Asymptotics of Approximation by Bivariate Linear Splines

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Abstract

An early work in the multivariate case of analysis of adaptive mesh generation was [Nadler 1985]. The setting is the approximation of a smooth bivariate function with polygonal domain by piecewise linear functions that are linear on each triangle in a triangulation of the domain, and the asymptotics as the number of triangles goes to infinity are considered. An asymptotic error estimate was obtained for best L_2 approximation in this setting, and used to characterize such an asymptotically optimal sequence of triangulations.

In this talk, the above results are reviewed and extended to the more useful cases of *continuous* linear (approximating) splines and *interpolating* (necessarily continuous) linear splines.

Outline

- Background
- Basics of the problem
- Triangle analysis
  - Quadratics
    - Approximation error
    - Best triangle shape
  - General smooth functions
- Triangulation analysis
  - Approximation error estimate
  - Optimal triangulation characterization
  - Algorithm remarks
- Approximation by *continuous* functions

Background

Dissertation 1985, summarized partially in:

"Piecewise Linear Best L_2 Approximation on Triangulations", in: Approximation Theory V, Chui, Schumaker, and Ward, eds., 1986.

Earlier work:

1-D optimal partitioning:

- de Boor 1972
- Burchard 1974
- McClure 1975
- Barrow & Smith 1978

2-D analysis techniques:

- Fejes Tóth 1959
- McClure 1976

Some subsequent work:

- D'Azevedo 1989+
- Rippa *et al* 1990+
- Field 1991+
- Elber 1996
- Berzins 1998+
- Garland *et al* 1999+
- Bertram *et al* 2000+
- Cao 2005+
- Babenko 2006+

...

Setting

Q a polygonal domain in $\mathbb{R}^2$, e.g., $[0,1]\times[0,1]$

$\triangle_n$: triangulation of Q with n triangles T_i

piecewise linear functions on $\triangle_n$:
$\mathcal{S}_1(\triangle_n) := \{f : Q \to \mathbb{R} : f|_{T_i} \in P_1\}$

$\mathcal{S}_1^0(\triangle_n) := \mathcal{S}_1(\triangle_n) \cap C^0(Q)$, *continuous* piecewise linear functions

$u \in C^3(Q)$: to be approximated by $\mathcal{S}_1^{(0)}(\triangle_n)$

error of best L_2 approximation of u on a given triangulation:
$\mathcal{E}(u, \triangle_n) := \mathrm{dist}_{L_2}(u, \mathcal{S}_1(\triangle_n))$

Objectives

Objective. Minimize $\mathcal{E}(u, \triangle_n)$ over all triangulations $\triangle_n$

Definition. A triangulation $\triangle_n^*$ is called *optimal* if

$$\mathcal{E}(u, \triangle_n^*) = \inf_{\triangle_n} \mathcal{E}(u, \triangle_n)$$

Theorem. $\triangle_n^*$ exists.

Geometric characterization of optimal triangulations $\triangle_n^*$ as $n \longrightarrow \infty$

Based upon asymptotic estimate for $\mathcal{E}(u, \triangle_n^*)$ as $n \longrightarrow \infty$

Approximation error of a quadratic function on a triangle

x_i : vertices of T

$z_i := x_k - x_j, (i, j, k)$ cyclic, i.e., vectors along sides of T

$d : d_i := z_i^T H z_i, i = 1, 2, 3$

Theorem. The error of best L_2 approximation by linear functions of a quadratic function with Hessian H on a triangle of area $\mathcal{A}$ is given by

$$\mathcal{E}^2 = \mathcal{A}\, d^T P d, \text{ with } P := \frac{1}{3600}\begin{bmatrix} 7 & -1 & -1 \\ -1 & 7 & -1 \\ -1 & -1 & 7 \end{bmatrix}$$

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Equivalently,

$$\mathcal{E}^2 = \underbrace{\mathcal{A}^3}_{\text{size}}\underbrace{v^T R v}_{\text{shape}}, \text{ with } v := (u_{xx}, u_{xy}, u_{yy})^T, R := \frac{1}{k^2} Q^T P Q,$$

$$Q := \begin{bmatrix} z_{1,1}^2 & 2z_{1,1}z_{1,2} & z_{1,2}^2 \\ z_{2,1}^2 & 2z_{2,1}z_{2,2} & z_{2,2}^2 \\ z_{3,1}^2 & 2z_{3,1}z_{3,2} & z_{3,2}^2 \end{bmatrix}$$

a form that will be used in subsequent analysis...

