Problem Set 2

18.904 Spring 2011

Instructions. Same as last time. Due: Friday, March 18.

Problem 1. Let S^n be the n-sphere and fix a base point $1 \in S^n$. For a pointed topological space (X, x_0) let $\pi_n(X, x_0)$ denote the set of homotopy classes of maps $(S^n, 1) \to (X, x_0)$.

(a) Suppose X is contractible. Prove that $\pi_n(X, x_0)$ is a one point set, for any n.

(b) Suppose $p : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ is a covering space. Show that $p_* : \pi_n(\widetilde{X}, \widetilde{x}_0) \to \pi_n(X, x_0)$ is a bijection for $n \geq 2$. [Here p_* is defined by $p_*(f) = [p \circ f]$; a standard argument shows this is well-defined.]

(c) Let T be a torus (i.e., $(S^1)^d$) and $t_0 \in T$ a basepoint. Prove that $\pi_n(T, t_0)$ is a one point set for $n \geq 2$. [Hint: what is the universal cover of T?] This is not at all visually obvious!

Remark. The set $\pi_0(X, x_0)$ is in fact the set of path components of X (convince yourself of this!). This set has no extra structure, such as that of a group. Of course, we know that $\pi_1(X, x_0)$ is a group, and can be any group. For $n \geq 2$, the sets $\pi_n(X, x_0)$ are in fact *abelian* groups in a natural way. These are the higher homotopy groups. Part (b) above says that the higher homotopy groups don't change when passing to covers, in constrast to the fundamental group.

Problem 2. As is well-known, there's no way to define a continuous square root function on the entire complex plane. More generally, one cannot always find a square root of a complex valued function on a given topological space. We'll show how covering spaces can be used to solve this problem. (In what follows, "function" means "continuous function.")

(a) Let X be a topological space and let $f : X \to \mathbf{C}$ be a function which is never equal to 0. Show that there exists a natural degree two covering space $p : \widetilde{X} \to X$ such that $p^* f$ has a square root, i.e., there exists a function $\widetilde{f} : \widetilde{X} \to \mathbf{C}$ such that $\widetilde{f}(x)^2 = f(p(x))$.

(b) Show that f has a square root if and only if $p : \widetilde{X} \to X$ is a trivial covering space, i.e., isomorphic to the covering space $X \amalg X \to X$.

(c) Establish analogues of (a) and (b) with logarithms taking the place of square roots.

Remark. Notice that if X is simply connected then any non-vanishing complex valued function on X has a square root and logarithm, since any covering space is then trivial.

Problem 3. Let $\mathcal{C}$ be a category and let A_1 and A_2 be two objects of $\mathcal{C}$. A triple (B, p_1, p_2) consisting of an object B of $\mathcal{C}$ and morphisms $p_1 : B \to A_1$ and $p_2 : B \to A_2$ is called a *product* of A_1 and A_2 if it satisfies the following condition: given any triple (T, f_1, f_2) consisting of an object T and morphisms $f_1 : T \to A_1$ and $f_2 : T \to A_2$, there exists a unique map $f : T \to B$ such that $f_1 = p_1 \circ f$ and $f_2 = p_2 \circ f$. We say that "$\mathcal{C}$ has products" if for every A_1 and A_2 there is a product (B, p_1, p_2).

(a) Suppose that (B, p_1, p_2) and (B', p_1', p_2') are two products of A_1 and A_2. Show that there exists a unique isomorphism $i : B \to B'$ such that $p_1 = p_1' \circ i$ and $p_2 = p_2' \circ i$.

(b) For each category $\mathcal{C}$ in the following list, say whether $\mathcal{C}$ has products or not. If it has products, describe the product of two general objects (proof not required). If not, give an example of two specific objects which do not have a product (with a reason, but not necessarily a formal proof).

(i) The category of topological spaces.
(ii) The category of pointed topological spaces.
(iii) The category of groups.
(iv) The category of covering spaces of a fixed space X.
(v) The category whose objects are sets and whose morphisms are bijections of sets.

As with most notions in category theoy, the notion of a product has a dual notion, that of a "coproduct," obtained by reversing all the arrows in the definition. Precisely, a *coproduct* of A_1 and A_2 is a triple (C, i_1, i_2) consisting of an object C and morphisms $i_1 : A_1 \to C$ and $i_2 : A_2 \to C$, with a universal property similar to that of the product.

(c) Carry out part (b) with coproducts in place of products.

