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The Notes Tell Us How to Tune

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J. S. Bach famously observed, "There is nothing remarkable about it. All one has to do is hit the notes at the right time, and the instrument plays itself."[1] In this simple and perhaps facetious statement about organ-playing, Bach suggests that if we merely "hit the notes" (and not some other wrong notes), playing his music is unproblematic. But what are these notes, and how can they inform appropriate choices of keyboard temperaments? In this article, I show that his notes teach us how to set up the scales to play harpsichords, clavichords, and organs correctly in tune.

In 2005, I presented a keyboard temperament derived directly from the enharmonic challenges within Bach's music, and catalyzed by his line drawing on the title page of *The Well-Tempered Clavier* Book I.[2] That "Bach-Lehman" temperament establishes the C-major scale of Ut-Re-Mi-Fa-Sol-La in the equally spaced (regular) way that was normal from contemporary practices with vocal pedagogy and other instruments.[3] This is to facilitate the duties of a keyboard player in solo and ensemble. It then finishes B and the sharps in a carefully extra-sharp manner, as compromises, so all those pitches can also serve with complete enharmonic flexibility as flats. Furthermore, when we get around to F serving as E♯, and C serving as B♯, the F and C do not need to be retuned. The F♯, C♯, G♯, D♯, and A♯ are all high enough that the E♯ and B♯ make sense with them.

[1] *The New Bach Reader: A Life of Johann Sebastian Bach in Letters and Documents,* ed. Hans T. David and Arthur Mendel, revised and expanded by Christoph Wolff (New York: Norton, 1998), no. 404.

[2] Bradley Lehman, "Bach's Extraordinary Temperament: Our Rosetta Stone," *Early Music* 33 (2005): 3–23 and 211–31 and sixty-six pages of "Supplementary Data" analysis as five Appendix files to download. The Appendix files and six listening examples are available as "Supplementary Data" at *Early Music*, https://academic.oup.com/em; and at Lehman, "Outlines of Academic Articles," http://www-personal.umich.edu/~bpl/larips/outline.html.

[3] The recipe is: Pythagorean $\frac{1}{6}$ comma fifths F-C-G-D-A-E; beatless fifths E-B-F♯-C♯; $\frac{1}{12}$ comma fifths C♯-G♯-D♯-A♯; residual A♯-F $\frac{1}{12}$ comma wide. In cents: C=0.0, C♯=98.0, D=196.1, D♯=298.0, E=392.2, F=502.0, F♯=596.1, G=698.0, G♯=798.0, A=894.1, A♯=998.0, B=1094.1, C=1200.0. Bradley Lehman, "Johann Sebastian Bach's Tuning," http://www.larips.com.

In my interpretation, Bach's eleven-loop drawing on the title page of *The Well-Tempered Clavier* was never a "code" or a pretext to calculate any numbers. Rather, I view it as a practical schematic or diagram of procedure, a picture of tuning a harpsichord by ear in ten minutes.[4] It shows how to set up the naturals first, balancing them with a consistent quality of controlled roughness or smudging in these fourths and fifths.[5] Getting to B and the first two sharps, it shows how to set those as beatless intervals from the preceding note. Finishing the last three sharps, it shows how to listen for the appropriate slighter smudging. Checking beatless octaves outward to finish the bass and treble, Bach's drawing is also a reference to show which intervening fourths and fifths ought to be either smudged or beatless, proving the octave. There is no time available in this process for counting beats in the ways piano tuners do; one quickly builds the appropriate character into the interval, and then moves on. When finished, the harpsichordist can play in any desired scale, because all the appropriate compromises for sharps and flats are available.

This enharmonic handling is the crux of the argument. Bach's music sets up stringent requirements when he uses notes far from the home scale, or when he spells pitches for more than one function within a composition, for instance, using E♭ and D♯ only a few beats apart. To give only one example, the published *Vater unser im Himmelreich* BWV 682 from the *Clavier-Übung III* (1739) needs A♭, G♯, E♭, D♯, B♭, A♯, F, and E♯. Bach does in fact show us which "notes" to "hit." Any temperament to play Bach's keyboard music must deliver all those notes.

Since 2005, many performers and scholars have welcomed the Bach-Lehman temperament into their practices and their theory. Some still disagree with the principle of accepting Bach's drawing as admissible corroborating evidence of any temperament. Others accept the drawing as relevant, but they interpret it in other ways (typically assuming it means some quarter-comma fifths and some

[4] As Johann Nikolaus Forkel noted, it never took Bach more than fifteen minutes to tune a harpsichord by ear. *The New Bach Reader,* 436.

[5] Beyond listening for only the properly smudged quality, I sometimes use my triplets-against-duplets technique to gauge accurately the amount of tempering among the fourths and fifths. See Bradley Lehman, "Tetrasecting an Angle," http://www-personal.umich.edu/~bpl/larips/tetrasect.html.

wide fifths, or a scheme of constant beat rates).[6] Most discussions concentrate on the graphology of the drawing, not on an understanding of the fundamental enharmonic evidence or its importance.[7] Counterarguments about Bach's tuning do not address the "Cinderella" pieces of music by him—those pieces which, like Cinderella, appear to be "dressed" poorly because they do not sound good by other methods of temperament—that had been awaiting convincing intonation to let their attractiveness emerge from the shadows. Such compositions have sections that are measurably harsh (with pitch errors of a comma or more) when played in unequal temperaments of demonstrably wrong shapes. It happens especially with Bach's keyboard music in key signatures of two or more flats or sharps. If any matters of musical taste are brought up, it is to assert that someone *prefers* what had been familiar from other tuning schemes (such as expecting intervals with flats to be farther out of tune than intervals with sharps) or *prefers* some degree of spicy tension (or not). This disregards the evidence of the notes in Bach's scores—the notes that were so self-explanatory to "hit."

In this article, I revisit my Bach-Lehman temperament afresh. I discuss the primary evidence in more depth because it shows the necessary restrictions on any proposed keyboard temperaments for Bach's music. I show that the evidence of Bach's music itself (with correct enharmonic spelling, the crux of the argument) requires a temperament with a profile similar to Bach-Lehman, regardless of using the drawing (or not) as admissible evidence.

Furthermore, this essay probes equal temperaments in general, but not merely the familiar scale that has only twelve notes in it. It is about evenly spaced scales more broadly than about keyboards: how singers and players of instruments like violins can orient their practical understanding of sharps and flats. It ties in principles of

[6] See Bradley Lehman, "Other 'Bach' Temperaments," http://www-personal.umich.edu/~bpl/larips/bachtemps.html. Sergio Martínez Ruiz carries this historical survey further in "Temperament in Bach's Well-Tempered Clavier" (PhD diss., Universitat Autònoma de Barcelona, 2011).

[7] For example, Peter Williams cites only section 1 of my article (not section 2 and the appendices) in his argument against the curlicues. He states that the *Well-Tempered Clavier* was "exceptional," while asserting that it does not require "great skill" to tune for it one key (scale) at a time. I argue below that these assessments are mistaken. Peter Williams, *J. S. Bach: A Life in Music* (Cambridge: Cambridge University Press, 2007), 333–38.

moveable-Do solfège to understand the ways in which scales work. Ultimately, the debate about which "notes" Bach said it was so easy to "hit" continues as people grapple with the challenges and expression of the music. We care to make the compositions sound enjoyable and fulfilling.

Assessing Keyboard Temperaments

In all my work, I follow the premise that, in order to assess historical or conjectural (pseudo-historical) keyboard temperaments for use in extant compositions, there are at least four important directions of investigation. A plausible solution ought to satisfy all of them:

1. Study all available historical clues from contemporary pedagogy, instrument construction, performance practices, and music theory to know what intonation practices were normal in given places and times. What did the composer expect people to know and do when learning and performing the music? Did theorists write down what people were actually doing? Were theoretical treatises used to prescribe practices? How can we reconstruct a reasonable description of practices?
2. Study the individual, correctly spelled notes within the compositions, and the way those notes are used within scales, to determine if a tuning scheme delivers those correctly spelled notes within reasonable tolerances. Enharmonic pairs such as E♭ and D♯ are different notes because they belong to different scales. These notes in scores are important evidence, going beyond key signatures. Axiomatically, all of these notes in the compositions had to sound reasonably acceptable to the people using and hearing them, or else they would not be there.
3. Study historical records of aesthetics contemporary with the music and its locale to know what errors of pitch would have been considered acceptable in practice. How much could those people tolerate playing or singing out of tune, along with a given keyboard scheme where misspelled notes are being played? What did listeners, collaborative musicians, or students consider correct and normal?[8]
4. Directly test the music on instruments that the composer knew and played: real harpsichords, clavichords,[9] and organs, not merely computer simulations with an electronically generated tone. Preferably,

[8] In the case of Bach, according to Marpurg, this was avoidance of errors as large as a syntonic comma, presented below. *The Bach Reader*, 449.

[9] On fretted clavichords, a change of temperament design requires bending some tangents sideways to hit the strings in different places, which is not an everyday operation.

tune those instruments by ear using only the physical devices and theoretical constructions available at the time and place the music was composed. This procedure tests the plausibility of the composer (and colleagues) having done it without any electronic devices.

Enharmonics and Equal Temperaments

Musical scores tell us directly which notes are to be played, and which notes must be available inside a temperament, whether the composer is Bach or someone else. That is an obvious general principle: when the score says D♯, do not perform E♭ or any other wrong note in its place. As we will see below, the requirements are matters of fact, not taste or preference. The notes are either present, or they are absent. When a pitch is as much as a comma out of tune from its expected spot within a scale, a wrong note is being played.[10] That is why the music sounds tense or discordant when filtered through temperaments that are mismatches for its requirements.

When a piece of music includes more than twelve differently named notes, we might assume that we must necessarily use twelve-note equal temperament (ET) to get the interchangeability of enharmonic note names. That assumption is mistaken. The distinctions among the note names might be important in musical function, just as solfège preserves distinctions between Me (flattened Mi) and Ri (sharpened Re). There are various ways to do this:

- Use instruments without fixed pitches (or voices) that can deliver the differently named notes as measurably different pitches, making a conscious effort to do so.
- Use a keyboard instrument with more than twelve key levers per octave, tuned differently for all the notes that are required.
- Use a keyboard with only twelve levers per octave, set up some other regular system (a meantone scheme), and simply let it be "wrong" when some of the notes are played (because the spelling does not match the pitch).[11] Count on listeners to enjoy that result or make some argument that they ought to do so. Or expect that some of the

[10] As we will see in the 55-division, all the notes are one comma apart from each other. Therefore, to miss the pitch of an intended note by more than a comma is to hit a measurably different note.

