Math 711 • Exercise Set 1

Problem 1. Give an example of a finite dimensional representation V of a group G such that V^G and V_G have different dimensions. For which coefficient fields does an example exist? For which fields does an example exist with G finite?

Problem 2. Let G be an infinite group. Show that some representation of G is not semi-simple.

Problem 3. Let H be a finite index subgroup of G. Show that induction and co-induction from H to G are naturally isomorphic.

Problem 4. Let H be an arbitrary subgroup of G. Show that induction and co-induction from H to G are exact functors (i.e., take exact sequences of H representations to exact sequences of G representations).

Problem 5. Let G be a finite group and let H be a subgroup.

(a) Let $\mathcal{C}$ be the space of H-biinvariant functions on G. Thus an element f of $\mathcal{C}$ is a function $G \to \mathbf{C}$ such that $f(hgh') = f(g)$ for all $h, h' \in H$ and $g \in G$. For $f, g \in \mathcal{C}$, define $f * g$ by

$$(f * g)(x) = \sum_{y \in G/H} f(y) g(y^{-1} x).$$

Show that $*$ is well-defined and that $\mathcal{C}$ is a unital ring under pointwise addition and $*$.

(b) (Gelfand's trick.) For $f \in \mathcal{C}$, define $f^\vee \in \mathcal{C}$ by $f^\vee(g) = f(g^{-1})$. Suppose that $f^\vee = f$ for all $f \in \mathcal{C}$. Show that $\mathcal{C}$ is commutative.

(c) Show that $\mathrm{End}_G(\mathbf{C}[G/H])$ is naturally isomorphic to $\mathcal{C}$.

(d) Let V be a representation of G. Show that V is multiplicity free (i.e., each irreducible has multiplicity at most one) if and only if $\mathrm{End}_G(V)$ is commutative.

(e) Let $G = S_{2n}$, let $\mathcal{M}$ be the set of perfect matchings on $2n$ vertices, and let $V = \mathbf{C}[\mathcal{M}]$. Show that V is a multiplicity free representation of G.