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For *interpolation* at the vertices of T, the L_2 error is of the same form:

$$\mathcal{E}_I^2 = \mathcal{A}\, d^T P_I d, \text{ with } P_I := \frac{1}{180}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

where P and P_I have the same eigenspaces.

Best L_2 approximation vs. interpolation at triangle vertices

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Eigenspaces of P and P_I : $\begin{cases} \mathcal{D}_1 : d_1 = d_2 = d_3 \\ \mathcal{D}_2 : d_1 + d_2 + d_3 = 0 \end{cases}$

Eigenvectors of P, P_I : $\left(\frac{1}{720}, \frac{1}{450}\right)$, $\left(\frac{1}{45}, \frac{1}{180}\right)$

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Minimize $d^T P d$ for constant triangle area $\mathcal{A}$

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$$\text{subject to} \quad d^T S d = \mathcal{A}^2 \det H, \quad S := \frac{1}{4}\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

S shares the eigenspaces $\mathcal{D}_1, \mathcal{D}_2$ of P and P_I.

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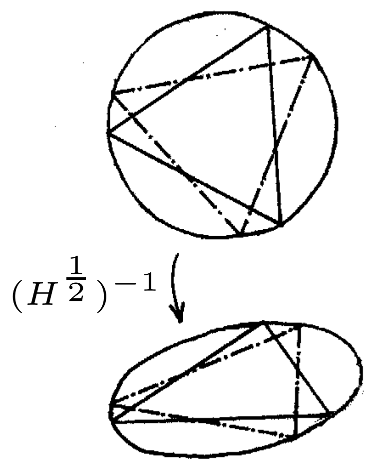
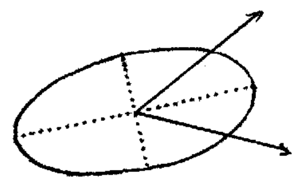
$\det H > 0$	$\det H < 0$
$d \in \mathcal{D}_1$	$d \in \mathcal{D}_2$
$\mathcal{E}^2 = \frac{1}{180}\mathcal{A}^3 \det H$	$\mathcal{E}^2 = \frac{1}{225}\mathcal{A}^3 \lvert\det H\rvert$
$z_i^T H z_i = \frac{4}{\sqrt{3}}\mathcal{A}(\det H)^{1/2} \ \forall i$	$z_i^T \lvert H\rvert z_i = \frac{4}{\sqrt{3}}\mathcal{A}\lvert\det H\rvert^{1/2} \quad \forall i$ where $\lvert H\rvert$ is H with all eigenvalues replaced by their absolute values
	$+$ stretch along axes of 0 curvature:





Best triangle shape for approximation of a quadratic *continued*

Theorem (Optimal triangle shape). The triangles T^* of area $\mathcal{A}$ for which the error of best L_2 approximation of a quadratic function with Hessian H by a linear function is minimized, are

$$T^* = \begin{cases} (H^{1/2})^{-1} T_{eq} \\ \mathcal{V}_p(|H|^{1/2})^{-1} T_{eq} \end{cases} \quad \text{with } \mathcal{E}^2 = \begin{cases} \frac{1}{180}\mathcal{A}^3 \det H & \det H > 0 \\ \frac{1}{225}\mathcal{A}^3 |\det H| & \det H < 0 \end{cases}$$

with matrix operators $\begin{cases} \cdot^{1/2} := M \text{ such that } M^T M = \cdot \\ |\cdot| : \text{ replaces all eigenvalues by their absolute values} \end{cases}$

and

where T_{eq} is an equilateral triangle in any orientation

$\mathcal{V}_p$ is a matrix stretching by $\{p, 1/p\}$ in the 2 directions of curvature 0

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And similarly for *interpolation* at the vertices of a triangle:

$$\mathcal{E}_I^2 = \begin{cases} \frac{8}{90}\mathcal{A}^3 \det H & \det H > 0 \\ \frac{1}{90}\mathcal{A}^3 |\det H| & \det H < 0 \end{cases}$$

Approximation of a smooth function on a triangle

Recall for a quadratic, $\mathcal{E}^2 = \mathcal{A}^3 \, v^T R v$,
$v = v(H), R$ a measure of triangle shape.