[11] There are some rare situations where minor sixths sound consonant in the 55-division and 43-division when the bass is playing a misspelled note. For example, the bass plays E♭ (pretending to be D♯) under a B. It sounds consonant, despite the misspelling, because that interval is coincidentally near 11:7. Such phenomena are

listeners will not notice, or will not care, that some of the pitches suddenly seem "out of tune."

- Disregard the problems or dismiss them as insoluble. Ignore the distinctions between note spellings in a score, or else argue that they are unimportant. Install an "all-purpose" irregular temperament and then treat it as an approximation to equal temperament, mostly ignoring intonation and its effects. Hope that it is adequate to please the listeners.
- Set up an irregular system on a twelve-lever keyboard where at least some of the pitches are compromised, on purpose, so they can sound "good enough" when played with more than one spelling. Treat this situation as either a grudging necessity or perhaps an expressive virtue (such as having D♯ sound more tense than E♭ in the first ten bars of *Vater unser im Himmelreich*, mentioned above). This is the approach in the Bach-Lehman temperament. With the irregular system, the errors associated with misspelled or respelled notes are more subtle than in regular temperaments.

Presumably, compositions were usually for performance opportunities at hand, not for some distant future of unknown tuning schemes. What does that mean for music that was composed long before twelve-note ET became standard?[12] Is it only about convenience and reading? Why do different scales name the notes differently?

A virtue of ET is its simple predictability. With only twelve possible spots to hit, musicians have a good chance to find those spots. Pianos and guitars reinforce those spots, because these instruments are usually tuned to hit twelve equally spaced pitches within each octave.

It is convenient to have the ability to rename notes without any consequences. We are so accustomed to this system, though, that it is hard to step outside it and see other possibilities. What if the notes with different names really ought to *sound* different from one another (measurably higher or lower in pitch, or giving different expressive characters to scales)?

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beyond the scope of this essay, because we cannot count on composers deliberately misspelling notes within their music.

[12] It was a slow and gradual process between about 1850 and the 1920s. For the history, see Ross W. Duffin, *How Equal Temperament Ruined Harmony (and Why You Should Care)* (New York: W. W. Norton, 2007).

Equal Temperaments with More Than Twelve Notes

There are some equally spaced scales besides twelve-note ET that preserve the distinctions of enharmonic pitches and names, but also retain that valuable convenience of evenness. They are ideal models for musicians' intonation, whether keyboards are involved or not. In our quest for such scales, let us begin with an expert on singing. Pier Francesco Tosi (1654–1732) had a long career as a singer, teacher, and diplomat. When he was about seventy, he wrote a book about the art of singing.[13] He looked back to practices of the 1680s and 1690s, explaining those principles along with more modern trends, showing us how scales and intonation were taught to pupils. His treatise is evidence for melodic models and practices where scalar whole steps were equally spaced, yet the semitones were *not* exactly half the size of those whole steps.

The two systems he describes work out to be the regular 55-division (55 equally spaced "commas" per octave), or the competing older 43-division.[14] He says that the octave (on a keyboard) has twelve semitones that are unequally spaced, although he admits that some theorists argue for having semitones spaced as exact halfway points within tones.[15] He explains that "major" semitones (the ones belonging to diatonic scales) take up either $\frac{5}{9}$ or $\frac{4}{7}$ of a whole step. "Minor" (chromatic) semitones are smaller, and generally not used in most melodic situations except for special effect. The minor (chromatic) semitone does not belong to a diatonic scale. If such motion occurs, the singer is either changing the syllable or the word, or the bass has moved to change the harmony. It is not big enough to be treated as a normal step between the nearby notes. In other words, appoggiaturas and similar ornamental figures use only the notes of whatever diatonic scale we are in, not chromatic alterations.

Tosi seems to have drawn some of his melodic profile and perspective from working with keyboards that were tuned near regular (i.e., evenly tempered) sixth- or fifth-comma temperaments.[16] With

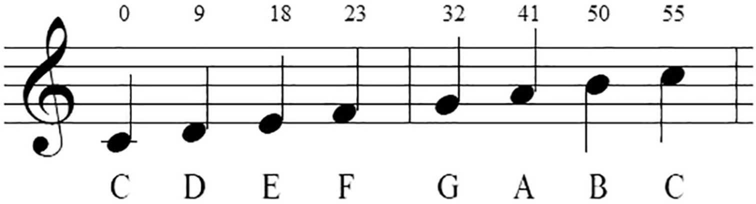
[13] Pier Francesco Tosi, trans. Johann Ernest Galliard, *Observations on the Florid Song; or, Sentiments on the Ancient and Modern Singers* (London: J. Wilcox, 1743).

[14] Tosi, *Observations on the Florid Song*, 17–21 and 33–38.

[15] Tosi, *Observations on the Florid Song*, 36.

[16] Typically, keyboards in such temperaments used the notes E♭-B♭-F-C-G-D-A-E-B-F♯-C♯-G♯. The note D♯ sometimes replaced E♭, being needed frequently for pieces in A minor or E minor.

Figure 1. Step sizes in the 55-division



every tone including nine small portions or "commas," or seven small portions, the notes of these schemes are in their expected positions within those regular temperaments (fig. 1).[17]

As a general definition of equally spaced temperaments, these represent "meantone" tuning schemes where all the fifths are narrowed by the same amount. "The same" means geometrically, a subtly perceptible quality of smudging, not a constant beat rate. "Meantone" means we have some size of major third made up of two consistently sized whole steps, like Do-Re-Mi or Fa-Sol-La. The note in the middle is "mean" (average), exactly halfway between the notes preceding and following it. Therefore, the whole step Do-Re is the exact same size (geometrically) as Re-Mi, and the same size as Fa-Sol, Sol-La, and La-Ti, as well.

Performers and listeners encountering such an equally spaced scale hear the predictable mechanical regularity of the model, knowing where each note is expected to be. If scales have too many different sizes of fifths, it is hard for non-keyboard instruments and voices to find where the notes belong. That is why modern keyboard schemes whose calculations are based on a premise of constant beat rates do not work: the fifths are all different sizes. This is historically wrong because people in Bach's time did not tune that way (arbitrarily choosing some bearing section to keep beat rates constant). This modern contrivance of constant beat rates disintegrates immediately whenever going into another neighboring octave on the keyboard: the beat rates are suddenly twice as fast or half as fast, instead of proceeding smoothly up the chromatic scale. Beyond being based on an ahistorical premise, such temperaments diverge too far from the virtues of the equal-spacing systems.

[17] For the 43-note division, the numbers are 0, 7, 14, 18, 25, 32, 39, 43.

In all the regular "meantone" tuning schemes (having consistent or "regular" size in narrowing all the fifths), the basic principle is: notes with sharps (like C♯) are *lower* in pitch than the nearby flat (D♭). Using consistently sized fifths, we get all the notes of the scales by tuning a sequence that puts each note into a relative position from the previous one. It is not calculation. Harpsichord tuners by ear do not have to calculate anything because we learn the correct amount of smudging by training and experience, apply it consistently, and all the notes are done. Rather, it is a process of listening while working: tuning the new note to a point where it sounds beatless with the previous note (a fifth lower or higher), and then carefully adjusting it to make the interval of the fifth narrower (slightly wavering in quality). Tuners also sometimes use fourths, making them slightly wide, corresponding to a narrowed fifth. Then, we copy one note or the other higher or lower by a beatless octave. Octaves are always beatless. It makes a tuning spiral, as seen in figure 2.

Figure 2. Tuning spiral by fifths. Excerpted from *How Equal Temperament Ruined Harmony (and Why You Should Care)*, 56 © 2007 by Ross W. Duffin. Used with permission of the publisher, W. W. Norton & Company, Inc. All rights reserved.

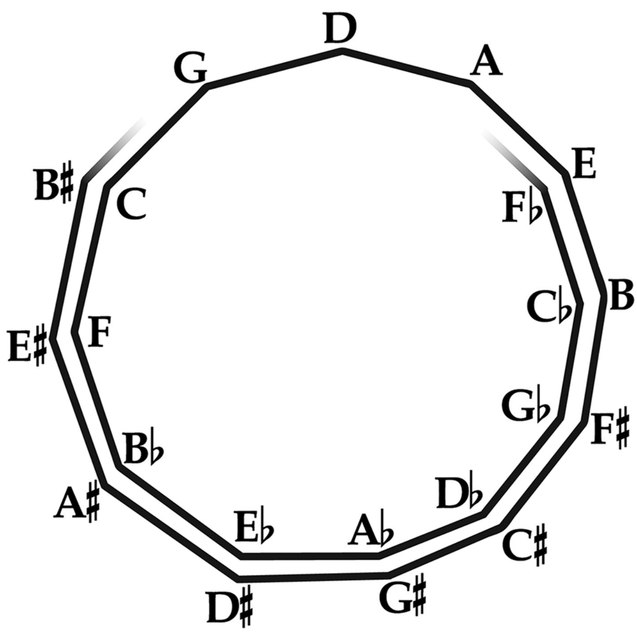
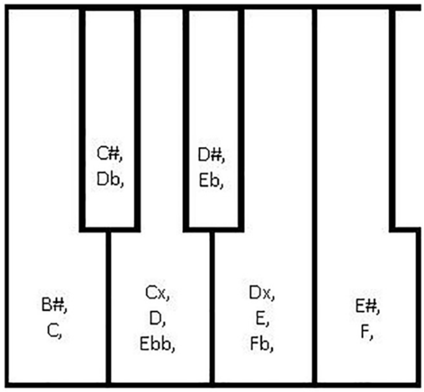


Figure 3. Fitting enharmonic notes onto a twelve-lever keyboard



We get a large set of differently named notes, all proceeding by consistently narrowed fifths from the start at C.[18] We then transpose all these notes up or down by octaves until they are near one another. They are collated into a sequence that is mostly alphabetical, rises in pitch, and (more or less) fits onto our keyboard (fig. 3).

How much narrowness or smudging in the fifths did we agree upon, before generating all these notes? That is what leads to the differently spaced equal temperaments.

- 12: If we use only the small amount of "twelfth-comma," which is a scarce and slow wobble, each round of twelve generated notes from the tuning spiral lands exactly at the same pitches from the previous round. The cycle of A♭-E♭-B♭-F-C-G-D-A-E-B-F♯-C♯-G♯ generated thirteen different notes, and with this size of narrowness in the fifths, the A♭ and the G♯ have exactly the same pitch. All the other notes in between those are evenly spaced as well. This is simple "equal temperament" of twelve notes, familiar to every piano student as "the chromatic scale." These are supposedly "all the notes," but they can each have more than one name in this system.
- 55: If we make the fifths narrow by twice that much, "sixth-comma" (more smudge, more wobble), the resulting pitches from the tuning spiral do not coincide (or nearly so) until we have generated 55

[18] For the mathematical theory of generating notes from consistently sized fifths, see Easley Blackwood, *The Structure of Recognizable Diatonic Tunings* (Princeton, NJ: Princeton University Press, 1985).

different notes.[19] This "55-division" preserves all the distinctions of differently named notes. It was an eighteenth-century standard for some prominent musicians, including Georg Philipp Telemann, Joseph Sauveur, both Wolfgang Amadeus Mozart and his father, and likely also for others who did not write about it because it was too normal to mention.[20] Tuning the keyboards, the fifths are all made narrow by the amount the musicians were accustomed to hearing. It was apparently the basis of Tosi's musical world.

