Let $\mathcal{C}$ be a "combinatorial category"

Ex:

- $\mathcal{C} = FI$: the category of finite sets and injections
- $\mathcal{C} = VI_L$: finite dim'l vector spaces / L and linear injections
- $\mathcal{C} = VIC_L$: finite dim'l vector spaces / L and split injections
- $\mathcal{C} = FS^{op}$: the opposite of the category of finite sets and surjections

Def: A $\mathcal{C}$-module over a commutative ring k is a functor $M : \mathcal{C} \longrightarrow \mathrm{Mod}\, k$:

$M_x \quad \forall\, x \in C$

$M_f : M_x \longrightarrow M_y \quad \forall\, f : x \longrightarrow y \in \mathcal{C}$

Def: M is finitely generated if there is a finite list of elts $m_i \in M_{x_i}$ that generate M_y $\forall y \in \mathcal{C}$ under the action of transition maps and linear combos.

M_f, $f: y \to x_i \in \mathcal{C}$

In our examples, every $\mathcal{C}$ module M has an underlying sequence of representations:

$$M_n \circlearrowleft G_n \quad \forall n \in \mathbb{N}, \text{ where}$$

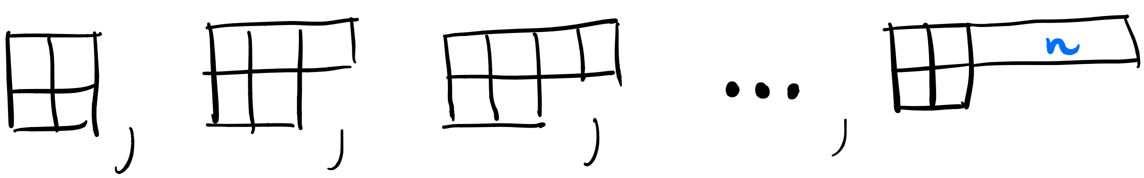
G_n is the automorphism group of the nth object of $\mathcal{C}$. $G = \mathrm{Aut}(n)$

Question: If M is a finitely generated $\mathcal{C}$ module, what does that imply about the sequence of representations M_n, $n \in \mathbb{N}$?

- **FI modules in char 0:**

Church–Ellenberg–Farb, Sam–Snowden

A finitely generated FI module is representation stable:



- Many other results for: VI, VIC, FI_G, FI_d, positive characteristic etc.

Results on Dimensions:

Let k be a field. Use $(\dim M_n)_{n \in \mathbb{N}}$ to build a function

Def: The Hilbert series of M is

$$h_M(t) := \sum_{n \in \mathbb{N}} \dim M_n \, t^n$$

Often $h_M(t)$ is rational:

$$h_M(t) = \frac{f(t)}{g(t)} \quad \text{for } f, g \text{ polynomials},$$

g taking a specific form.

Ex:

- $\mathcal{C} = \mathrm{FI}$: C-E-F, S-(S), C-E-F-N

  $h_M(t)$ is rational w/ denominator $(1-t)^d$

  (dim M_n is eventually polynomial)

- $\mathcal{C} = \mathrm{VI}_{\mathbb{F}_q}$: Nagpal in non-describing characteristic

$h_M(t)$ is rational w/ denominator $\prod_{j=0}^{d-1} (1-q^j t)$

($\dim M_n$ is eventually q-polynomial)

- $\mathcal{C} = FS^{op}$: Sam-Snowden

$h_M(t)$ is rational w/ denominator a power of $\prod_{j=0}^{d-1} (1-jt)$

In general, Sam-Snowden Gröbner theory is the most general method for proving "rationality Thms."

Drawback: Breaks symmetry, so only tells us about the dimension sequence

Goal for today: Introduce a construction that helps to unify these results, and that is able to prove equivariant versions

A coincidence:

Thm: (Arnold) The Poincaré polynomial of

$$\mathrm{Conf}_d\,\mathbb{C} = \{ (x_i) \in \mathbb{C}^d \mid x_i \neq x_j \text{ if } i \neq j \}$$

is $$\prod_{j=0}^{d-1} (1 + jt)$$

Goresky - Macpherson: interpret this result in terms of posets / poset homology.

