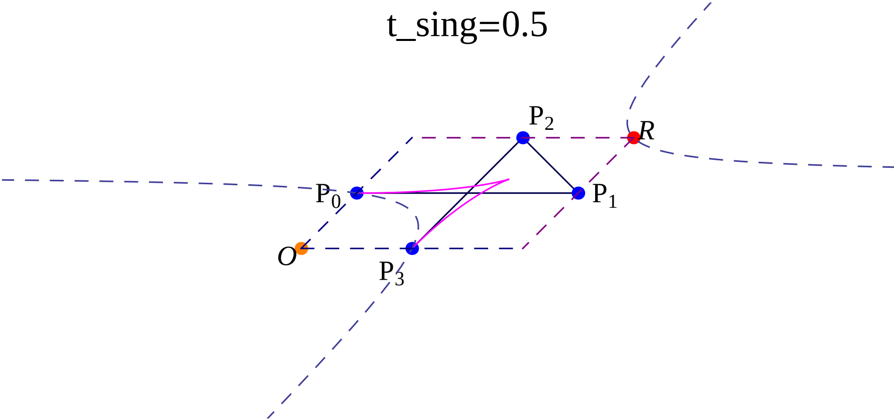

A complete description cubic curve shapes in x-y space is given in [Su & Liu 1990].

Singularity of Bézier Curve of degree 3 – part 6a

Interval **interior like endpoints** – coincident end-control points

Seen $t^* = 0 \Longrightarrow P_0 = P_1\,;\; t^* = 1 \Longrightarrow P_2 = P_3$.
For $t^* \in (0,1)$, can also regard singularity as coincident end-control points (from e.g., Farin & Hansford 2000). . .

$\mathcal{C} = \mathcal{C}^- \cup \mathcal{C}^+$, with domains $[0, \hat{t}]$ & $[\hat{t}, 1]$,
control points $\{P_i^-\}$ & $\{P_i^+\}, i = 0, 1, 2, 3$, respectively,
with $P_3^- = P_0^+ = \mathcal{C}(\hat{t})$

If $\hat{t} = t^*$, then, also, $P_2^- = P_3^-$ & $P_1^+ = P_0^+ \Longrightarrow P_2^- = P_1^+$

de Casteljau algorithm, revisited:
Subdivision:
$\{P_i^-\} = \{P_0, P_{01}, P_{012}, P_{0123}\}$
$\{P_i^+\} = \{P_{0123}, P_{123}, P_{23}, P_3\}$
$P_{0123} = \mathcal{C}(t^*)$

$$P_2^- = P_1^+ \Longrightarrow P_{012} = P_{123}:$$

$$[(1-t^*)P_{01} + t^* P_{12}] = [(1-t^*)P_{12} + t^* P_{23}]$$

$$\ldots \quad \mathcal{C}[\{P\}]'(t^*) = \vec{0}$$

Singularity of Bézier Curve of degree 3 – part 6a

Interval **interior like endpoints** – coincident end-control points

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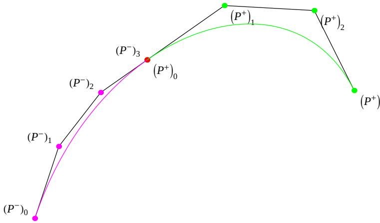
$P_{0123} = \mathcal{C}(t^*)$



Singularity of Bézier Curve of degree 3 – part 6a

Interval **interior like endpoints** – coincident end-control points

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de Casteljau algorithm, revisited:
Subdivision:
$\{P_i^-\} = \{P_0, P_{01}, P_{012}, P_{0123}\}$
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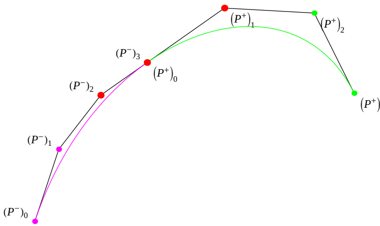

Singularity of Bézier Curve of degree 3 – part 6a

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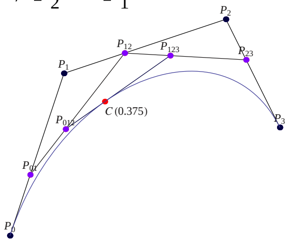

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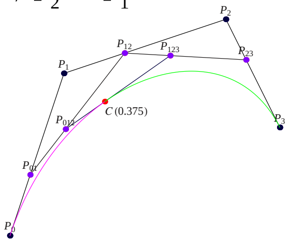

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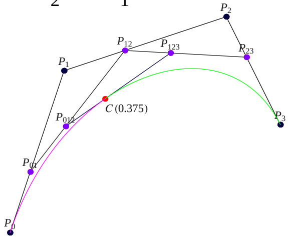