- 43: If we make the fifths yet narrower, "fifth-comma," it keeps spiraling until we get a practical approximation of a 43-division.[21] This amount of tempering had some use in the sixteenth and seventeenth centuries. It too generates all the notes commonly used in musical compositions: naturals, sharps, flats, a few of the double sharps, and a few of the double flats. See again figure 2, where the notes and their names come from a "spiral of fifths" and its tuning procedure.
- 31: If we make the fifths as narrow as "quarter-comma," it approximates a 31-division. This is the "meantone" temperament for people who prefer beatless major thirds, at the expense of fifths that sound more wobbly. With this few notes in an octave that are named differently from each other, having only 31 equally spaced pitches, the pitch differences between sharps and flats are obvious.[22]
- 19: We could make the fifths narrower yet with "third-comma," and approximate a 19-division.[23] With fifths this narrow, major thirds are also narrow, but minor thirds are beatless (with their frequencies in ratio 6:5). "Minor" semitones are only half the size of "Major" semitones.

[19] The 55-division was described by Joseph Sauveur in 1701; regular sixth-comma was described by Nicolaus Ramarin in 1650. Sauveur wrote in 1711 that the 55-division was the practice of ordinary musicians. These sources are discussed in Dominique Devie, *Le Tempérament musical: Philosophie, histoire, théorie et pratique* (Société de musicologie du Languedoc: Béziers, 1990), 67–77. For Sauveur on the 55-division and 43-division, see also Mark Lindley, "Innovations in Temperament and Harmony in French Harpsichord Music," *Early Music* 61 (2013): 403–20.

[20] For an explanation of the 55-division and its historical use, see Ross W. Duffin, "'Propriety and Justness' in the Eighteenth Century," *Historical Performance* 2 (2019): 55–90; and Duffin, *How Equal Temperament Ruined Harmony*.

[21] The 43-division was described by Joseph Sauveur in 1701; regular fifth-comma was described by Abraham Verheijen in 1600 and Lemme Rossi in 1666. See Devie, *Le Tempérament musical,* 67–77.

[22] The 31-division was described by Nicola Vicentino in 1555; regular quarter-comma was described by Pietro Aron, Gioseffo Zarlino, and others in the early 1500s. See Devie, *Le Tempérament musical,* 67–77. Regular quarter-comma (with or without modification) continued well into the eighteenth century, especially on organs.

[23] The 19-division and regular third-comma were both described by Francisco de Salinas in 1577. See Devie, *Le Tempérament musical,* 67–77.

Figure 4. Regular divisions of 55, 43, 31, or 19 steps

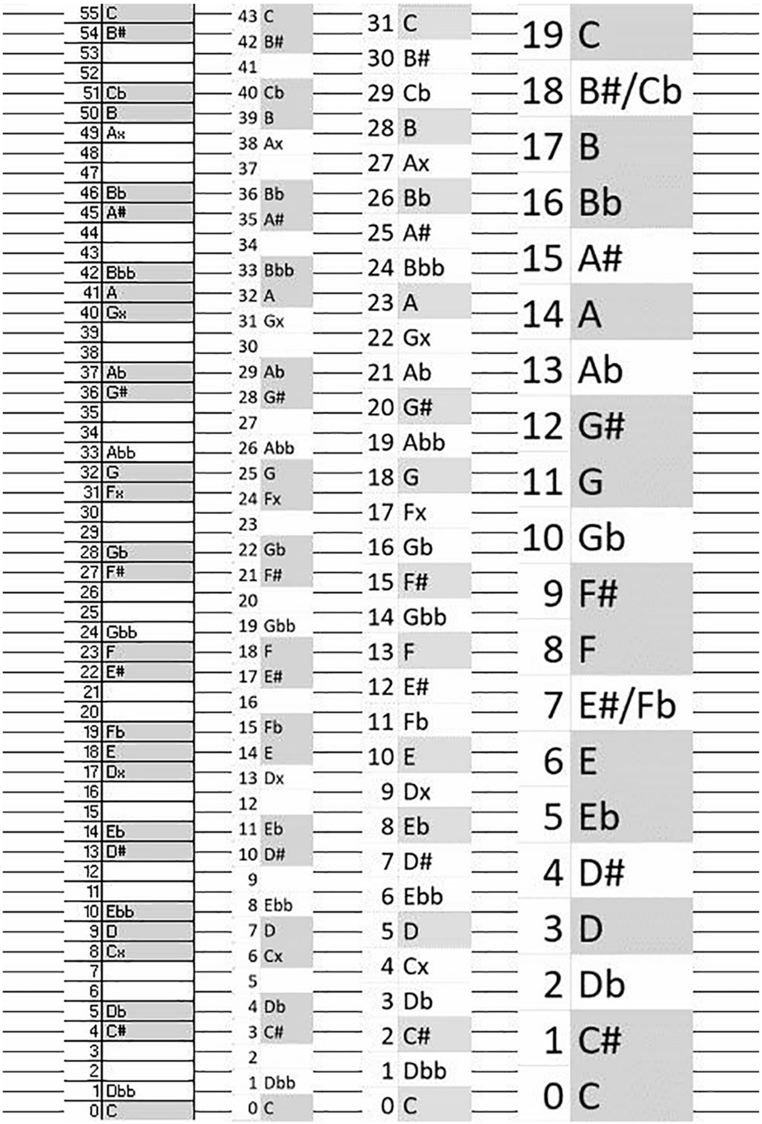


Figure 4 provides a map of these notes and their spacing. The columns map the entire octave, dividing it into 55, 43, 31, or 19 equal pieces.

The columns are:

- 55-division (equal temperament with 55 notes), corresponding in practice to Pythagorean sixth-comma or syntonic sixth-comma regular tempering in the fifths. The difference between Pythagorean and syntonic here is negligible, because those two commas are almost the same size.[24] The 55-division has the useful feature that every note is one comma away from the next; 55 commas of this size make an octave.[25] The diatonic semitones are $\frac{5}{9}$ of a whole step, and thechromatic semitones are the other $\frac{4}{9}$. That concurs with Tosi. Notice, for example, that there are four commas from B♭ to B, and five commas from B to C.
- 43-division (equal temperament with 43 notes), corresponding in practice to syntonic fifth-comma regular tempering in the fifths. The diatonic semitones are $\frac{4}{7}$ of a whole step, and the chromatic semitones are the other $\frac{3}{7}$. Tosi, again, includes this in his alternative system.
- 31-division (equal temperament with 31 notes), corresponding in practice to syntonic quarter-comma regular temperament in the fifths, or the classic “meantone” that has beatless major thirds. The diatonic semitones are $\frac{3}{5}$ of a whole step, and the chromatic semitones are the other $\frac{2}{5}$: much smaller.
- 19-division (equal temperament with 19 notes), corresponding in practice to syntonic third-comma regular temperament in the fifths. The diatonic semitones are $\frac{2}{3}$ of a whole step, and twice the size of the chromatic semitones.

The background grid shows the midpoint of each note within the first column, the 55-division. When a pitch is tuned to that frequency (whatever it might be), it is sounding as a “pure” (uncompromised) representation of that note.

Listening to keyboard instruments tuned in these regular systems builds strong expectations of intervals’ distances within diatonic scales. Observe that the consistently sized minor thirds (belonging

[24] 21.5 cents for the syntonic comma, 23.5 for the Pythagorean comma. See also Duffin, “‘Propriety and Justness’ in the Eighteenth Century,” 62–63.

[25] 1200 cents divided by 55 = 21.8, almost identical to the syntonic comma at 21.51. Kyle Gann, *The Arithmetic of Listening: Tuning Theory and History for the Impractical Musician* (Urbana: University of Illinois Press, 2019), 176–77, gives the cent values for all the notes in the 55-division. He points out that we do not have names for the gaps unless we go into triple sharps and triple flats.

to the diatonic scale at Re-Fa, Mi-Sol, and La-Do) are bigger than augmented seconds (non-diatonic).[26] Diminished fifths (Ti-Fa) are bigger than augmented fourths (Fa-Ti).[27] The "tritone," built from a sequence of three equally sized whole steps (three tones), is at Fa to Ti in the diatonic scale.

Modulation and Transposition

Measure any interval on the equal-spacing grid lines in figure 4, either with a ruler or by making marks (Do, Re, Mi, etc.) on a scrap of paper to be used as a sliding reference. What are the distances of the diatonic semitone from Ti to Do, or the fourth from Do to Fa? Observe those identical distances when you start somewhere else, transposing the interval (moving your measuring device). Whenever the notes are correctly spelled, the size of any given interval stays the same within any scale. You can see and hear exactly where to find any note in its "pure" form.

This illustrates how to transpose a composition or a section uniformly, simply by starting elsewhere but continuing to use the proper distances for scale motion. When we play Bach's Prelude in A-flat Major from the *Well-Tempered Clavier* Book II and get to the last line, the complete B𝄫 major scale played from Sol to Sol does not sound odd. It sounds like a keyboard nicely in tune in A major, but sounding a comma lower than its notated pitch.

What notes does Bach use in his pieces? I have marked them with darker shading within the 55-division (fig. 4). In modulations within the *Well-Tempered Clavier* he uses scales of D♯, A♯, and E♯ major (where the key signatures would be nine, ten, and eleven sharps!). He requires the notes C𝄪, G𝄪, D𝄪, and A𝄪.[28] Notice the distances between D𝄪, E, and F♭. On the flat side, he gets as far as B𝄫 and E𝄫. Notice the distances between E𝄫, D, C𝄪, and D♯.[29]

[26] Minor thirds in the 55-division: fourteen commas. Augmented seconds: thirteen commas. See figure 4.

[27] In the 55-division, the diminished fifth is a distinctive and resonant sound very near a beatless 45:32. It gives a strong harmonic center of gravity (Ti-Fa, usually needing to resolve to Do-Mi), and this made it attractive for eighteenth-century composers to use diminished-seventh chords.

[28] The A𝄪 is the rarest note, occurring in only one phrase of one piece in this survey of Bach's keyboard music: a cadence into E-sharp minor within the Fugue in C-sharp Major, *Well-Tempered Clavier* Book I. Accordingly, I have not marked it with the dark "frequently" shading.

[29] E𝄫 and D♯ are 100 cents apart in the 12-division, but only about 65 cents apart in the 55-division.

