AN IMPERFECT RING WITH A TRIVIAL COTANGENT COMPLEX

BHARGAV BHATT

Fix a perfect field k of characteristic p. Recall the following well-known fact:

Proposition 0.1. *If A is a perfect k-algebra, then $L_{A/k} \simeq L_A \simeq 0$.*

Proof. The Frobenius $F : A \to A$ induces the 0 map $F_* : L_A \to L_A$ (which is true for any $\mathbf{F}_p$-algebra A). On the other hand, since A is perfect, F is an isomorphism, so F_* is also an isomorphism, and thus $L_A \simeq 0$. Finally, since k is perfect, $L_A \simeq L_{A/k}$ as $L_k = 0$. □

The goal of this note is to record a counterexample to the converse statement; the idea of forcing variables to become products, rather than powers, used below was suggested to me by Gabber, and I grateful to him for this suggestion.

Proposition 0.2 (Gabber)**.** *There exists a non-reduced k-algebra A such that $L_{A/k} \simeq L_A \simeq 0$.*

Proof. For $i \geq 0$, let $B_i = k[x_{i,1}, x_{i,2}, \ldots, x_{i,2^i}]$ be be the polynomial algebra on the displayed 2^i generators. Write $I_i \subset B_i$ for the ideal spanned by the variables, and set

$$A_i := B_{i,\mathrm{perf}}/J_i,$$

where $B_{i,\mathrm{perf}}$ is the perfection of B_i (i.e., the direct limit along Frobenius), and $J_i = I_i \cdot B_{i,\mathrm{perf}}$. Then $L_{B_{i,\mathrm{perf}}} = 0$. As J_i is defined by a regular sequence, it is standard to see that

$$L_{A_i} \simeq J_i/J_i^2[1]$$

is a free module on 2^i generators, placed in homological degree 1. Now define maps $A_i \to A_{i+1}$ given by

$$x_{i,j}^{\frac{1}{p^n}} \mapsto \left(x_{i+1,2j} \cdot x_{i+1,2j+1}\right)^{\frac{1}{p^n}}.$$

In other words, each variable in A_i becomes a product of two variables in A_{i+1}. Set $A = \operatorname{colim} A_i$. Then we claim that $L_A = 0$, and yet A is non-reduced. To see $L_A = 0$, note that

$$L_A \simeq \operatorname*{colim}_i L_{A_i} \simeq \operatorname*{colim}_i J_i/J_i^2[1],$$

as the formation of the cotangent complex commutes with filtered colimits. Now it is enough to observe that the natural map

$$J_i/J_i^2 \to J_{i+1}/J_{i+1}^2$$

is the 0 map, since each variable in J_i becomes a product of two variables in J_{i+1}. To see that A is not perfect, set $\alpha := x_{0,1}^{\frac{1}{p}} \in A_0$. Then $\alpha^p = 0$ (since $x_{0,1} \in J_0$). On the other hand, the image of α in A_i is given by

$$\prod_{j=1}^{2^i} x_{i,j}^{\frac{1}{p}},$$

which is non-zero (as it does not lie in J_i). Thus, α gives a nilpotent non-zero element in A, so A is not reduced. □

We end by raising a question about the characteristic 0 analog:

Question 0.3. Let E be a field of characteristic 0. Does there exist a E-algebra A such that $L_{A/E} \simeq 0$, yet A is not ind-étale over E?