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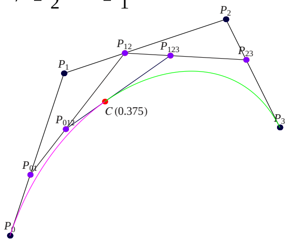

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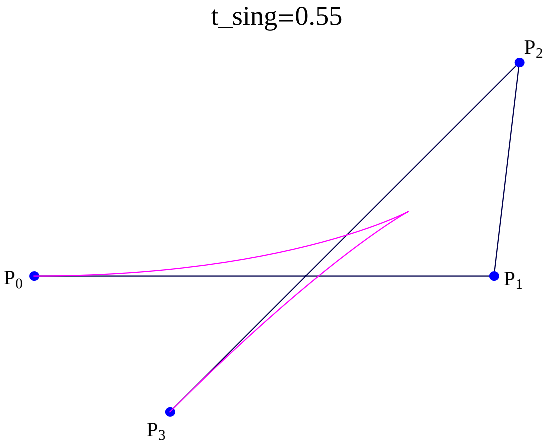
Singularity of Bézier Curve of degree 3 – part 6b

Example: Interval interior like endpoints

Convergence of de Casteljau points P_{012} and P_{123} as $t \longrightarrow t^*$:



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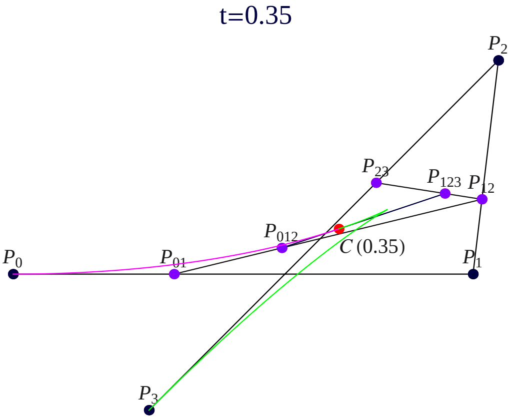

Singularity of Bézier Curve of degree 3 – part 6b

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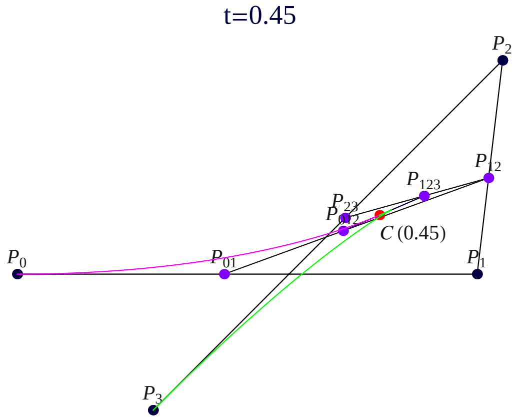

Singularity of Bézier Curve of degree 3 – part 6b

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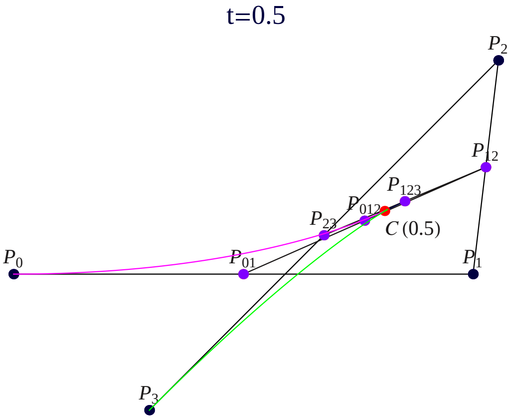

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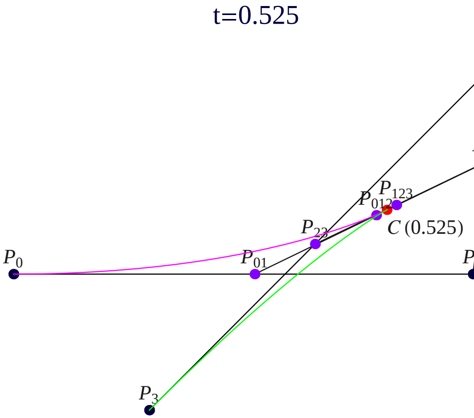

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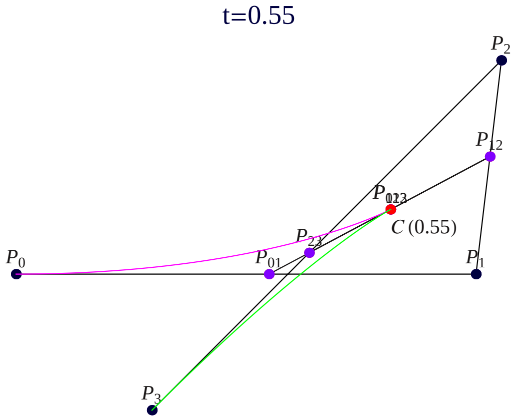

