Numerical Investigations of Convection

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Abstract

We report on high resolution numerical studies of infinite Prandtl number convection using a simplified model with relevance to the motion of the Earth's mantle. The simulations are done using pseudospectral Fourier (x-direction) and Chebyshev methods (z-direction).The model uses the incompressible Navier-Stokes equations with the Boussinesq approximation and free-slip velocity boundary conditions that is driven solely by internal heating. We examine the transition from conduction to steady convection, to unsteady laminar convection, and lastly to chaotic convection.

Introduction

Convection is driven primarily by temperature changes that are introduced either through the boundaries or internally generated. The temperature variations give rise to variations in density, which can cause fluid motion. The Navier-Stokes equations with the Boussinesq approximation are:

$$\nabla \cdot \boldsymbol{u} = 0 \tag{1}$$

$$\boldsymbol{u}_t + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \mathbf{u} + g\alpha T \hat{\boldsymbol{k}} \tag{2}$$

$$T_t + \boldsymbol{u} \cdot \nabla T = \kappa \nabla^2 T + H \tag{3}$$

We consider a two dimensional idealized fluid layer with height h that has free-slip boundary conditions at the vertical boundaries. The velocity vector is $\boldsymbol{u} = (u, 0, w)$, p is the pressure field, ν is the kinematic viscosity, g is the acceleration of gravity that points in the $\boldsymbol{k}$ direction, α is the thermal expansion coefficient, T is temperature, and κ is the thermal diffusivity . $T = 0$ at the top and bottom boundaries and all variables are periodic in the x direction.

We choose a time-scale of h^2/κ, length scale of h (height of the box), and a temperature scale of Hh^2/κ. After scaling, Equation 2 becomes

$$\frac{1}{\mathrm{Pr}}(\boldsymbol{u}_t + \mathbf{u} \cdot \nabla \ \ \mathbf{u}) = -\nabla p + \nabla^2 \mathbf{u} + \mathrm{R}T\hat{\mathbf{k}} \tag{4}$$

and Equation 3 becomes

$$T_t + \mathbf{u} \cdot \nabla T = \nabla^2 T + 1 \tag{5}$$

where R $=\frac{g\alpha H h^5}{\nu\kappa^2}$ is called the **Rayleigh number** and Pr $= \frac{\nu}{\kappa}$ is the **Prandtl number** (both dimensionless).

The Pr number for the Earth's mantle is $\approx 10^{25}$, thus an infinite Pr number is a good approximation for our simulations. Looking at the Equation 4 we can see that setting Pr to infinity drops the term on the left side:

$$\nabla p = \nabla^2 \mathbf{u} + \mathrm{R}T\hat{\mathbf{k}} \tag{6}$$

The final simplification that we use is to introduce the **stream function** in order to eliminate the continuity equation. This is done by writing the three vector equations in two dimensional component form, where x-direction is horizontal and z-direction is vertical. The velocity vector is $\boldsymbol{u} = (u, 0, w)$ and our stream function is defined as $\boldsymbol{\psi} = (0, \psi, 0)$. Writing u and w in terms of ψ yields,

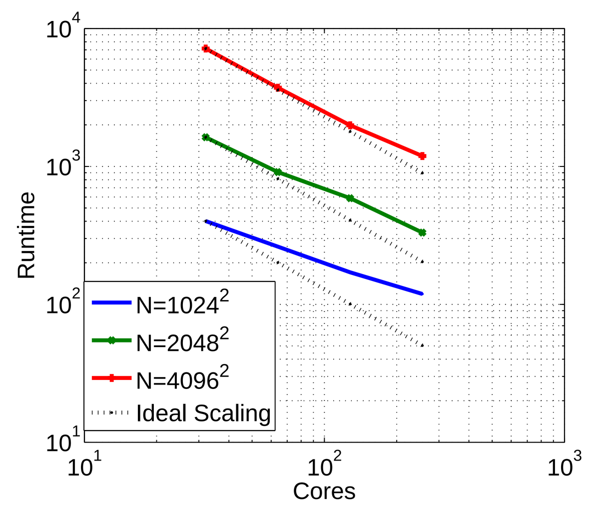
$$u = \psi_z \qquad w = -\psi_x \tag{7}$$