What scale has the note G𝄪 in it? It is Ti of the A♯ major scale, the correct distance from Ti of G𝄪 up to the Do of A♯. What scale has B𝄫 in it? It is Fa of F-flat major, the correct distance of a fourth above F♭.[30] We can always do this with Ti on a sharp note (or double sharp), or with Fa on a flat note (or double flat), finding which scales contain them. In my article "Fa, Ti, and Modulation," I show how the scale degrees Fa and Ti participate in modulation around the spiral of fifths.[31] These scale degrees are the places where new flats or sharps appear in (or disappear from) key signatures.

Does the whole instrument go sharp or flat when we modulate to get to exotic notes such as F♭? No. Within this theoretical model we can compose in any scale without a problem. We can also transpose a passage or the whole piece to a different scale, simply by assigning Do at a different place and then adjusting all the other scale degrees to be the properly spelled notes appearing (sounding) at the correct distances from Do.

Purity and Compromises

Going from theory to practice, we set up the keyboard instrument so its pitches hit most of these evenly spaced notes (especially the naturals) "nearly enough." Any irregularities or impurities are introduced carefully so they will scarcely be noticeable: playing some pitches slightly higher or lower than the "pure" correct spelling, as a concession toward a nearby enharmonic note that also must sound credible.[32] Such compromises are made gradually, so all the relationships sound correct enough. The whole piece might get played a comma too high or low, but it is transposed consistently enough and gradually enough among all the intervals that it does not ever sound wrong or "out of tune." With a temperament that is pure enough (i.e., consistent enough to seem evenly spaced through all possible scales), the composer can adhere purely to the model, confident that the music will always sound good enough.

Indeed, Bach showed directly in the *Well-Tempered Clavier* Book I that there is no need to retune for a different key signature. He

[30] Prelude in E-flat Minor, *Well-Tempered Clavier* Book I.

[31] Bradley Lehman, "Fa, Ti, and Modulation," http://www-personal.umich.edu/~bpl/larips/fati.html.

[32] These are the darker-shaded notes in the figure, the spellings we see used in scores.

paired the Prelude in E-flat Minor with the Fugue in D-sharp Minor, which ought to be a comma lower. It would be absurd to retune the harpsichord uniformly sharp or flat by a comma. Reportedly, he played this entire book in lessons for a student, without retuning.[33]

I use the word "pure" (*rein*) in the way Carl Philipp Emanuel Bach did in 1753.[34] For me, purity means doing something that is consistent and unproblematic, such as playing pitches exactly where we would expect them to be. It is avoiding anything that would seem like contamination. This contrasts to the nomenclature in much of the tuning literature, where "pure" describes beatless intervals whose pitch frequencies are in simple ratios, such as 3:2, 4:3, or 5:4. When these intervals are played together, the overtones of such a "pure" interval align without conflict. When a slightly "impure" interval is played, the overtones create a quiet but noticeable wobble or "beating." Tuners and musicians use this wobble quality or its speed to assess various amounts of such "impurity," gauging how far out of tune it is. My point is: such use of the word "pure" might mislead us into expectations of beatless intervals, instead of the different "purity" of adhering consistently to a model (such as the 55-division). Therefore, to avoid confusion, I use the word "beatless" when referring to that use of simple ratios.

Ensemble with Non-Keyboards

Let us not overlook the scales of our colleagues who do not play keyboards. What about a violinist's open strings? Should they be tuned somewhat narrower than beatless fifths, so they too will hit the notes of the 55-division or 43-division accurately? This would help orient everything.

For advice to string and wind players, and to the keyboardists playing with them, the experience of flautist Johann Joachim Quantz (1697–1773) is valuable. He worked with orchestras early in his career. In 1728 he began teaching flute to Prince Frederick of Prussia, who eventually became king in 1740 and brought Quantz into service at his court (which also employed Carl Philipp Emanuel Bach as the

[33] This was Heinrich Nikolaus Gerber. The recollection is by Gerber's son Ernst Ludwig, in 1791. *The New Bach Reader*, No. 315.

[34] See Bradley Lehman, "Carl Philipp Emanuel Bach," http://www-personal.umich.edu/~bpl/larips/cpeb.html. This sense of the word *rein* as "right" or "proper" is also presented in Duffin, "'Propriety and Justness' in the Eighteenth Century."

keyboardist). In 1752 he published a comprehensive manual of musical practices, based on these experiences.[35]

In Quantz's chapter about the ensemble that accompanies soloists, he suggested that the violin's fifths (and presumably everyone else's open strings in the ensemble) ought to be tuned as narrowed fifths, to agree with the keyboard.[36] A few pages later, he explicitly said that these ensemble musicians should carefully play the flats a comma higher than the sharps.[37] He showed how to do that: for string players by stretching to a different place on the fingerboard, or for wind players by turning the instrument (flute) or pressing the reed differently (oboe and bassoon). For keyboardists, he said the sharps and flats must be tempered as compromises to sound endurable either way.[38] By fingering carefully, violinists can play different pitches for sharps versus flats. Keyboard players cannot.[39] The keyboard instrument is tuned before the performance begins, and it does not change during the piece. In the preceding section for keyboardists, Quantz wrote also about sharps and flats being different by a comma.[40] If the player's keyboard does not have good enough (or any) compromises, Quantz's practical advice is to hide the wrongly tuned note by playing it low in the texture, or by omitting it and letting the other musicians play it in tune.[41] I believe this indicates that the 55-division was standard practice, including narrowed fifths for strings and a keyboard temperament with some compromises.

Ordinary Keyboard Tempering

For keyboard players, where should we put the pitches, now that we know that purely placed sharps and flats are theoretically "that much" (a comma) different from one another, that is, entirely different notes? How closely can we stay to the models of the 55-division, 43-division, or 31-division? That is what the Bach-Lehman

[35] Johann Joachim Quantz, trans. Edward R. Reilly, *On Playing the Flute* (1752; Boston: Northeastern University Press, 2001).

[36] Quantz, *On Playing the Flute*, 267.

[37] Quantz, *On Playing the Flute*, 270.

[38] Quantz, *On Playing the Flute*, 261.

[39] We are disregarding the uncommon situation where the keyboard instrument includes split keys, or *subsemitonia*, with the front half and the back half of the key playing strings tuned differently.

[40] Quantz, *On Playing the Flute*, 46, 260.

[41] Quantz, *On Playing the Flute*, 261.

temperament is: first building the naturals from the purity of the 55-division and using "ordinary" seventeenth-century tempering principles to adjust B and the remaining notes minimally. It is what is called a *tempérament ordinaire*.

Here are some basic principles. An effective keyboard temperament should sound easy enough for singers and players of other instruments to get along with. It should fit within an organizing system that makes sense, like one of these equally spaced scales. It should facilitate the abilities and freedom of all the musicians to express emotion, not unduly distracted by technical details. These musicians do not have to match every pitch, but they should not feel they are fighting against the keyboard instrument.[42] Axiomatically, we ought to make the notes sound "best" within the scales that are used the most. Traditionally, this is a bias toward C major and D minor, the naturals on the keyboard (C, D, E, F, G, A, and B, the same as the home notes for Tosi and his colleagues, Do/Ut, Re, Mi, Fa, Sol, La, plus the Ti/Si borrowed from the French). B♭ and F♯ are near C major, as well, which we saw earlier. They are the first new notes generated by the spiral of fifths, moving outward from the C-major scale in both directions (fig. 2), and the first notes that arrive in key signatures. The rarest notes, logically, are the ones that arrive in key signatures much farther away from C major, and the double flats and double sharps.[43]

"Ordinary" keyboard tempering, *tempérament ordinaire*, is a method used by the French and Italians of the seventeenth and eighteenth centuries.[44] This is a procedure of starting from one of the

[42] Extended sixth-comma was an orchestral standard in the eighteenth century. Musicians with non-keyboard instruments do not perform "in" a temperament, but they adjust themselves to reasonable standards to make it work. The 55-division is that practical standard, by this model. See Bruce Haynes, "Beyond Temperament: Non-Keyboard Intonation in the 17th and 18th Centuries," *Early Music* 19 (1991): 357–81.

[43] See Lehman, "Fa, Ti, and Modulation."

[44] Examples include writings by Lambert Chaumont in 1695, Antoine Vincent in 1712, Jean-Philippe Rameau in 1726 before he went to the twelve-division equal temperament, Jean Edme Gallimard explaining Rameau in 1754, an anonymous writer in Caen in 1746, and Jean le Rond d'Alembert in 1752. See Devie, *Le Tempérament musical,* 95–113. In 1756, Friedrich Wilhelm Marpurg also described an ordinary temperament he liked. Marpurg, *Principes du Clavecin* (Berlin: Haude et Spener, 1756). In Italian practice, there was a pre-1650 "Siracusa" temperament discovered by Andrea Grisendi in the archives of the cathedral of Siracusa. It is described in Francesco Cera's liner notes to Bernardo Storace, *Selva di varie compo-sizioni d'intavolatura per cimbalo et organo, Venezia, 1664—part I,* with Francesco

above regularly spaced schemes ("meantone," 55-division, 43-division, et al.), and then tastefully adjusting some of the pitches to compromised places slightly higher or lower. The procedure is:

- Keep all or most of the naturals (the C-major scale) in their regular positions.
- Make the sharps slightly higher in pitch from their tempered positions in the equally spaced model (regular division), so they can also tastefully function as flats.
- Make the flats slightly lower in pitch from their tempered positions in the equally spaced model, so they can also tastefully function as sharps.
- At the transition point between naturals and flats, that is, when tuning B♭ from F, consider also making F slightly lower than its regular spot. It makes the note E♯ work better when it is occasionally needed, plus it helps get the flats lower.
- At the transition point between naturals and sharps, that is, when tuning F♯ from B, consider also making B slightly higher than its regular spot. This helps C♭, and it helps get the sharps higher.[45]
- When the sharps and the flats meet somewhere, all twelve key levers on the keyboard have been tuned, and we are done. Test it with the music that is to be played.
- If you like the sound, write down the recipe so you or someone else can do it again. My Bach-Lehman temperament is a *tempérament ordinaire* of this type, diagrammed in Bach's book about tuning.[46]

Cera (harpsichord and spinet), 2000, Tactus 601901, compact disc. Other Italian examples of temperaments with the ordinary procedure include one by Giordano Riccati in 1751 (with approximately sixth-comma in the naturals), and the "Venetian" common symmetrical temperament based on sixth-comma and described by Giuseppe Tartini, Francesco Antonio Vallotti, and Alessandro Barca. See Devie, *Le Tempérament musical,* 183–96. In Germany, Andreas Werckmeister had described a similarly patterned symmetrical temperament in his *Orgel-Probe* of 1681, as mentioned by Devie, *Le Tempérament musical*, 124, 166, 275.