Remark. Due to part (a), products are essentially unique, and there is no harm in speaking of "the" product of two objects A_1 and A_2 (when it exists). This is usually denoted $A_1 \times A_2$. Similar remarks apply to the coproduct; the typical notation is $A_1 \amalg A_2$. One of the wonders of category theory is that, after defining a category, all these things like product and coproduct come "for free" — you don't need to give new definitions for each category.

Problem 4. In this problem, we'll examine covering spaces of topological groups.

(a) Let G be a path-connected topological group and let $p : \widetilde{G} \to G$ be a covering map with $\widetilde{G}$ path-connected. Let $\widetilde{1}$ be an element of $\widetilde{G}$ mapping to 1 under p. Show that there is a unique group law on $\widetilde{G}$ such that $\widetilde{1}$ is the identity, p is a homomorphism and multiplication and inversion are continuous. [Hint: use path lifting!]

(b) Let $G = \mathrm{SL}(2, \mathbf{R})$. In the last problem set, we saw that $\pi_1(G, 1) = \mathbf{Z}$. By the Galois correspondence, we therefore have a unique connected degree two covering space $\widetilde{G} \to G$ (up to isomorphism), and by (a) we have a canonical group structure on $\widetilde{G}$ after choosing $\widetilde{1}$. Give a description of $\widetilde{G}$, as a topological group. [Hint: look up "metaplectic group" on Wikipedia.]

Part (b) is really hard, don't feel bad if you cannot get it (but do try)!

Problem 5. Let X be a topological space such that every point has a neighborhood basis of contractible open sets. We'll show how the groups $\pi_1(X, x)$, for x varying, can be put together to form a covering space of X. The construction is similar to the that of the universal cover.

Let $\Pi(X)$ denote the set of all homotopy classes of loops in X, i.e., the set of all classes $[\gamma]$ where $\gamma : I \to X$ satisfies $\gamma(0) = \gamma(1)$. Given $[\gamma] \in \Pi(X)$ and a contractible open neighborhood U of $\gamma(0)$, let $U_{[\gamma]}$ consist of all loops of the form $[\eta\gamma\eta^{-1}]$ where η is a path in U with $\eta(1) = \gamma(0)$. We topologize $\Pi(X)$ by taking the $U_{[\gamma]}$'s to be a basis. Let $p : \Pi(X) \to X$ be defined by $[\gamma] \mapsto \gamma(0)$.

(a) Show that p is a covering space map.

(b) Construct a canonical bijection $f_x : \pi_1(X, x) \to p^{-1}(x)$ for any $x \in X$.

(c) Let h be a path in X from x to y. Let $i_h : \pi_1(X, x) \to \pi_1(X, y)$ be the usual isomorphism. Define a map $i'_h : \pi_1(X, x) \to \pi_1(X, y)$ as follows. Given $[\gamma] \in \pi_1(X, x)$, regard $[\gamma]$ as an element of $p^{-1}(x)$ via the isomorphism f_x. Let $\widetilde{h} : I \to \Pi(X)$ be a lift of h with $\widetilde{h}(0) = [\gamma]$. Define $i'_h([\gamma])$ to be $\widetilde{h}(1) \in p^{-1}(x_1)$, regarded as an element of $\pi_1(X, x_1)$ via f_y^{-1}. Show that $i_h = i'_h$.

(d) Show that $\Pi(X)$ is path-connected if and only if X is simply connected.

(e) Suppose that X is path-connected and let $x_0 \in X$ be a basepoint. Let $q : X \times \pi_1(X, x_0) \to X$ be the trivial covering map, given by $q(x, [\gamma]) = x$. Show that $(\Pi_1(X), p)$ is isomorphic to $(X \times \pi_1(X, x_0), q)$ as covering spaces if and only if $\pi_1(X, x_0)$ is abelian. [Hint: use the result from the first problem set that $i_h = i_{h'}$ for any two paths h and h' if $\pi_1(X, x_0)$ is abelian. You may also use its converse, without proof. It may also be useful to consider the categorical form of the Galois correspondence.]

Remark. The covering space $\Pi(X)$ is in some ways more natural than the fundamental group, since it does not depend on a base point. There is a natural multiplication map $\Pi(X) \times_X \Pi(X) \to \Pi(X)$, where $\times_X$ denotes the product in the category of covering spaces of X, which gives each fiber $p^{-1}(x)$ a group law in such a way that each f_x is an isomorphism of groups. The covering space $\Pi(X)$ is closely related to the "fundamental groupoid."