§ 5.1-5.3 Handout

Moral. The theme of today's lecture is that you can "undo" the operation of taking the derivative of f by computing the area under the graph of f.

Example. R2-D2 is trying to escape from some stormtroopers. His velocity (in m/s) as a function of time (measured in seconds) is summarized in the following table.

t	0	1	2	3	4	5
v	3	7	8	5	9	6

How can we estimate the distance R2 traveled? Find an estimate (it may help to draw a picture) and explain a better estimate could be made with additional info.

Given the graph of velocity vs. time, the total distance traveled can be determined by calculating the area under the curve (which is positive when the graph is above the x-axis and negative below). This area can be estimated by making rectangles which touch the curve on the right corner (right-hand sum) or the left corner (left-hand sum). See your lecture notes or textbook for a graphical representation of this.

Note: The left-hand sum is an underestimate if the function is increasing, and an overestimate if the function is decreasing (opposite for right-hand).

Recall that velocity is the time derivative of distance traveled. In a similar way, given any derivative function, we can calculate how much the original function changed by finding the area under the curve. In math terms, the idea of finding the area under the curve is called *integration.*

Definition: The *definite integral* of f from a to b, written

$$\int_a^b f(t)dt$$

is defined as the "area under the curve" of $f(x)$ from a to b. In this class we usually require that f be a continuous function. Here $f(t)$ is called the *integrand* and a and b are the *limits of integration.* Note that you input a function to a definite integral, and get a number as a result.

Summation Notation: This is a notation which may be unfamiliar to you. It is a way to represent sums of many terms (even infinitely many terms) easily. If you want to sum the terms $x_1, x_2, ...x_{10}$, then this is written as

$$\sum_{i=1}^{10} x_i = x_1 + x_2 + \cdots + x_{10}$$

Here the letter i is the *index* of the sum, which starts at the number underneath the Sigma (1) and ends at the number above the Sigma (10). The number after the Sigma is called the summand, and is a different term of the sum for each index.

Example. $\sum_{i=1}^{5} i = 1+2+3+4+5 = 15$ $\qquad$ $\sum_{i=0}^{3} i^2 = 0^2 + 1^2 + 2^2 + 3^2 = 14.$

Exercise. We can estimate area under the curve by making rectangles. Draw an example below of a left-hand sum with rectangles of equal widths (for any function you like). Recall that for a left-hand sum, the upper left corners of the rectangles must touch the curve.

On your graph, label the width of the rectangles as Δt. Label the t-values of the rectangles as $a = t_0, t_1, t_2, ..., t_{n-1}, b$, where n is the number of rectangles. The heights of these rectangles are $f(t_0), f(t_1), ...f(t_{n-1})$. Note that we must have (why is this?)

$$\Delta t = \frac{b-a}{n}$$

Then to estimate the area under the curve, you add up the rectangles (area = width∗height), so

$$\int_a^b f(t)dt \approx f(t_0)\Delta t + f(t_1)\Delta t + ... + f(t_{n-1})\Delta t = \sum_{i=0}^{n-1} f(t_i)\Delta t$$

Note that if we instead did a right-hand sum, the only change would be that we would use $f(b)$ instead of $f(a)$, since the rectangles would touch the curve on the upper right. Now, we can improve the approximation by increasing the number of rectangles, which also corresponds to decreasing the width of the rectangles. Therefore, we want to make Δt as small as possible. In fact, the left- or right-hand sum will become exact as Δt approaches 0. Therefore,

$$\int_a^b f(t)dt = \lim_{\Delta t \to 0} \sum_{i=0}^{n-1} f(t_i)\Delta t$$

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Exercise. Estimate $\int_1^2 \frac{1}{t}dt$ by left- and right- hand sums with $n = 2$ and with $n = 4$. Which are upper bounds and lower bounds? Give an inequality of the form lower bound $<$ integral $<$ upper bound for each. This is similar to how we estimated limits and derivatives using tables of values.

Exercise. Look at the graph of $\sin(t)$ to determine the exact value (without doing any calculations) for $\int_0^{2\pi} \sin(t)dt$.

Note. You'll want to carefully read about "general Riemann Sums" from the textbook, in which rectangles are not all the same size.

Error in Riemann sums: We can estimate the error introduced by doing a Riemann sum by taking the difference between the left- and right-hand sums. **This will only make sense if one is and underestimate and the other an overestimate.** Subtract the two formulas to get a simple expression for the error (most terms will cancel out immediately as most rectangles are in both sums). Notice the error depends directly on Δt so we can control the size of the error by choosing the value of Δt.

The Fundamental Theorem of Calculus (FTC)

If $F(x)$ is a function such that $F'(x)$ is continuous on some interval $[a, b]$, then

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

In words, this says that the definite integral of the rate of change gives the total change.

Example: We already know that finding the area under the curve of velocity (the rate of change of position) results in the change in position.

Another way of writing the FTC is that if $f(x) = F'(x)$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Practical Interpretations of the Integral

For practical interpretations, just state what the integral means in terms of the problem, using the FTC. You do not need to say approximately for these, as there will not be any approximations here, and you should as always avoid math terms like "integral" or "derivative" in your answer. You do need to include units, which are always the units of the function times the units of "dx", (often time or distance).

Example: Suppose water is entering a lake at a rate of $f(t)$ cubic meters per second. Then $\int_0^{60} f(t)dt = 1000$ means that between 0 and 60 seconds, 1000 cubic meters of water entered the lake.

Problems

(1) If $f(t)$ is measured in dollars per year and t is measured in years, what are the units of $\int_a^b f(t)\,dt$?

(2) If $f(x)$ is measured in pounds and x is measured in feet, what are the units of $\int_a^b f(x)\,dx$?

(3) For the following, let $f(t) = F'(t)$. Write down the integral $\int_a^b f(t)\,dt$ and evaluate it using FTC.

(a) $F(t) = t^2$, $a = 1$, $b = 3$

(b) $F(t) = \ln t$, $a = 1$, $b = 5$

(4) Let $F(t) = \dfrac{1}{2}\sin^2(t)$. Find $F'(t)$.

(a) Approximate $\int_{0.2}^{0.4} \sin(t)\cos(t)\,dt$ using a Riemann sum with 4 subintervals (rectangles).

(b) Evaluate the above integral exactly using FTC and compare to your approximation.

(5) A cup of coffee at $90°C$ is put into a $20°C$ room when $t = 0$. The coffee's temperature is changing at a rate of $r(t) = -7e^{-0.1t}\,°C$ per minute, where t is in minutes. Estimate the coffee's temperature when $t = 10$.