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Recall the **covering homotopy property**: Given a covering space $p : \tilde{X} \to X$, a homotopy $f_t : Y \to X$ and a map $\tilde{f}_0 : Y \to \tilde{X}$ lifting f_0, then there exists a unique homotopy $\tilde{f}_t : Y \to \tilde{X}$ of $\tilde{f}_0$ lifting f_t.

In particular, taking f_0 to be a constant map, and f_t to be a path — that is, a homotopy between to constant maps — we get that paths between points have unique lifts as long as those paths lie in the image of the covering space.

We use this to show the **lifting criterion** (Hatcher, Prop. 1.33): Suppose $p(\tilde{X}, \tilde{x}_0) \to (X, x_0)$ a covering space, $f : (Y, y_0), (X, x_0)$ a map with Y path-connected and locally path-connected. Then a lift $\tilde{f} : (Y, y_0) \to (\tilde{X}, \tilde{x}_0)$ of f exists iff $f_*(\pi_1(Y, y_0)) \subset p_*(\pi_1(\tilde{X}, \tilde{x}_0)$

One direction: Suppose there is a lift $\tilde{f}$ of f. Then $f = p\tilde{f}$, so that $f_* = p_*\tilde{f}_*$ and hence the image of f_* is a subset of the image of p_*, as desired.
Suppose $f_*(\pi_1(Y, y_0)) \subset p_*(\pi_1(\tilde{X}, \tilde{x}_0))$.

For each $y \in Y$, we take a path γ from y_0 to y.
Then $f\gamma$ has a unique lift $\tilde{f\gamma}$ starting at $\tilde{x}_0$. Take $\tilde{f}(y)$ to be the second endpoint of this lifted path. We need to show that this is well defined, continuous. That, if it is, it is a lift follows immediately from teh fact that $\tilde{f\gamma}$ is a lift of γ with second endpoint y

Let γ' be any other path from y_0 to y. Then $(f\gamma') \cdot (\overline{f\gamma})$ is a loop h_0 in X, being the concatonation of a path from x_0 to $f(y)$ and a path from $f(y)$ to x_0.

In particular, $[h_0]$ is an element of $f_*(\pi_1(Y, y_0))$ since this is $f(\gamma' \cdot \overline{\gamma})$.

So $[h_0] \in p_*(\pi_1(\tilde{X}, \tilde{x}_0)$. h_0 is therefore homotopic with some loop h_1 which lifts under p to $\tilde{h}_1$ with homotopy h_t lifting to $\tilde{h}_t$. This then gives us a lift $\tilde{h}_0$. By the uniqueness of lifted paths, we have $\tilde{h}_0|_{[0,.5]} = \tilde{f\gamma'}$, $\tilde{h}_0|_{[.5,1]} = \overline{\tilde{f\gamma}}$. So $\tilde{f\gamma}$ and $\tilde{f\gamma'}$ have the same end point, and hence provide consistent definitions of $\tilde{f}$.

We then need to show that $\tilde{f}$, as defined is continuous

Let $U \subset X$ an open neighborhood of $f(y)$, with a lift $\tilde{U} \subset \tilde{X}$ containing $\tilde{f}(y)$, $p : \tilde{U} \to U$ a homeomorphism. Then there is a path-connected open neighborhood V of y with $f(V) \subset U$, since Y locally path-connected. We observe that $\tilde{f}$, by the above, is independent of choice of γ, so that for points $y' \in V$ we can define $\tilde{f}$ interms of the concatonation of some fixed path γ and paths η from y to y' in V. Then there is a unique inverse $p^{-1} : U \to \tilde{U}$ onto the sheet of the covering space containing U, so this inverse lifts η to $\tilde{f\eta} = p^{-1}f\eta$. Then $(\tilde{f\gamma}) \cdot (\tilde{f\eta})$ lifts $(f\gamma) \cdot (f\eta)$. $\tilde{f}(V) \subset \tilde{U}$, and $\tilde{f}|V = p{-}1(f)$. Since $p|\tilde{U}$ a homeomorphism, $p^{-1}f$ is therefore continuous, so that $\tilde{f}$ is continuous on V, and hence at y. Since this applies for each y, we have $\tilde{f}$ continuous, and hence a lift of f, as desired.

We now show the **unique lifting property** (*Hatcher, Proposition 1.34*): Given a covering space $p : \tilde{X} \to X$ and a map $f : Y \to X$, then if Y connected and $\tilde{f}_1, \tilde{f}_2$ are lifts of f which agree at some point $y \in Y$, then $\tilde{f}_1, \tilde{f}_2$ agree on all of Y.

To see why, let $y \in Y$, and let U an open neighborhood such that $p^{-1}(U)$ is a disjoint union of $\tilde{U}_\alpha$ homeomorphic to U. by p. Let $\tilde{U}_1$, $\tilde{U}_2$ be the $\tilde{U}'_\alpha s$ containing $\tilde{f}_1(y), \tilde{f}_2(y)$.

Then there is some neigborhood N of y mapped by f into U, so that $\tilde{f}_i$ map N into $\tilde{U}_i$. In particular, if $\tilde{f}_1(y) = \tilde{f}_2(y)$, then $\tilde{U}_1 = \tilde{U}_2$. But then $p^{-1}f$ lifts f in N, so that $\tilde{f}_1, \tilde{f}_2$ agree on N. On the other hand, if $\tilde{f}_1(y) \neq \tilde{f}_2(y)$ then $\tilde{U}_1 \cap \tilde{U}_2 = \emptyset$, so that $\tilde{f}_1(N) \cap \tilde{f}_2(N) \subset \emptyset$. We therefore have that the set of points where $\tilde{f}_1$,$\tilde{f}_2$ agree is open and closed. But if Y connected the only subsets of Y that are both open and closed are Y and $\emptyset$, so that $\tilde{f}_1, \tilde{f}_2$ agree everywhere or nowhere, as desired.