Exercises from February 16th

18.904 Spring 2011

These are exercises from Rafael's and Gabriels's talks. Solutions are on the following page.

Exercise 1 (Rafael)**.** Read carefully the following attempt to prove that $\pi_1(S^1 \vee S^1) \approx F_2$.

"Let F_2 be the free group on two generators. Giving a homomorphism from F_2 to any group G is the same as giving two homomorphisms from $\mathbb{Z}$ to G, which is equivalent to giving two elements of G. In the same way, if (X, x_0) is a pointed topological space and if we have two elements of $\pi_1(X, x_0)$ – we can think of these two elements as maps from $S^1 \to X$ – then we get a map $S^1 \vee S^1 \to X$ and consequently a map $\pi_1(S^1 \vee S^1) \to \pi_1(X, x_0)$.

Since we just showed that $\pi_1(S^1 \vee S^1)$ satisfies the universal property of F_2, namely, giving a homomorphism $\pi_1(S^1 \vee S^1) \to \pi_1(X, x_0)$ is equivalent to giving two elements of $\pi_1(X, x_0)$, we can conclude that these two groups are isomorphic."

This seems to be a very nice proof that $\pi_1(S^1 \vee S^1) \approx F_2$. However, this proof is not correct. Explain. (Keep in mind that this is a pre van Kampen proof, and cannot implicitly make use of that theorem or its consequences.)

Exercise 2 (Gabriel)**.** Complete the example given in lecture by showing that $\mathbb{Z}/2\mathbb{Z} = \mathbb{Z}/2\mathbb{Z} \ltimes \mathbb{Z}$ where the semi-direct product is taken over the homomorphism family $\phi_i : \mathbb{Z} \to \mathbb{Z}$ defined. $\phi_0(x) = x$, $\phi_1(x) = -x$

Exercise 3 (Gabriel)**.** Let A be the free product on $a_1 ... a_n$, B be the free product on $b_1 ... b_m$, C the free product on $c_1 ... c_k$ with $k \leq n, m$.

Take homomorphisms $\varphi : c_i \to a_i$ $\psi : c_i \to b_i$

Prove that $A *_C B$ is the free product on $n + m - k$ elements.

Solution to Exercise 1: The flaw in this proof is that it assumes that every group occurs as a fundamental group of a topological space. It is true that every group occurs as a fundamental group, but one needs van Kampen's theorem to prove this result.

To make this proof correct, we also need to show that every group occurs as a fundamental group of a pointed topological space. If we show this, which will be done later in the course, then this fact together with the proof above makes it a very nice (and correct) proof that $\pi_1(S^1 \vee S^1) \approx F_2$.

Solution to Exercise 2: We take ρ_1, ρ_2 to be differently labeled generators of $\mathbb{Z}/2\mathbb{Z}$. Let

$$\varphi \begin{cases} e \to (0,0) \\ \rho_1 \to (1,0) \\ \rho_2 \to (1,1) \end{cases}$$

Then this generates a homomorphism from $A = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} \to G = \mathbb{Z}/2\mathbb{Z} \ltimes \mathbb{Z}$. To see why, we observe that the only cancelation in A under multiplication is that $\rho_1^2 = e, \rho_2^2 = e$, and this cancelation applies under the image as well, since $\phi(\rho_1), \phi(\rho_2)$ are both of order 2, so that ϕ is operation preserving.

We then wish to show that this is an isomorphism:
Injective: we observe that every reduced word is $\rho_2^a(\rho_1\rho_2)^b\rho_1^c$ for some choice of a, b, c with $a, b \in \{0, 1\}$, $c \geq 0$ Let w a word in ρ_1, ρ_2 reduced, and suppose $\phi(w) = (0,0)$. Then $\phi(\rho_2)^a\phi(\rho_1\rho_2)^b\phi(\rho_1)^c = (0,0)$. $(1,1)^a(0,1)^b(1,0)^c = (0,0)$. But this is $(1,1)^a(0,-1)^b(1,0)^c$. In particular, for all possible choices of a, c this produces elements in $\{(0,-b), (0, 1+b), (1,-b), (1, 1+b)\}$, equal to $(0,0)$ iff $a, b, c = 0$. In particular, this means that ϕ is injective, since no non-identity element is taken to the identity.

Surjective: In particular, we also have that given $n > 0$ $\phi((\rho_1\rho_2)^n) = (0, -n)$, $\phi(\rho_2(\rho_1\rho_2)^{n-1}) = (1, n)$, $\phi(\rho_2(\rho_1\rho_2)^{n-1}\rho_1) = (0, n)$, $\phi((\rho_1\rho_2)^n\rho_1) = (1, -n)$, so that we can generate any combination (m, n) for $m \in \{0, 1\}$, $n \in \mathbb{Z}$. This shows ϕ surjective, so that ϕ is a bijective homomorphism, and hence an isomorphism from $A = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \ltimes \mathbb{Z}$, as desired.

Solution to Exercise 3: The result follows by a fairly direct application of the Fundamental Principle of Amalgamated Free Products. Take the free group generated by $a_1 ... a_n, b_{k+1} ... b_m$ to be D. Let G a group, $f : A \to G$, $g : B \to G$ homomorphisms such that the maps $C \to A \to G, C \to B \to G$ agree. Then in particular, we may define a homomorphism on D, h, by $h(a_i) = f(a_i) = g(b_i)$ for $i \leq k$. $h(a_i) = f(a_i)$ and $h(b_i) = g(b_i)$ for $i > k$. Then for each f, g, G satisfying the above we get a unique map from $\to G$. On the other hand, given any group G, homomorphism $h : A *_C B \to G$, we can define $f : A \to G$ by $f(a_i) = h(a_i)$, $g(b_i) = h(a_i)$ if $i \leq k$ and $g(b_i) = h(b_i)$ otherwise. Then $h(\varphi(c)) = h(\varphi(c))$, so that homomorphisms from D to G are equivalent to homomorphisms from A to G and B to G which agree on the image of C.

In particular, this means that D satisfies the fundamental principle of amalgamated free groups, so that $A *_C B \equiv D \equiv$ the free group on $n + m - k$ elements, as desired.