Extend this to $u \in C^3$

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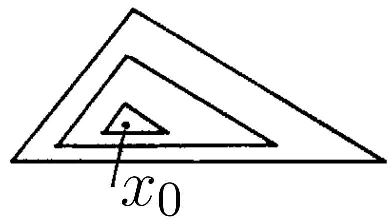
$T_h := x_0 + h(T - x_0)$, with area $\mathcal{A}_h = h^2 \mathcal{A}$,

Expand u in a Taylor series at $x_0 \in T$...



Theorem. For any point x_0 in a triangle T of area $\mathcal{A}$, the error of best L_2 approximation of $u \in C^3(Q)$ by a linear function on triangle T_h is given by

$$\mathcal{E}^2(T_h) = \mathcal{A}_h^3\, v^T(x_0)\, R\, v(x_0)\big(1 + O(h)\big)$$

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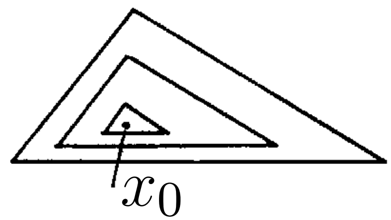

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$J(x) := v^T(x)\, R\, v(x) = \lim_{h\to 0} \frac{\mathcal{E}^2(T_h)}{\mathcal{A}_h^3}$
acts as an "error density"

Approximation of a smooth function on a triangle *continued*

From the optimal shape analysis define

$$J^*(x) := \begin{cases} \frac{1}{180}\det H(x) & \det H(x) > 0 \\ \frac{1}{225}|\det H(x)| & \det H(x) < 0 \end{cases}$$

$J \geq J^*$, with equality iff the triangle *conforms* to H., i.e., is optimal in the sense of the *Optimal triangle shape* theorem.

Definition. A triangle is said to be *well-shaped* if it conforms to H at some point within it.

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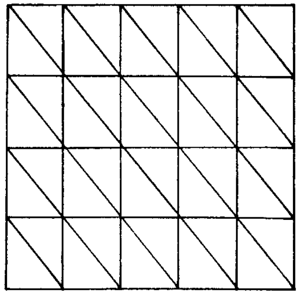
Optimal triangulation - introduction

Regular triangulation with n triangles, obtained obtained by linear transformation by $(H^{1/2})^{-1}$ of a grid of equilateral triangles

$$\mathcal{E}^2 = \sum \mathcal{E}_i^2 \sim \sum \mathcal{A}_i^3 J$$

from which easily follows

$$\lim_{n\to\infty} n^2 \mathcal{E}_n^2 = \mathcal{A}^2 \int_Q J$$



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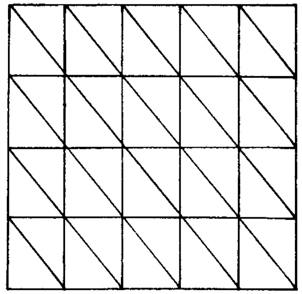

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1-D analogue: approximation on an interval

equal partition: $\lim_{n\to\infty} n^4 \mathcal{E}_n^2 = \frac{1}{720}\text{Length}^4 \int u''^2$

optimal partition: $\lim_{n\to\infty} n^4 \mathcal{E}_n^{*2} = \frac{1}{720}\left(\int u''^{\frac{2}{5}}\right)^5$ [McClure 1975]

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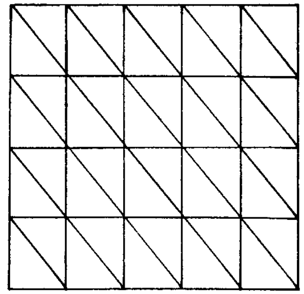

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By analogy with 1-D case, expect 2-D estimate of the form

$$\lim_{n\to\infty} n^2 \mathcal{E}_n^{*2} = \left(\int_Q J^{*p}\right)^q$$

Must restrict attention to u with $\det H$ bounded away from 0 on Q.

