# Exercises from February 25th

18.904 Spring 2011

These are exercises from Noah's and Gabriels's talks. Solutions are on the following page.

**Exercise 1** (Noah, from Munkres, 54.8)**.** Let $p : \tilde{X} \to X$ be a covering map. Show that if $\tilde{X}$ is path connected and $X$ is simply connected, then $p$ is a homeomorphism.

**Exercise 2** (Gabriel)**.** Let $X = S^1$, $\tilde{X} = \{(e^{ix}, x) | x \in \mathbb{R}\}$ for some continuous function $f : \mathbb{R} \to \mathbb{R}$. Show that the standard projection $p : (x, y) \to x$ is a covering space of $X$ only if $f$ monotonic, no-where constant.

*Solution to Exercise 1*: As $p$ is a covering map, it suffices to prove that $p$ is injective. Suppose $x_0 \in X$, and $\tilde{x_0}, \tilde{x_1} \in p^{-1}(x_0)$. As $X$ is path connected, there exists some path $\tilde{f_0}$ in $\tilde{X}$ from $\tilde{x_0}$ to $\tilde{x_1}$. Then as $X$ is simply connected, there exists a path homotopy between the loop $p\tilde{f_0}$ based at $x_0$ and the trivial loop based at $x_0$. By the path homotopy lifting property, there exists a path homotopy between the loop $\tilde{f_0}$ and a lift of the trivial loop. But the only loop of the trivial loop is the trivial loop, so we must have $\tilde{x}_0 = \tilde{f}_0(0) = \tilde{f}_0(1) = \tilde{x}_1$, completing the proof.

*Solution to Exercise 2*: There are several possible ways of pursuing this result. We may, however, choose a fairly simple one: suppose $f$ somewhere constant, that is, $f[a, b] = \tilde{x}$. Then $[a, b] \in p^{-1}(p(\tilde{x}))$, so that $p^{-1}p\tilde{x}$ not discrete, and hence $p : \tilde{X} \to X$ not a covering space.

Suppose, that $f$ has an extremal point — without loss of generality a local maximum. Then there exists a point $a \in \mathbb{R}$, neighborhood $U \ni a$ such that $f(U)$ simply connected and for $b \in U$ $f(b)$ clockwise of $f(a)$ in the standard orientation. In particular, this means we can find $b_1 < a < b_2 \in U$ such that $f(b_1) = f(b_2)$. We consider $y_0 = p(e^{if(a)}, a), y_1 = p(e^{if(b_1)}, b_1) = p(e^{if(b_2)}, b_2)$. Choose a path $\gamma$ from $y_0$ to $y_1$. Then this lifts to a path $\gamma'$ from $(e^{if(a)}, a)$ to $(e^{if(b_1)}, b_1)$, but it also lifts to a path $\gamma''$ from $(e^{if(a)}, f(a))$ to $(e^{if(b_2)}, b_2)$. But these are two lifts of the same path which intersect at $(e^{if(a)}, a)$ and no-where else. But this is not possible, by Proposition 1.34, the unique lifting property.

So $p\tilde{X} \to X$ cannot be a covering space.

In particular, this shows that our standard model of covering spaces (of the circle) as helices is in some sense justified: we cannot have an infinitely sheeted covering space which, in some sense, bends back on itself. It is worth noting that the infinite-sheeted nature of this covering was not actually important to the proof; we could just as well have chosen to let $\tilde{X}$ be $\{(e^{if(x)}, g(x))\}$. The proof would then proceed similarly, with the requirement that $\tilde{X}$ proceed strictly clockwise or strictly counterclockwise as $x$ increases.