Numerical Investigation of Convection

Application of Chebyshev Integration

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Overview

Assumptions

- Convection driven solely by internal heating infinite Pr (similar to the Earth's mantle)
- Start with dimensionless two dimensional Navier-Stokes equations with the Boussinesq approximation.
- Choose time-scale of h^2/κ, length scale of h and temperature scale of Hh^2/κ. Where h is height, κ is thermal diffusivity, and H is heating.

$$\Delta^2\psi = RT_x \tag{1}$$

$$T_t + \psi_z T_x - \psi_x T_z = T_{xx} + T_{zz} + 1 \tag{2}$$

$$\psi|_{z=-1,1} = 0 \quad \psi_z|_{z=-1,1} = 0 \tag{3}$$

$$T|_{z=-1,1} = 0 \tag{4}$$

- $T(x,z)$ - Temperature. $\psi(x,z)$ - Streamfunction, $u(x,z) = (u,w)$

Biharmonic Equation

- Decompose biharmonic into two equations, one for the stream function and the other for the vorticity

$$\Delta^2\psi = f(x,z)$$

$$\Delta\psi = \omega \quad (5)$$

$$\Delta\omega = f(x,z) \quad (6)$$

- Easy to do with free slip boundary conditions
- Four order problem with N nodes implies dividing by N^4
- N=4096, $\frac{1}{N^4} \sim 3.5 \times 10^{-15}$, $\frac{1}{N^2} \sim 6 \times 10^{-8}$ (edge of double percision)

Chebyshev polynomials

- Consider a problem on the interval $[-1, 1]$
- Space is discretized using Chebyshev polynomials

$$T_n(z) := \cos n \cos^{-1} z \tag{7}$$

- with x evaluated at Chebyshev points

$$z_i := \cos \frac{\pi i}{N} \qquad i = 0, ..., N \tag{8}$$

- Discretization allows for the use of Fast Fourier Transform to calculate integrals and derivatives

Fourier Space

- FFT in the x-direction and rewriting derivatives [1]

$$(ik_x)^2\hat{\omega} + \hat{\omega}_{zz} = \widehat{f(x,z)} \quad (9)$$

- $\omega(x,z)$ and $f(x,z)$ are periodic functions on the interval $[-1,1]$.

[1] Properties of FFT allow $\frac{\partial^n f(x)}{\partial x^n} = (ik)^n f(\hat{x})$

Chebyshev Integration

- Chebyshev integration matrix method amounts to solving for the highest order derivative by expanding as a summation of Chebyshev polynomials in z-direction.

$$((ik)^2 I_2 + I_0 + \widehat{LBC})\hat{\psi}_{zz} = \widehat{f(x,z)} + \widehat{RBC} \qquad (10)$$

- $\widehat{LBC}$ and $\widehat{RBC}$ represent boundary conditions
- $\hat{\omega}_{zz}$ is a vector of the truncated series expansion for ω_{zz}.

Finding Numerical Solution

- Given Boundary conditions, fix two coefficients from the indefinite integral of ω_{zz}.
- Linear system is solved to find $\hat{\omega}_{zz}$ and then integrated to find $\hat{\omega}$.
- Lastly we can use IFFT to convert back to real space and find $\omega(x, z)$.

Finding I_0 and I_2

- Suppose, where b_n are Chebyshev series expansion coefficients for f

$$\omega_{zz} = \sum_{n=1}^{\infty} b_n T_n(z) \tag{11}$$

- We use the following indefinite integral identities

$$\int T_0(z) = T_1(z), \quad \int T_1(z) = \frac{T_2(z)}{4}$$
$$\int T_n(z) = \frac{T_{n+1}(z)}{2(n+1)} - \frac{T_{n-1}(z)}{2(n-1)} \tag{12}$$

- We can use these integral identities in write integral matricies of ω_{zz}, ω_z, and ω

Using Indefinite Integral Identities

- Truncated to N+3 modes

$$U_x = e_1 + (b_0 - \frac{b_2}{2})T_1(x) + \sum_{n=2}^{N+3} (\frac{b_{n-1} - b_{n+1}}{2n} T_n(x)$$

$$U = e_0 + (e_1 - \frac{b_1}{8} + \frac{b_3}{8})T_1(x) + (\frac{b_0}{4} - \frac{b_2}{6} + \frac{b_4}{24})T_2(x) + \ldots$$

$$+ \sum_{n=3}^{N+3} \left(\frac{b_{n-2}}{4n(n-1)} - \frac{b_n}{2(n-1)(n+1)} + \frac{b_{n+2}}{4n(n+1)} \right) T_n(x)$$

Integration Matricies Explicitly

- Resulting system of equations are

$$((ik)^2 I_2 + I_0 + LBC + \widehat{LBC})\hat{\psi}_{zzzz} = \widehat{f(x,z)} + \widehat{RBC} \quad (13)$$

$$(ik)^2 b_0 + e_0 = f_0 \quad (14)$$

$$(ik)^2 b_1 + (e1 - \frac{b_1}{8} + \frac{b_3}{8}) = f_1 \quad (15)$$

$$(ik)^2 b_2 + \left(\frac{b_0}{4} - \frac{b_2}{6} + \frac{b_4}{24}\right) = f_2 \quad (16)$$