Optimal triangulation - developing the estimate

Estimate suggested by series of lower bounds on a quantity asymptotically equal to $\mathcal{E}^2$

$$\sum_{i=1}^{n} J_i(\xi_i)\mathcal{A}_i^3 \;\geq\; \sum_{i=1}^{n} J^*(\xi_i)\mathcal{A}_i^3 \;\geq\; \frac{1}{n^2}\Big(\sum_{i=1}^{n} J^*(\xi_i)^{\frac{1}{3}}\mathcal{A}_i\Big)^3$$

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Equality is achieved in the first inequality iff each triangle is *well-shaped*, and in the second iff $\mathcal{A}_i \propto J^*(\xi_i)^{-\frac{1}{3}}$

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$\mathcal{E}(\triangle_n) \to 0$ implies $\mathcal{A}_i \to 0$ *uniformly*, from which follows

Lemma. The sequence $\mathcal{E}_n^* := \mathcal{E}(u, \triangle_n^*)$ satisfies

$$\liminf_{n\to\infty} n^2 \mathcal{E}_n^{*2} \geq \left(\int_Q J^*(x)^{\frac{1}{3}}\right)^3$$

Optimal triangulation - the estimate

Construct a sequence of triangulations for which $\limsup_{n\to\infty} n^2 \mathcal{E}_n^{*2} \le (\int_Q J^*(x)^{\frac{1}{3}})^3$ to establish our *main result*:

Theorem (Optimal triangulation estimate)**.** The sequence $\mathcal{E}_n^*$ of optimal errors for best L_2 approximation by $\mathcal{S}_1(\triangle_n)$ satisfies

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Characterization of optimal triangulations

- The *Optimal triangulation estimate* says that the lower bounds in the inequalities are attainable, in an asymptotic sense.
- The conditions for equality in these two inequalities essentially provide geometric characterizations of optimal triangulations.
- In the following
  - $\{\triangle_n^*\}$: sequence of triangulations satisfying $(*)$
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Result: $\mathcal{A}_i \propto J^*(x)^{-\frac{1}{3}}$ asymptotically. Formally...

Theorem (Triangle *size* characterization)**.** Let R be an open subset of Q. Then the fractional number of triangles contained in R approaches the following limit as $n \to \infty$

$$\frac{\int_R J^*(x)^{\frac{1}{3}}}{\int_Q J^*(x)^{\frac{1}{3}}}$$

Characterization of optimal triangulations *continued*

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Theorem (Triangle *shape* characterization)**.** Let $r_i := \mathcal{E}^2(u, T_i)/\mathcal{A}_i^3 J^*(x_i)$, and let $m_n(t)$ be the total area of triangles in $\triangle_n$ with $r < t$. Then for any $t > 1, m_n(t) \to \mathcal{A}$, the total area of Q.

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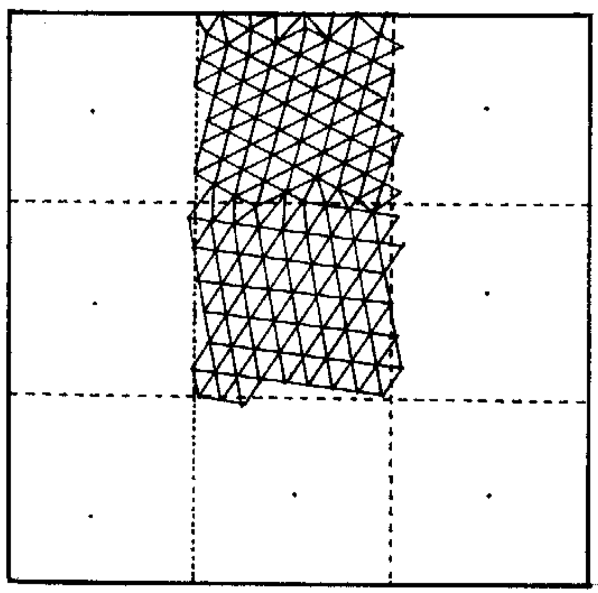
The above shape characterization, combined with the previous size characterization, $\mathcal{A}_i \propto J^*(x)^{-\frac{1}{3}}$ asymptotically, says that $\mathcal{E}$ is asymptotically *balanced* over the triangles in $\triangle_n^*$