[45] I released two flowcharts in 2013 to show the "ordinary" tempering procedure in action, making methodical adjustments to build the compromises. Those flowcharts show this process of listening and further adjustment. Whenever a pitch makes a big move from its spot in the regular division, it is worth considering adjustments to the notes on each side of it (by the spiral of fifths) as well. See Bradley Lehman, "'Ordinary' Temperament Strategy: Overview," http://www-personal.umich.edu/~bpl/larips/tuning-flow-overview.pdf; and Lehman, "'Ordinary' Temperament Strategy," http://www-personal.umich.edu/~bpl/larips/tuning-flow.pdf.

[46] *Well-Tempered Clavier*. For a description of the Bach-Lehman temperament, see n3. This temperament ensures that no major thirds anywhere are as wide as Pythagorean (as we would get from four consecutive beatless fifths). No notes are as far off the 55-division as a full comma, in any scales.

Starting with C-G-D-A-E, I notate the ordinary tempering procedure as follows.[47] The flats and sharps overlap or meet somewhere (fig. 2).

- **C** * **G** * **D** * **A** * **E**
- E ? B (B tuned as some size of fifth above E)
- B + *F♯* + *C♯* + *G♯* + *D♯* . . . *(building the sharps as fifths above existing notes)*
- C ? F (F tuned as some size of fifth below the existing C)
- F + *B♭* + *E♭* + *A♭* . . . *(building the flats as fifths below existing notes)*

The general result is, therefore, a series of fifths like this:
A♭ + *E♭* + *B♭* + F ? **C** * **G** * **D** * **A** * **E** ? B + *F♯* + *C♯* + *G♯* + *D♯*

- The * between C-G-D-A-E indicates "temper these fifths narrow by the agreed consistent amount," as if we were going to make a completely regular system. How much of the C-major scale do we want to keep exactly in the spots where we expect those notes to be?
- The ? between F-C and between E-B indicates "maybe use that same amount, or maybe less," that is, closer to a beatless fifth. The F and/or the B might be slightly off their spots from the regular system, but subtly enough that scarcely anyone will notice.
- The + between the others indicates "maybe use yet less narrowing here, or none, or let the fifth go slightly wide(!), but still with subtlety."
- Meet somewhere so all the compromised flats and all the compromised sharps sound "good enough" in every scale that contains them.
- Play through those scales and play some two-voiced or three-voiced harmony in those scales (perhaps some François Couperin, or Bach's Inventions and Sinfonias), to confirm that nothing is grossly dissonant either melodically or harmonically.

To generate a good *tempérament ordinaire*, the questions are: Which sharps or flats ought to be compromised? Which notes can be left at or near their "pure" spots in a regular scale, instead of "tastefully moved"? How do we know?

[47] These are the same notes needed by violinists, violists, and cellists to tune their open strings.

Scores Show Us Which Notes We Need

An effective *tempérament ordinaire* answers these questions by examining the scores we plan to play. Which sharps and which flats are required? Examine the entire piece, not merely the key signatures. Do we need to have a "good enough" D♯ and E♭ installed on the keyboard at the same time? Do we need both A♯ and B♭ in the pieces that will be played? Does the composer require the more exotic notes like G♭, C♭, F♭, B𝄫, E♯, B♯, or F𝄪 anywhere? Make a list of all the notes we need (with sharps and flats always reckoned separately).

In 2005 I tabulated these notes for all the pieces of the *Well-Tempered Clavier* Books I and II.[48] Even before any expectation of playing both books without retuning, almost all the individual prelude-fugue pairs already need more than twelve differently named notes. Only six prelude-fugue pairs in Book I (and none in Book II) use as few as twelve notes, and these notes are different for each.[49] All of Book I needs twenty-seven notes (B𝄫-A𝄪), and Book II needs a different twenty-seven notes (E𝄫-D𝄪).

Within Bach's music, we can see that the *Well-Tempered Clavier* is not an outlier in its requirements with adventurous modulations. A survey of most of his keyboard music makes it obvious that it is not rare to go beyond twelve notes within a single piece.[50] He went beyond twelve notes in all stages of his career.[51]

Since 2005, I have added to my list more than 400 other keyboard solo pieces and chamber-ensemble pieces by Bach. I include everything that harpsichordists and organists play as "standard" Bach

[48] Bradley Lehman, "'Temperament-Killer' Tests," http://www-personal.umich.edu/~bpl/larips/testpieces.html.

[49] The twelve-note ranges are D♭-F♯, E♭-G♯, A♭-C♯, G♭-B, G-B♯, and D-F𝄪.

[50] Lehman, "'Temperament-Killer' Tests." Beyond that list, it is also important to study the keyboard requirements for Bach's vocal works, in the keys of the original organ continuo parts. For example, if a movement in a cantata was in E-flat major, the organist was usually playing from a part in D-flat major and needed all the notes near that scale. Elsewhere I pointed out that the organ pitched at *Chorton* behaved like a transposing instrument. Lehman, "Bach's Extraordinary Temperament," 17–18.

[51] For example, one of the earliest pieces in my list is the Toccata in F-sharp Minor BWV 910, which has been dated to 1707 when Bach was twenty-two. It needs the fifteen notes F, G, D, A, E, B, F♯, C♯, G♯, D♯, A♯, E♯, B♯, F𝄪, and C𝄪. A yet earlier piece is the Capriccio in B-flat Major BWV 992, which needs the different fifteen notes from G♭ to G♯. It is from 1704, when Bach was eighteen or nineteen.

repertoire. The chamber pieces have the keyboardist playing with one to three other instruments. Bach's pieces in these other genres need the same notes as the *Well-Tempered Clavier*, except for the extreme double sharp at A𝄪. Bach frequently requires flats out to G♭ or beyond (C♭, F♭, B𝄫), or sharps out to A♯or beyond (E♯, B♯, F𝄪, C𝄪). D♭ and E♯ are common notes, even in short pieces. Nor do the organ pieces show simpler enharmonic ranges than the harpsichord or clavichord repertoire. Every keyboard (and player) had to be capable of all the sharps and flats. Other than one-page chorales or pedagogical pieces for beginners, it is rare to find any pieces that are confined within the old-fashioned meantone range of E♭-G♯, that is, not using any A♭ or D♯ (or beyond). Evidently, Bach considered the meantone range of E♭-G♯ to be suitable only for beginners at tuning and playing. Whenever the home key is anything but C major, D minor, or F major, the piece almost surely goes beyond E♭ or G♯.

As seen within the *Well-Tempered Clavier*, pieces frequently need thirteen to sixteen notes. The Fantasia in A Minor BWV 922 is another example that needs sixteen. We get occasionally to seventeen notes in pieces with especially adventurous modulations (Flute Sonata in B Minor BWV 1030, *Die Kunst der Fuge*, the three-voiced Ricercar from the *Musical Offering*, and Concerto and Fugue in C Minor BWV 909). The four Duetti from the *Clavier-Übung III*, taken together, need eighteen (G♭-E♯) within a book that needs twenty-one (F♭-B♯).[52] Prelude and Fugue in E minor BWV 548 "The Wedge" needs a different eighteen. The *Goldberg Variations* need nineteen, as does the Fantasia and Fugue in G Minor BWV 542. The *Kleine harmonisches Labyrinth* BWV 591 needs twenty.[53] Some of the most extravagant examples in the list need twenty-one (Organ Concerto in C Major BWV 594) or twenty-two (Fugue in D Minor BWV 948). The *Chromatic Fantasia and Fugue* needs twenty-two or

[52] The Duetti might serve as a short test to play the whole *Clavier-Übung III*. They expose all the most important intervals with nowhere to hide an organ's temperament problems. Lehman, "Bach's Extraordinary Temperament," Appendix 5, 4.

[53] This piece is attributed variously to Bach or Johann David Heinichen. It includes at least six occasions with direct enharmonic swaps of perception or notation (a held or restruck key gets reinterpreted as a different note), such as C changing to B♯. It is approaching a note in one way with diatonic expectation, but exiting that note in a different way, like navigating a labyrinth. Lehman, "Bach's Extraordinary Temperament," Appendix 5, 3.

twenty-one, which depends on choosing a manuscript source that includes B𝄫 or not.[54] The whole book of Inventions and Sinfonias together needs twenty-four, on the presumption that we will not retune the harpsichord or clavichord.

As described above, with its extra length and the thematic enterprise of playing in all keys, each book of the *Well-Tempered Clavier* needs twenty-seven. The Prelude and Fugue in E-flat minor/D-sharp minor, Book I, needs twenty-five by itself. The Prelude and Fugue in B Minor, Book I, needs seventeen, and the fugue subject alone has thirteen![55] Most of these pieces cannot be played on a meantone organ, harpsichord, or clavichord, even if we allow for transposing either the piece or the temperament, because of the ranges beyond twelve notes. When this happens, there will always be notes at the overlap points that stick out, being wrong (more than a comma out of tune) in one spelling or another.[56] Those levers on the keyboard must be able to play two differently spelled notes, giving a pitch that (presumably) sounds acceptable either way.

From the tuning spiral, a temperament to play Bach's music must certainly be able to play the sharps D♯, A♯, E♯, and B♯, plus the flats A♭, D♭, G♭, C♭, and F♭. These are common notes in Bach's keyboard pieces. Nor should we neglect the first several double sharps and double flats. We can observe that these are the stretching points beyond old meantone schemes that went only as far as G♯ and E♭ (fig. 2).

Which Equal Division Should be Used?

If we are to accompany Tosi's ideal voice pupil, join Quantz's ensembles, or play any of the keyboard music by Bach or his

[54] Johann Sebastian Bach, *Fantasien, Präludien und Fugen*, Critical Report, ed. Georg von Dadelsen and Klaus Rönnau (Munich: Henle Verlag, 1973), 145.

[55] Arnold Schoenberg remarked that this B-minor fugue has "a *Dux* in which all twelve tones appear." Schoenberg, "Bach," in *Style and Idea: Selected Writings of Arnold Schoenberg*, ed. Leonard Stein (New York, 1975), 393–97. That statement is framed, however, assuming the interchangeability of notes in equal temperament. This *Dux* needs thirteen, including both C and B♯.

[56] For example, in his recording of the French Suites, Christopher Hogwood experimented with a customized temperament for each suite. For Suite No. 4 in E-flat Major, he used regular fifth-comma, transposed so the wolf was moved to D♭-F♯. The occurrences of G♭ (the extra note) were still wrong. Johann Sebastian Bach, *French Suites*, with Christopher Hogwood (clavecin), 1983, Decca B00004TTIH, compact disc.

predecessors, where can we put the keyboard's notes so they will be the least confusing and the most helpful? Where will they fit best into all their responsibilities as Fa, Ti, Mi, and all the other scale degrees?