- And for $2 < n \leq N$, use formula were $b_n = 0$ for $n > N$ and f_n are Cheby expansion coefficients

Impose Boundary Conditions

- All that is left is to impose the two boundary that will fix the last 2 coefficients
- Using the following boundary conditions, $\omega(\pm 1)$ we have,

$$\omega(\pm 1) = e_0 \pm (e_1 - \frac{b_1}{8} + \frac{b_3}{8}) + (\frac{b_0}{4} - \frac{b_2}{6} \pm \frac{b_4}{24} + \dots \quad (17)$$

$$+ \sum_{n=3}^{\infty} (\pm 1)^n \left(\frac{b_{n-2}}{4n(n-1)} - \frac{b_n}{2(n-1)(n+1)} + \frac{b_{n+2}}{4n(n+1)} \right) T_n(x) \quad (18)$$

$$\omega_x(\pm 1) = e_1 \pm (b_0 - \frac{b_2}{2}) + \sum_{n=2}^{\infty} (\pm 1)^n (\frac{b_{n-1} - b_{n+1}}{2n} T_n(x) \quad (19)$$

$$\omega_{xx}(\pm 1) = \sum_{n=1}^{N} (\pm 1)^n b_n \quad (20)$$

Timestepping: Implicit Midpoint Rule

- Fixed point iteration we solve

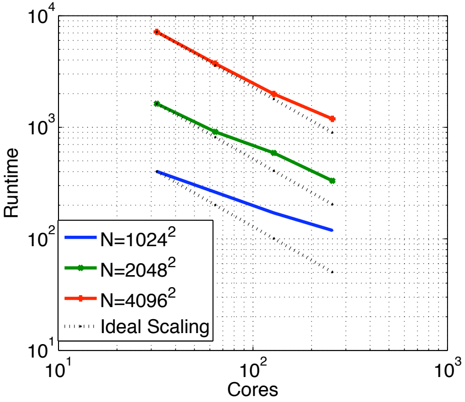
$$\frac{T^{n+1} - T^n}{dt} + \frac{1}{2}u^{n+1} \cdot \nabla T^{n+1} + \frac{1}{2}u^n \cdot \nabla T^n = \frac{1}{2}(\Delta T^{n+1} + \Delta T^n)$$
$$\Delta^2 \psi^{n+1} = f(x,z)^{n+1}$$
$$\Delta \omega^{n+1} = f(x,z)^{n+1}$$
$$\Delta \psi^{n+1} = \omega^{n+1}$$

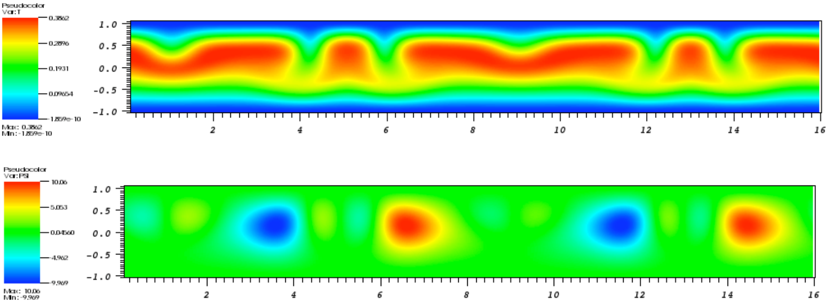
- Initial conditions: perturbation of conductive solution

$$T(x,z) = \frac{1}{2}(1 - z^2) + .1 \sin\left(\frac{2\pi x}{x_{max}}\right) \sin(.5\pi(1+z))$$

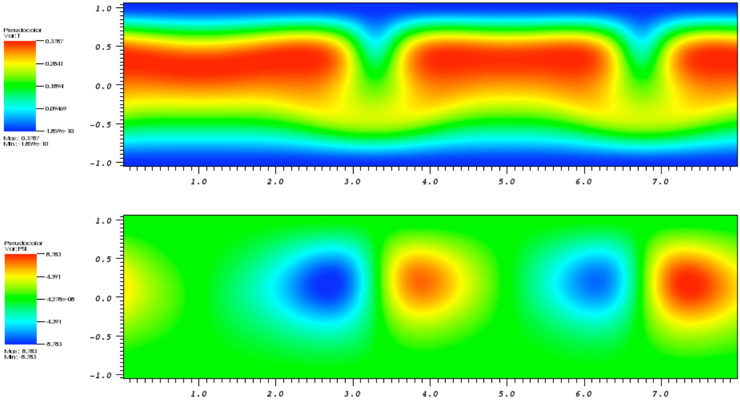
Scaling

Parrallel code run Trestles and Lonestar. Thanks to Teragrid/Xsede resources supported by award TG-CTS1100010.





Aspect Ratio: 8, Rayleigh number:10^5, top is temperature field and bottom is the streamfunction.



Aspect Ratio: 4, Rayleigh number:10^5, top is temperature field and bottom is the streamfunction.

Video

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