Characterization $\Longrightarrow$ triangulation algorithm

One idea: (used in proof of *Optimal triangulation estimate*):
Coarsely partition Q into subregions Q_i, and triangulate as much of Q_i as possible by a regular grid of triangles that conform to $H(x_i), x_i \in Q_i$, "fixed up" near the boundaries of Q_i



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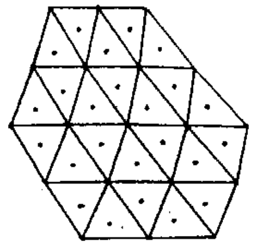
Better idea:

Continuous variation in triangle size and shape to reflect $H(x)$.



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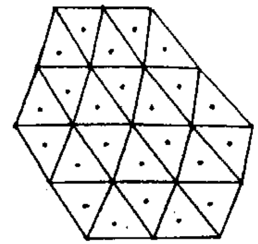

- Generalize distribution function method [Sacks & Ylvisaker 1970] of 1-D, in which a function determines a partition by applying its inverse to even partition.
- In 2-D, from the size & shape characterizations, $|\det H|^{-\frac{1}{12}}|H|^{\frac{1}{2}}$ acts like a density function; its inverse applied to a fixed small equilateral approximates a triangle of $\triangle_n^*$.
- May be possible to transform a grid of equilateral triangles to satisfy this density function
- [D'Azevedo 1991] does something similar for a related problem.

Extension to approximation by continuous functions

- In 1-D, best L^2 approximation by piecewise linear functions with optimal knots is *automatically continuous* [Barrow & Smith 1978].
- In 2-D, it turns out that best L^2 approximation *of quadratics* by piecewise linear functions on an optimal triangulation, i.e., an equilateral grid transformed by $\left(|H|^{\frac{1}{2}}\right)^{-1}$, is automatically continuous.

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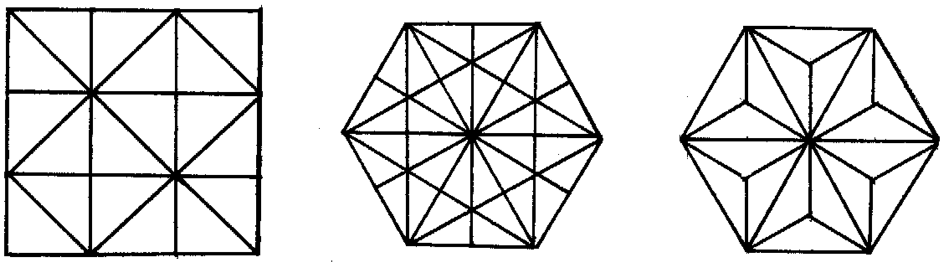
- Moreover, the same holds for the sub-optimal triangulations that are transformations by $\left(|H|^{\frac{1}{2}}\right)^{-1}$ of the following three regular ones.



- These four triangular triangulations are the ones that fill the plane with their reflections.

Extension to approximation by continuous functions *continued*

- For general smooth functions u, the best L^2 approximant by piecewise linear functions on a sequence of optimal triangulations $\triangle_n^*$ are asymptotically "nearly continuous".
- Consider the continuous piecewise linear approximant obtained by selecting a value at each vertex in the range of the values of the not-necessarily continuous approximant from the triangles sharing the vertex.
- The continuous function so constructed satisfies $(*)$.

Hence

Theorem. The main results for *Optimal triangulation estimate* and *Triangle size & shape characterization* hold for approximation by *continuous* piecewise linear functions.

Summary

- Review old results
  - approximation by linear functions on triangles
  - asymptotic results for approximation by piecewise linear functions on triangulations
- New results
  - optimal *interpolation* by linear functions on triangles
  - asymptotic results for *interpolation* by piecewise linear functions on triangulations
  - asymptotic results for approximation by *continuous* piecewise linear functions on triangulations