Singularity of Bézier Curve of degree 3 – part 6c

Interval **endpoints like interior** – cusp

Seen how singularity in the *interior* of the parameter interval $[0, 1]$ is like one on the *ends*: characterized by coincident end control points (of the two control polygons of de Casteljau subdivision)

Also, singularity at *ends* is like one in the *interior*: exhibits a *cusp* $\cdots$ if parameter interval is extended *beyond* $[0, 1]$:

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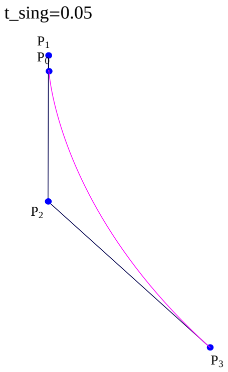

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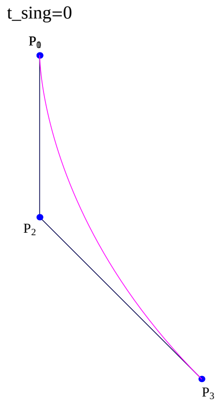

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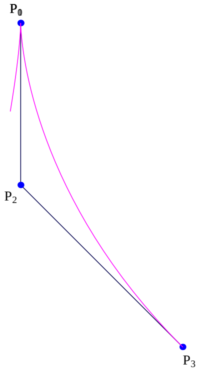

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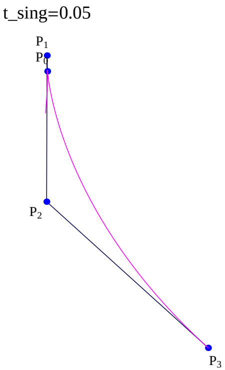

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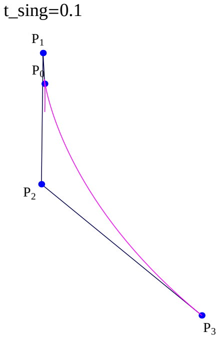

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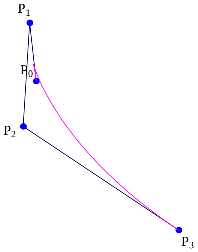

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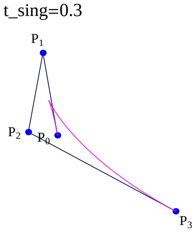

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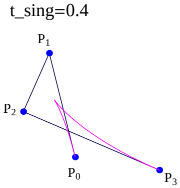

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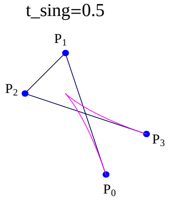

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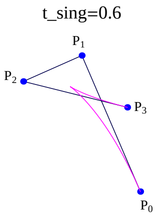

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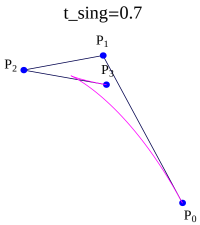

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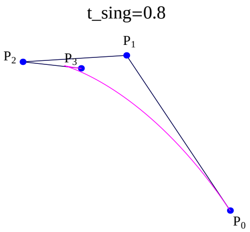

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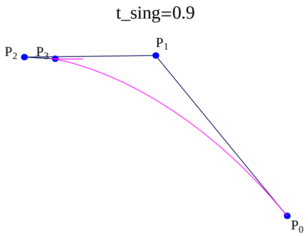

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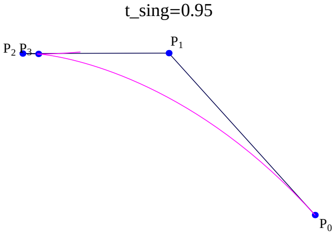

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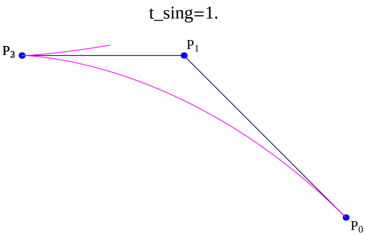

Summary

- Parametric polynomial curves of degree 3 are useful.
- Need to understand their singularities
- Bézier form is the best way to represent a parametric polynomial curve.
- Use Bézier form to describe singularities of parametric polynomial curves of degrees 1,2,3.

- Current and future related work
  - Curvature of curves
  - Singularity of surfaces
  - G^1 surface fitting in the presence of T-junction

For Further Reading I

- P.J. Davis
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- G. Farin, D. Hansford
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  Academic Press, 1990