Different music will often require a different temperament, especially if several contrasting pieces are to be performed together. Having decided how many of the more exotic and rare notes we need, we start from some regular system and then convert it to a *tempérament ordinaire* including those required notes. When building a *tempérament ordinaire* to play other eighteenth-century pieces, often going far beyond needing only twelve notes, we should usually start from either the 55-division (regular sixth-comma) or the 43-division (regular fifth-comma) (fig. 4). For seventeenth-century pieces, the 31-division (regular quarter-comma) with its beatless major thirds might be tempting. Its beatless major thirds make the music sound resonant, but also arguably somewhat static and lacking tension. This can, however, lead to problems. If we try to compromise G♯ because we also need A♭, the resulting pitch is so far away from both of them (between them) that the A♭ must be a comma or more too flat, or the G♯ must be a comma or more too sharp. That is how wide the gap is, where the difference between G♯ and A♭ is almost as much as two commas.[57]

The 43-division will therefore often be a better choice for seventeenth-century music, offering the necessary maneuvering space for easier compromises. For example, in the harpsichord music of Dieterich Buxtehude and Georg Böhm, the range of required notes is G♭ to B♯ (nineteen notes). In 2008, as an exercise for myself to play some seventeenth-century Germanic music, I developed a new, all-purpose *tempérament ordinaire* based on the 43-division (fifth-comma). It is suitable for music of Buxtehude, Böhm, Johann Jacob Froberger, Johann Pachelbel, Johann Kuhnau, et al.[58] Its recipe is:

[57] 1200 cents divided by 31 = 38.7 cents, which is almost as big as 2 x 21.51 (the syntonic comma). See the section "No Beatless Major Thirds," below.

[58] See Bradley Lehman, "Bonus 5: Lehman ordinary temperament for Buxtehude, Reincken, Froberger, Kuhnau, Böhm, et al.," http://www-personal.umich.edu/~bpl/larips/practical.html#17c_germanic; and my explanation in "HPSCHD-L Archives," https://list.uiowa.edu/scripts/wa.exe?A2=ind0806&L=HPSCHD-L&P=R9048. Pachelbel, who knew Bach's family in Eisenach and Erfurt, required C♭ to B♯ (twenty notes) in his organ music. The 1683 set of keyboard suites attributed to him uses twenty-four notes: D𝄫 and all of E𝄫-B♯. Pachelbel taught some of Bach's older siblings in Erfurt.

C-G-D-A-E in regular 43-division spots; F lowered to become beatless with C; B raised to be beatless with E; B-F♯-C♯ slightly narrow (twelfth-comma); C♯-G♯ beatless; G♯-D♯-A♯ similarly slightly narrow. The leftover diminished sixth A♯-F is only a twelfth-comma wide, almost beatless.[59] There are no Pythagorean major thirds anywhere. I presented this theory of tabulating sharps and flats again in 2009, showing how to assess the enharmonic requirements of any given score, and how to make good practical choices based on that information.[60] I found that this 43-division temperament also works well for François Couperin's and earlier French music.

Fa, Ti, Mi, and Their Relationships

The *tempérament ordinaire* procedures (and both the Bach-Lehman temperament and my later 2008 temperament for seventeenth-century music) are supported further by observing the way the degrees of Fa, Ti, Mi, et al. behave within tonal music elsewhere.[61] The flats must be flattened enough, and the sharps sharpened enough, so the resulting pitches can serve both sides: as Fa with all its relationships, and as Ti with all its relationships.

Ti enters diatonic scales as the last sharp added into the key signature, or by deleting the last flat. Ti always must be reasonably consonant with Sol (major third), Mi (fifth), and Re (minor sixth). Furthermore, the whole tone from La to Ti cannot sound grossly different from other whole-tone steps of Fa-Sol, Sol-La, Do-Re, or Re-Mi. Scale spelling and intonation go together. The major third Sol-Ti and the fifth Mi-Ti are the most obvious intervals to test for any new sharp when tuning by ear: no major

[59] Cent values are: C=0.0, C♯=92.2, D=195.1, E♭=294.1, E=390.2, F=498.0, F♯=592.2, G=697.6, G♯=794.1, A=892.7, B♭=994.1, B=1092.2, C=1200.0. This temperament sounds like a more intensely tonal version of Bach-Lehman, with stronger contrasts.

[60] Bradley Lehman, review of Claudio di Veroli, "Unequal Temperaments: Theory, History and Practice—Scales, Tuning and Intonation in Musical Practice," *The Viola da Gamba Society Journal* 3 (2009): 137–63. I later recast selected parts of that review as Lehman, "Why John Barnes Was Wrong to Invent a 'Bach' Temperament Using Fake Statistical Methods," http://www-personal.umich.edu/~bpl/larips/barnes.html. It includes case studies from violin sonatas by Jean-Marie Leclair and Arcangelo Corelli, and keyboard pieces by Henry Purcell, Böhm, and Bach.

[61] Lehman, "Fa, Ti, and Modulation."

third that sounds like a diminished fourth, and no fifth that is really a misspelled wolf.[62]

Now consider the flats. Flats enter as Fa, which is again familiar from key signatures as we go around the spiral of fifths in the other direction. The new tonic is a fourth below the last flat in the signature. A♭ is Fa of E-flat major, D♭ is Fa of A-flat major, and G♭ is Fa of D-flat major. Fa must always be reasonably consonant with La (major third), Re (minor third), and a good fourth or fifth with Do. The Fa-Sol whole step cannot sound grossly different from the other whole steps in the scale.

To play this music without a full comma of error in any of these intervals (i.e., without ever hitting a wrong neighboring note within the regular division), there cannot exist any Pythagorean major thirds. Pythagorean thirds come from tuning four beatless fifths in succession around the spiral. Thirds of that size are a full comma too wide, perceived as a too-sharp Mi within Do-Mi, or a too-sharp Ti within Sol-Ti. The similar problem on the flat side is Fa that is too flat under La, or serving as a new Do under Mi, again making a major third Fa-La or Do-Mi that is a comma too wide.

Fa and Mi are related melodically, as well. When Fa appears in tonal music (typically as an accidental flat intruding into a passage that had fewer or no flats), we expect it to resolve downward to Mi. If that Mi-Fa semitone is small, the effect is that Mi is too sharp for this scale context, even if Do is not being played under it. Beyond its melodic behaviors in the Re-Mi whole step and the Mi-Fa diatonic semitone, Mi must be harmonious with all of Do, Sol, La, and Ti, making no misspelled intervals anywhere.

The easiest procedure to solve all the problems is to build a quick rise of pitch into B and the early sharps (putting B-F♯-C♯ sharper than their regular-division positions). I learned this in 2004 from working at my harpsichord guided by the shapes in Bach's drawing, following what I took to be his diagram of step-by-step instructions, but it could have been deduced without that drawing as a catalyst. It is simply a good way to accommodate all the necessary intervallic

[62] In the 55-division we can listen also for the consonant Fa-Ti tritone, the 45:32 borrowed from just intonation. The 43-division has the consonant Do-Ti major seventh, 15:8. The 31-division has the consonant Sol-Ti (and all other correctly spelled major thirds) at 5:4, a seductive feature for that system harmonically, while less desirable for melodic motion.

relationships with Ti and Fa for tonal music. Beyond the direct improvements as Fa compromises to play C♭-G♭-D♭, this sharpness in B-F♯-C♯ makes it easy to handle D♯-A♯-E♯ as Ti or Mi major thirds above them, given that those pitches must also serve as compromises to play E♭-B♭-F.

No Beatless Major Thirds

Let us look more closely at the beatless major thirds within regular quarter-comma or the 31-division (fig. 4, third column). They create a technical problem that troubled Bach's contemporaries and continues to challenge today's performers. If we were to tune regular quarter-comma meantone, four of the "major thirds" would actually be diminished fourths that are extremely wide; they are almost two commas wider than the major thirds. If we try to use a diminished fourth as a major third, we must deal with closing the gap by which this interval is out of tune (i.e., it is too wide to sound like a major third) (fig. 5).

In reconciling the major thirds and diminished fourths, we cannot have all these results at the same time:

- A beatless major third, such as C-E
- Another major third (E-G♯) and a diminished fourth (G♯-C) with sizes similar to that, completing the octave C-C
- A place to put a compromised G♯/A♭ (between E and the next higher C) that sounds consonant for both

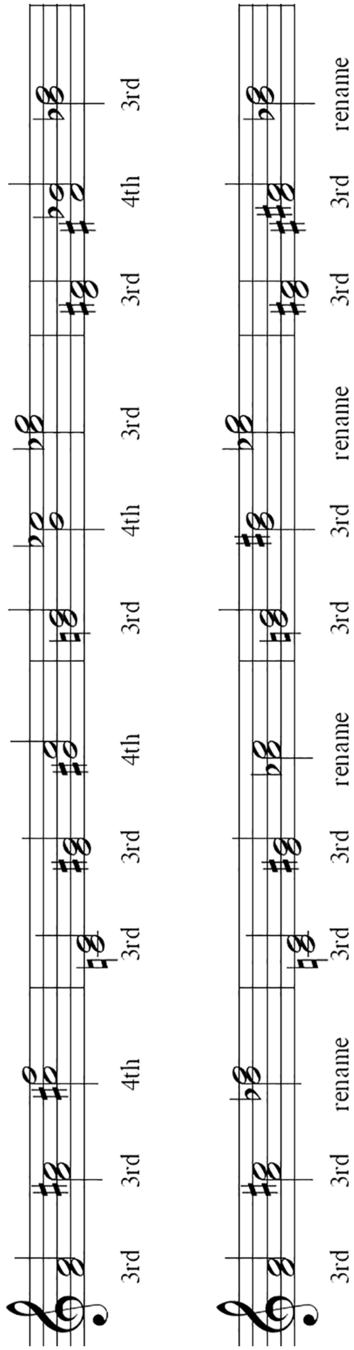
If C-E is that small, then (because of the wide gap of nearly two commas) either the G♯ or the A♭ must be a comma (or more) out of tune from their expected spots.

This problem engaged theorists during Bach's time. For example, Georg Andreas Sorge explained this problem with C-E-G♯-C in 1744, showing how a start with beatless C-E ruins any possibility of a usable G♯/A♭.[63] He also presented the temperament that Johann Philipp Kirnberger would experiment with privately thirty-five years later, the one we now know as "Kirnberger III."[64] Sorge called it

[63] Georg Andreas Sorge, *Anweisung zur Stimmung und Temperatur sowohl der Orgelwerke, als auch anderer Instrumente, sonderlich des Claviers* (Hamburg, 1744), 16–17.