We substitute into our three dimensionless equations of motion and simplify, elimination p by taking the curl:

$$\psi_{xxxx} + 2\psi_{xxzz} + \psi_{zzzz} = RT_x \tag{8}$$

$$T_t + \psi_z T_x - \psi_x T_z = T_{xx} + T_{zz} + 1 \tag{9}$$

Scaling Results on Trestles



Simulation Results

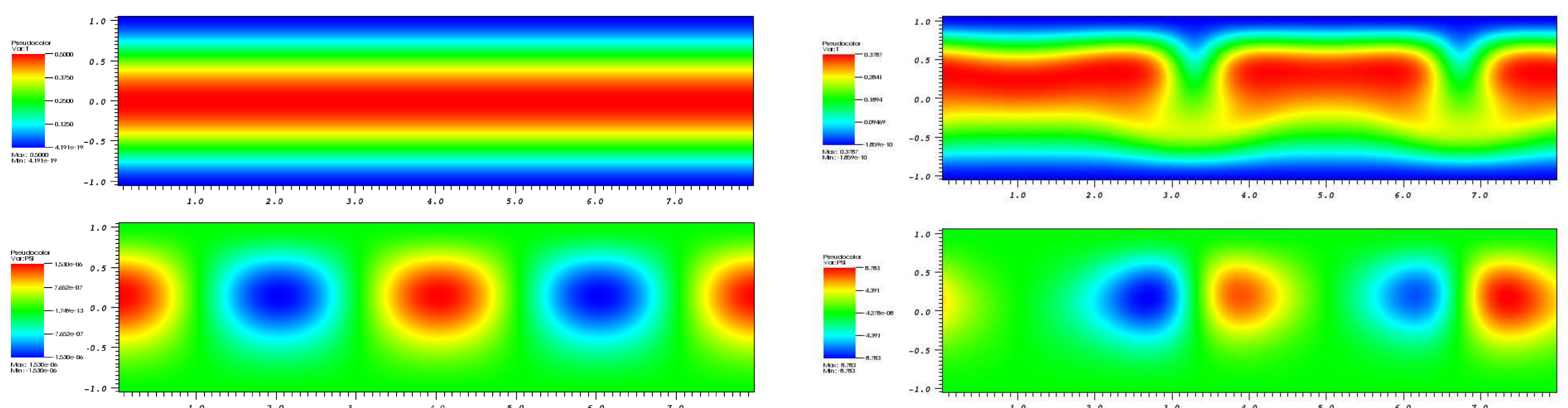


Figure 1: Aspect ratio 4. Left shows Ra=10^4 steady state conduction. Right shows Ra=10^5 steady convection. In all figures, top shows temperature, bottom shows stream function.

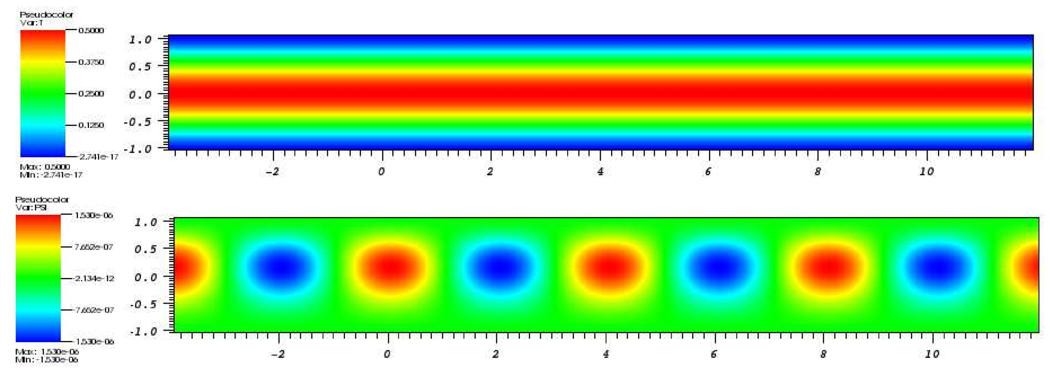


Figure 2: Aspect ratio 8 with Ra=10^4.