[64] Sorge, *Anweisung*, 27. This is the scheme with quarter-comma tempering in C-G-D-A-E, and other fifths beatless or nearly so. Kirnberger's presentation was in

Figure 5. Major thirds and diminished fourths



clumsy (*ungeschickte*), advising that it sounds bad when playing in the major keys of E, A-flat, B, C-sharp, or F-sharp, or the minor keys of C, E-flat, B-flat, C-sharp, F, or G-sharp.[65] He said earlier in the book that he cannot tolerate listening to Pythagorean thirds or sixths, and he would rather hear cats and dogs attempting music.[66] Similarly to C-E, if F-A, G-B, and D-F♯ are made small enough to become beatless, there is nowhere to put compromises of C♯/D♭, D♯/E♭, and A♯/B♭ within the octaves of F-F, G-G, and D-D. Something will be dissonant, a comma or more out of tune. All four of those sets of major thirds set up their own gap that must be handled whenever notes are renamed (fig. 5).

In his published theoretical quarrel with Kirnberger in the late 1770s, Friedrich Wilhelm Marpurg made a similar point: temperaments containing beatless major thirds do not work well.[67] Kirnberger was promoting his extreme system of ratios where he kept as many beatless intervals as possible. It was an obsession of his, for his book about the art of pure composition, trying to achieve some measure of "purity."[68] Notice that this has nothing to do with adherence to any of the equal temperament schemes. "Pure" for Kirnberger meant beatless intervals that came from simple ratios of frequencies among the notes. Defending music against Kirnberger's standards of intonation, Marpurg cited the reputation of Bach as his expert witness on intonation. Marpurg reminded readers that Kirnberger's celebrated teacher, Bach, had taught him directly not to tune beatless major thirds.[69] Marpurg asserted that Bach, "whose ear was not spoiled by

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a private letter to Forkel, 1779, and he did not say where he got it. "The letter from Johann Philipp Kirnberger to Johann Nikolaus Forkel," http://harpsichords.pbworks.com/f/Kirn_1871.html.

[65] Sorge, *Anweisung*, 27.

[66] Sorge, *Anweisung*, 12. "Ich lieber die Katzen und Hunde wolte musiciren hören, als einen Clavieristen mit seinem auf solche Weise gestimmten Clavier oder Orgel."

[67] Friedrich Wilhelm Marpurg, *Versuch über die musikalische Temperatur* (Breslau: Johann Friedrich Korn, 1776), 208–16.

[68] Johann Philipp Kirnberger, *Die Kunst des reinen Satzen in der Musik* (Berlin, 1771); and Johann Philipp Kirnberger, *Die Kunst des reinen Satzen in der Musik*, 2nd ed. (Berlin and Königsberg, 1776, 1777, 1779).

[69] Marpurg pointed out that Kirnberger had studied with Bach when young, and that he said Bach had taught him some scheme(s) where all major thirds are required to be somewhat sharp. They cannot be beatless. Marpurg, *Versuch über die musikalische Temperatur*, 213.

bad calculation," could not have tolerated anything as far out of tune as a comma too sharp or too flat. As he put it, such notes were spoiled by the amount of 81:80, the syntonic comma.[70] The argument between the two men started to disintegrate after that, but Marpurg's theoretical point is valid.[71] So is Sorge's. To avoid all errors of as much as a comma, we must start from scales where all the major thirds are somewhat sharp, like in the 55-division or the 43-division.

The compromises of pitch to deliver the required notes are measurable as shapes and distances, corresponding to the smoothness (or not) of the scale motion within various keys. We can see and hear "how well" or "how badly" any note in any given temperament fits within the scales that include it. Taste in intonation can thus be measurable and knowable; Marpurg articulated a measurement by which Bach's taste was different from Kirnberger's, and one can articulate measurements by which various temperaments adhere to or contradict the 55-division model, as I do below.

As we can see from figure 4, any pitch that misses its "pure" target by a comma or more is a wrong note. Kirnberger's major third of A♭-C was Pythagorean, a full syntonic comma too flat from the spot where a beatless A♭ would be, relative to C. At the same time, the misspelled "G♯" above E was also almost a comma too sharp from the spot where a beatless G♯ would be. The notes A♭ and G♯ were both, therefore, unacceptably placed.[72] They were not "pure" to anything, except for having beatless fifths above and below them. No good compromise for this pitch exists, starting from Kirnberger's premise of a beatless C-E. The resulting gap between E and the next higher C is so wide, it cannot be filled by two reasonable-sounding major thirds. They are dissonant, as can be demonstrated easily at a harpsichord.

In 1756, twenty years before his quarrels with Kirnberger, Marpurg's own stated preference was to set up equal temperament by Sorge's method: set C-E-G♯-C first, each sounding good enough, and then make series of evenly tempered fifths to get there.[73] For

[70] Marpurg, *Versuch über die musikalische Temperatur*, 213.

[71] For more on the debate between Marpurg and Kirnberger, see *The Bach Reader*, 447–50; and Rita Steblin, *A History of Key Characteristics in the Eighteenth and Early Nineteenth Centuries*, 2nd ed. (Rochester, NY: University of Rochester Press, 1996), 73–95.

[72] Marpurg, *Versuch über die musikalische Temperatur*, 211.

[73] Marpurg, *Principes du Clavecin*, 3–7.

readers not ready to embrace this preference for equal temperament, he gave an alternative “best unequal in use” scheme showing that he also understood *tempérament ordinaire* practice and its viability for practicing harpsichordists.[74] His unequal temperament starts with seven evenly tempered fifths in sequence from F to F♯, and the other five “less tempered” fifths to end, that is, raising C♯ and the next sharps toward flats while checking major triads with each newly available note.[75] He was demonstrating the “ordinary” procedure here: build all the naturals first (putting the best major thirds at F-A, C-E, G-B, and D-F♯), and then fit the remaining pitches into places where they make good compromises.

Kirnberger’s Melodic Errors

The melodic properties of temperaments are important and measurable. They determine the ways that the steps sound when playing in various diatonic scales. This is another reason to stay as close to an evenly spaced scale as is feasible: so none of the scales to be used will sound implausible in their steps of Do-Re-Mi-Fa-Sol-La-Ti-Do. Because of Kirnberger’s obsession with the selective “purity” of beatless intervals and simple mathematical ratios, he disregarded that evenness and went his own way. All the notes except A in Kirnberger’s published temperament were beatless to some other note(s).[76] Kirnberger had a beatless C-E, beatless C-F-B♭-E♭-A♭-D♭, beatless E-B-F♯, beatless C-G-D, and then he was stuck with a leftover D-A-E where both of those fifths D-A and A-E were tempered by half a syntonic comma each to be extremely narrow. There is a negligible rift where F♯ and D♭ do not quite meet as a fifth.[77] This obsession with beatlessness creates melodic problems, too. Kirnberger’s whole

[74] “la meilleure des partitions inégales qui soient en usage.” Marpurg, *Principes du Clavecin*, 6.

[75] Devie’s analysis of this Marpurg temperament is mistaken to assume a quarter-comma start, or any wide or beatless fifths later. Marpurg said only to make the first fifths *foible*, and the last fifths *moins foible* (i.e., less tempered than the starting fifths). Marpurg’s word for beatless intervals (e.g., octaves) was *juste*. Devie, *Le Tempérament musical,* 104.

[76] The cent values for this “Kirnberger II”: C=0, D♭=90.2, D=203.9, E♭=294.1, E=386.3, F=498.0, F♯=590.2, G=702.0, A♭=792.2, A=895.1, B♭=996.1, B=1088.3, C=1200. See Lehman, “Bach’s Extraordinary Temperament,” Appendix 2, 22.

[77] In Kirnberger’s temperament, the interval from F♯ to D♭ is about $\frac{1}{11}$ syntonic comma narrow from a beatless fifth.

steps of C-D and F-G are the same size, but D-E is a different size (smaller), and G-A and A-B are yet another size.[78] This makes a lumpy diatonic scale, having nothing to do with Tosi's ideals of regularity among the tones (whole steps). Figure 6 shows how this looks against our grid-line map of the 55-division, with the Kirnberger marks variously hitting or missing the centers of the notes, because they are tuned flatter or sharper than expected.

Against the regular spacing of the 55-division, these are the sounds of the pitches in Kirnberger's temperament:

- C, F♯, and A are where we would expect them to be.
- G and D are each sharp from their proper positions, so much that the fifth from D to A is grotesquely narrow. It is a dissonance, not a consonance.
- E and B are each flat from their positions, so much that the fifth from A to E is also grotesquely narrow (again, a dissonance). E is so far flat that it has become a beatless major third with C.
- F, B♭, and E♭ are each flat. E♭ is so far flat that it gets almost halfway to D♯.
- A♭ is so far flat that it is closer to G♯ than A♭, but not close enough to G♯ that the major third from E to G♯ would meet expectations.
- D♭ is so far flat that it has become a very slightly sharp C♯.

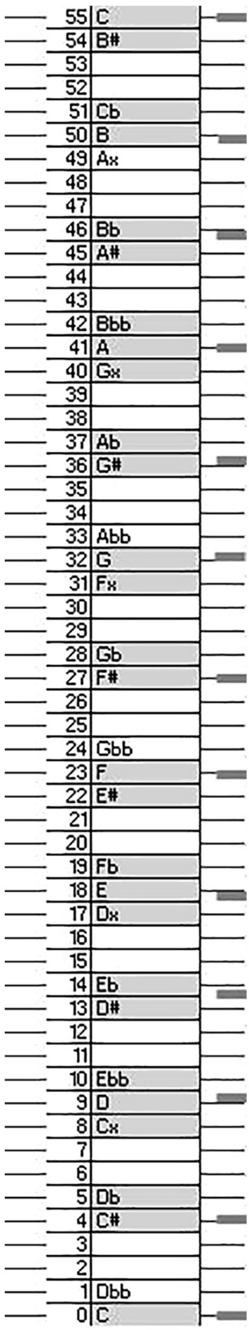
As we saw with figure 4, we can measure the consistent distances that the intervals are expected to have within a regularly spaced model. We know how far everything ought to be from Do and Sol, et al. Then, transposing the scale to play the tones and semitones starting from a different Do, observe how far Kirnberger's pitches miss those marks. For example:

- In E-flat major: Do and Sol sound acceptable, but Re and La are too sharp, Mi and Ti are yet sharper than that, and Fa is too low.
- In G major: Do, Sol, and Ti are good, but Re, Mi, Fa, and La are all too flat.

Readers might try some scales, either on paper or at a harpsichord, keeping track of which scale degrees are farthest out of tune from the expectation of the 55-division's regularity.

[78] C-D and F-G are 204 cents, D-E is 182 cents, and G-A and A-B are 193 cents. There are also some 202-cent steps elsewhere.