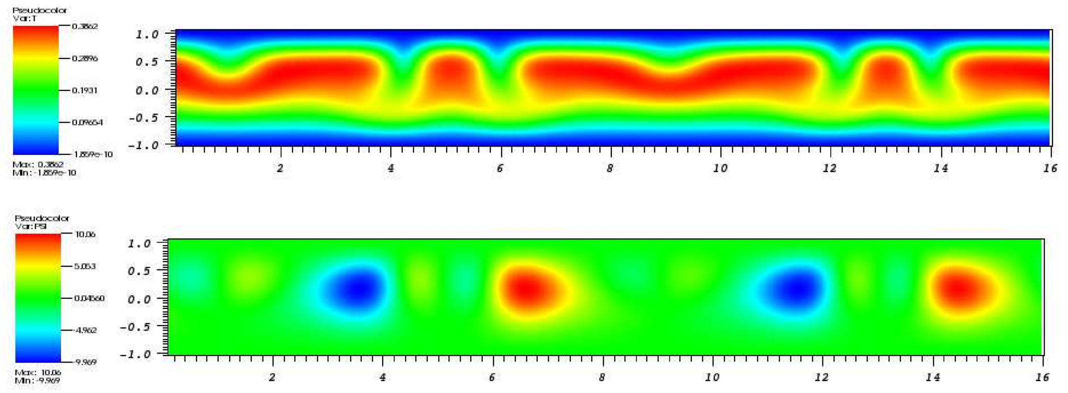


Figure 3: Aspect ratio 8 with Ra=10^5.

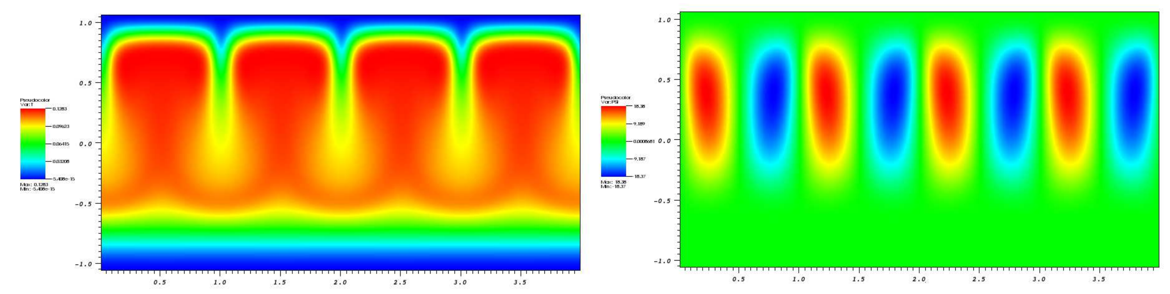


Figure 4: Aspect ratio 2 with Ra=10^6.

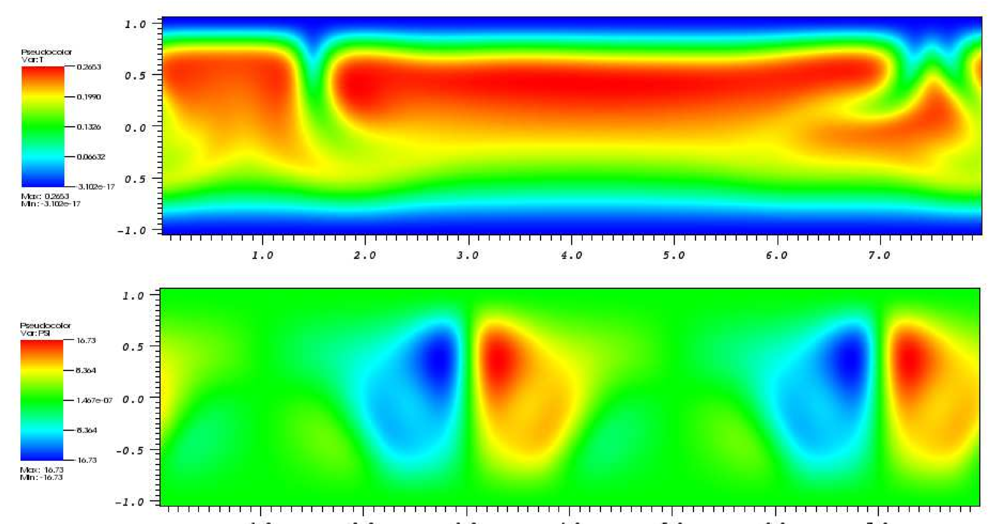


Figure 5: Aspect ratio 4 with Ra=10^6.

Conclusion

We conclude that our Fortran code is working and have been able to compute high resolution images that are qualitatively correct. We found that for small aspect ratios, Rayleigh numbers with values of 10^4, 10^5, and 10^6 reached a steady state in two dimensions as the average scaled temperature decreased. For larger aspect ratios, we found Rayleigh numbers of 10^6 leading to unsteady convection.

Further Work

Recent work shows that the average temperature has a lower bound of the form $\langle T \rangle \approx c * (R^{-5/17})$, where c is an unknown constant[2]. Further work will determine if the data supports this scaling. Our current data is for moderate Rayleigh numbers. By the end of the summer, we hope to simulate Rayleigh numbers on the order of 10^{11}.

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References

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