Figure 6. Positions of notes in Kirnberger II against grid lines representing the 55-division



Kirnberger's scales were vastly different from one another, organizing the consonances and dissonances haphazardly. He presented this irregularity as a virtue of variety, being able to express different emotions by choosing different transpositions. The errors help the listeners recognize which scales are in use. One person's sales point can be another person's reason to hate the product, as Marpurg did. Kirnberger's A is at a bizarrely impure spot from another perspective, too. It is not near a place where violinists and cellists would want to put it, with respect to reasonable fifths on open strings from either D or E.[79]

Despite Marpurg's objections, Kirnberger's misshapen temperament and his theory about key categories were repeated and reprinted for another seventy years after his death. It could have been a marketing coup, or a case of people naively reproducing scientific-looking material that they did not understand or use. When the scale steps of Do-Re-Mi-Fa-Sol-La-Ti-Do are so unevenly sized in unpredictable ways, the temperament draws attention to its own oddities. In the way it makes melodies and harmonies sound faulty, it is a distraction from listening to the music.

Other Temperaments

Even if we set up a temperament less extreme than Kirnberger's published half-comma scheme, the problems of wrong shapes are similarly observable in other temperaments. If we ever stray too far from the expected spots of regularity, our pitch becomes unrecognizable. We can abide some small errors in this mistuning, but not as far as a comma. This range of tolerance comes from Marpurg's analysis: a usable pitch must be less than a full comma out of tune, and no Pythagorean thirds are admissible.[80] It comes from Tosi, as well: in deciding what semitone to sing within your scale, the big "major"

[79] Kirnberger dealt with this problem near the end of his life, in a private letter to Forkel in 1779, suggesting that the naturals C-G-D-A-E ought to be evened out to approximately a quarter-comma each. Steblin, *A History of Key Characteristics*, 74, 86. As I mentioned above, Sorge published and disliked this "clumsy" temperament thirty-five years earlier. It still has the same problem of having nowhere to put a good G♯/A♭, or to get rid of any of the Pythagorean major thirds. Those problematic intervals occur whenever music modulates past B-flat major (flatward) or D major (sharpward).

[80] Marpurg, *Versuch über die musikalische Temperatur*, 211.

semitone (diatonic, such as C to D♭) is one comma wider than the small "minor" semitone (chromatic, such as C to C♯) (fig. 4).

I have illustrated this point elsewhere, using percentages. I compared forty-four keyboard temperaments, aligning them to have C exactly on its expected spot in the 55-division (as in figs. 4 and 6).[81] Against that background, I then measured the compromises for all twenty-four of the most common notes, from B𝄫 around the spiral of fifths to get to C𝄪. This was to show all the enharmonic pairs considered together, the amount of compromise toward the pure and correct pitch of one or the other. The notes had error measurements expressed in percentages of a comma, showing how far off they were. Reckoning all the sharps and flats separately, the percent measurement for each note showed how far the pitch on the keyboard is shifted away from that note in the 55-division, and toward a nearby note that shares the same lever on the keyboard.

I presented the reasons why some temperaments commonly used today, such as Werckmeister III and "Vallotti," are demonstrably wrong and crude.[82] These temperaments put commonly needed notes as far as a comma out of tune, especially noticeable in phrases where the bass line has flats. For instance, Werckmeister III's most offensive notes melodically (having the biggest errors from the 55-division) are G♭, C♭, and D♭. This is the reason why phrases and pieces using these notes can sound sour when played in Werckmeister III. Bach's pieces in G minor and C minor usually need at least D♭ and G♭. Even Thomas Young's best temperament from 1800 does not solve all the comma errors, because it has a Pythagorean major third at F♯-A♯.[83]

Johann Georg Neidhardt's four organ temperaments from 1724 and 1732 for a big city (*Grosse Stadt*), small city (*Kleine Stadt*), and village (*Dorf*) all do well enough at dodging Pythagorean major

[81] Eight in Lehman, "Bach's Extraordinary Temperament," 214–18; and forty-four (including those eight) in that article's Appendix 2.

[82] Lehman, "Bach's Extraordinary Temperament," 214–18.

[83] This is "Young I," not the inferior "Young II," which is sometimes presented as "Vallotti transposed up a fifth." Six notes of "Young II" are different from "Vallotti's," and the differences do not solve any of the syntonic comma problems. Furthermore, Young (working independently) did not say that he was aware of Vallotti's work. See Ross W. Duffin, "Why I Hate Vallotti (or Is It Young?)," *Historical Performance Online* (2000), https://casfaculty.case.edu/ross-duffin/why-i-hate-vallotti-or-is-it-young.

thirds.[84] Their widest major thirds are $\frac{10}{11}$ of a syntonic comma for 1732 *Dorf*, and $\frac{9}{11}$ comma for the other three.[85] Watch out for errors within the Neidhardt recipes as published in modern references.[86] Except for the one that served as Neidhardt's 1724 *Dorf* and 1732 *Kleine Stadt*, these temperaments all alter the regularity of C-G-D-A-E from the 55-division by slightly raising E. Sorge's good *Chorton* temperament from 1758 raises E, also.[87]

If using any of these, remember that the organ temperaments of Werckmeister, Neidhardt, and Sorge were not designed for the timbres of harpsichords or clavichords, or for day-to-day work. Nor are the tuning formulas in those organ treatises presented as practical hands-on methods working by ear at stringed instruments. Finally, eighteenth-century musicians were unlikely to have read those treatises of mathematical formulas unless they were organ builders. If they read music books at all, it would have been primers like Marpurg's from 1756, telling them to balance the *foible* qualities of tempered fifths by ear.[88] Smudge the fifth appropriately and then move on, without counting beats.

Conclusion

In review: a part of our goal for a keyboard temperament is to make collaboration with singers and players of non-keyboard instruments easy. When we put the notes into nearly regular spots, they are easy to find, and the scales do not contain steps of wildly different sizes. The

[84] From 1724 to 1732, he discarded 1724's *Grosse Stadt*, promoted *Kleine Stadt* to become *Grosse Stadt*, promoted *Dorf* to become *Kleine Stadt*, and made a new 1732 *Dorf*. Bradley Lehman, "Errata and Clarifications," http://www-personal.umich.edu/~bpl/larips/errata.html.

[85] The widest third in these is always A♭-C, sometimes uniquely, or sometimes sharing its size with one to four others. By contrast, in my Bach-Lehman temperament, the widest third is E-G♯.

[86] For details of the errors, see Bradley Lehman, "Errata and Clarifications," http://www-personal.umich.edu/~bpl/larips/errata.html.

[87] Georg Andreas Sorge, *Zuverlässige Anweisung Clavier und Orgeln zu temperiren und zu stimmen* (Lobenstein and Leipzig, 1758). The recipe is: $\frac{1}{6}$ Pythagorean comma C-G-D-A; A-E $\frac{1}{12}$; E-B beatless; B-F♯-C♯ $\frac{1}{12}$ each; C♯-G♯ beatless; F-C beatless; F-B♭-E♭-A♭ $\frac{1}{12}$ each. It fits the model of *tempérament ordinaire* procedure, but with a slightly higher E on the way toward building the sharps. Eight of Sorge's twelve notes are at the same places as in Bach-Lehman. His F and C♯ are lower, and his E and B are higher. Lehman, "Bach's Extraordinary Temperament," 16.

[88] Marpurg, *Principes du Clavecin*.

same principles of regularity that make it easy to find ensemble notes also make solo pieces sound unproblematic.

Anyone proposing to tune for Bach's keyboard solos and ensemble keyboard parts ought to consider the following summary:

- All diatonic scales must be usable. It is not sufficient merely to check root-position triads. Melodic motion and counterpoint demand scales whose steps make sense with enough consistency.
- Notes from G♭ to E♯ are commonplace in Bach's keyboard music, and he went beyond these (in both directions) throughout his career.
- Recognize that errors of a full syntonic comma (or more) are intolerable. Such errors "sound bad" because (measurably) we are playing the wrong note.
- Eliminate all Pythagorean (or wider) major thirds and diminished fourths because they have that full comma error, as Sorge demonstrated and derided. In other words, there can never be four beatless fifths consecutively in the tuning sequence.
- Eliminate all beatless major thirds, because they lead inevitably to the above problems.
- If unwilling to abide by these restrictions, in preference for a sound that is "more colorful" or "spicier" in differences among the keys, that preference needs a strong defense. Why should Marpurg's assertion about Bach's taste (i.e., no errors of a full comma) be disregarded?

Studying the music we plan to play, which of the levers on our keyboard must play pitches reasonably close to two different note names? Where can we put such a pitch as compromise? Tabulate the requirements and work out a practical solution, whether playing Bach's music or someone else's. The survey of these enharmonic requirements, coupled with *tempérament ordinaire* procedures, has led to the Bach-Lehman temperament as explained in 2005 and again here. The notes of the scores tell us how to tune them.

Abstract

Notes have different names (such as D♭ vs C♯) because they belong to different diatonic scales. From the sixteenth century forward, there were equally spaced systems of intonation holding a general principle where sharps are pitched lower than nearby flats. The naming distinctions matter because these notes were at least a comma apart from one another, differently pitched to fit into the scales. To play the correct notes within ensembles, and to play keyboard solos, keyboards

with only twelve key levers presented a practical problem. It was necessary to compromise (tastefully adjust) some of their sharp or flat pitches toward one another to play acceptable approximations.

Bach's keyboard parts and solos show that he required more than twelve differently named notes per composition throughout his career. For example, he frequently used both G♯ and A♭ within the same piece. He did this with such impunity and flexibility that he obviously had practical ways to make these notes sound acceptable in their different scale functions. *The Well-Tempered Clavier* Book I (1722) was not exceptional in its requirements of such enharmonic overlapping. A temperament to play Bach's keyboard music therefore must be able to play the sharps D♯, A♯, E♯, and B♯, along with the flats A♭, D♭, G♭, C♭, and F♭. These notes are primary evidence for compromised keyboard tuning. Temperaments are demonstrably wrong where they do not provide notes such as D♭ and E♯ that are far from the C-major scale.

This article presents a close look at this evidence of the required notes in Bach's music, with more than 400 pieces beyond the *Well-Tempered Clavier* Book I. Measured within eighteenth-century expectations of expert practice, enharmonically flexible temperaments can have no notes out of tune by as far as a comma, and no beatless major thirds. This rules out quarter-comma temperaments. With these technical constraints on keyboard temperaments to play this repertoire, this article proposes that Bach required a sixth-comma *tempérament ordinaire* to play his sharps and flats. That is the same practical tuning procedure and set of principles presented by this author in 2005. The background and enharmonic measurements from 2005 are explained more thoroughly in this article.

Documents from Pier Francesco Tosi, Johann Joachim Quantz, Georg Andreas Sorge, Friedrich Marpurg, et al. provide context in recognizing the scale requirements, regular systems of intonation, avoidance of Pythagorean thirds, and matters of taste. An equal-spacing illustration shows how to transpose music and observe the regularity or irregularities of temperaments. The example of Johann Philipp Kirnberger's published temperament shows why Kirnberger was unreliable as a witness for Bach, because the behavior of melodic motion is crucial in assessing a temperament's suitability.