Möbius functions

Let P be a locally finite poset, we can consider P as a category:

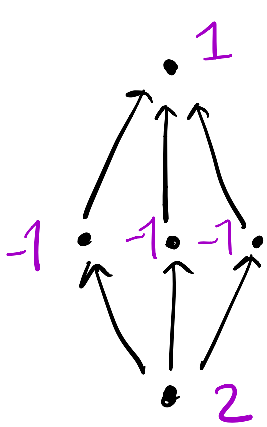
- objects $p \in P$
- morphisms: $\exists!$ morphism $p \to q$ if $p \leq q$

Def: Let P have a top element $\hat{1} \in P$. The Möbius function, $\mu : P \to \mathbb{Z}$, is the unique function satisfying

$$\sum_{p \geq q} \mu(p) = \begin{cases} 1 & \text{if } q = \hat{1} \\ 0 & \text{otherwise} \end{cases}$$

(more properly $\mu(p, \hat{1})$)

Ex



Thm: (Hall) $\mu(p) = \sum_i (-1)^i \#\{\hat{1} > p_1 > \cdots > p_i = p\}$

Def: When P is graded, we let $r(p)$ be the length of a maximal chain

chain $p = p_0 < p_1 < \cdots < p_{r(p)} = \hat{1}$

or "characteristic"

The Whitney polynomial of P is

$$W_P(t) := \sum_{p \in P} \mu(p)\, t^{r(p)}$$

(when it exists)

Def: Given $d \in \mathcal{C}$, the overcategory, $\mathcal{C}/d$, is the category with

Objects: $f : x \longrightarrow d$, $x \in \mathcal{C}$

Morphisms:

$$\begin{tikzcd} x \arrow[rr] \arrow[dr, "f"'] & & x' \arrow[dl, "f'"] \\ & d & \end{tikzcd}$$

In Rep Stability often $\mathcal{C}/d$ is equivalent to a graded poset.

Examples:

- $\mathcal{C} = FI$

$FI/X \simeq$ poset of subsets of X + containment

Whitney poly: $(1-t)^{\#X} = \sum_{S \subseteq X} (-t)^{\#X-S}$

- $\mathcal{C} = VI_{\mathbb{F}_q}$

$VI_{\mathbb{F}_q}/W \simeq$ poset of subspaces of W

<u>Whitney Poly:</u>

$$\prod_{j=0}^{\dim W - 1} (1-q^j t) = \sum_{V \subseteq W} q^{\binom{\operatorname{codim} V}{2}} (-t)^{\operatorname{codim} V}$$

- $\mathcal{C} = FS^{op}$

$FS^{op}/X \simeq (X/FS)^{op} \simeq$ poset of set partitions of X. + refinement

<u>Whitney poly:</u>

$$\prod_{j=0}^{\#X-1} (1-jt) = \sum_{p \text{ a set partition of } X} (r(p)-1)! \, (-t)^{r(p)}$$

- $\mathcal{C} = VIC$

$VIC/W \simeq$ poset of subspaces + splittings (introduced by Charney)

<u>Whitney poly:</u> ?

Main Construction: Let $\mathcal{C}$ be one of the categories above, $d \in \mathbb{N} \leftrightarrow \mathcal{C}$

Let $\mathcal{C}/_d \simeq P$.

Build a complex $K_d(M)$, $\forall M \in \mathrm{Mod}(\mathcal{C})$

$$\underset{0}{\Sigma^d M} \leftarrow \underset{1}{\bigoplus_{\substack{p \in P \\ r(p)=1}} \Sigma^{d-1} M^{\oplus N(p)}} \leftarrow \cdots \leftarrow \underset{d}{\bigoplus_{r(p)=d} M^{\oplus N(p)}}$$

Where $(\Sigma^i M)_x := M_{x+i}$.

Prop: If $K_d(M)_m$ is exact $\forall m \gg 0$, $h_M(t)$ is rational w/ denominator $W_p(t)$.

Pf: Write $W_p(t) = \sum_i c_i (-t)^i$. In degree $m = n+d$ we have $K_d(M)_m$

$$M_n \leftarrow M_{n-1}^{\oplus c_1} \leftarrow M_{n-2}^{\oplus c_2} \leftarrow \cdots \leftarrow M_{n-d}^{\oplus c_d}$$

$$\sum (-1)^i \; c_i \; \dim M_{n-i} = 0 \quad \forall \quad n \gg 0$$

$$\Longrightarrow$$

$$\sum_n (-1)^i \; c_i \; \dim M_{n-i} \; t^n = w_p(t) \; h_M(t)$$

is a polynomial, $f(t)$. $\quad \Rightarrow \quad h_M(t) = \dfrac{f(t)}{w_p(t)}$

Exactness categorifies rationality

Proto-Thm: (T.) Let $\mathcal{C} = FI$, $VI_{\mathbb{F}_q}$, or FS^{op}.

Let M be a f.g. $\mathcal{C}$ module.

Then $\exists \; d, s \in \mathbb{N}$ such that $K_d^{\,os}(M)_n$ is exact $\forall \; n \gg 0$.

$h_M(t)$ has denom $w_p(t)^s$

Construction of Chain Complexes

Let $(\mathcal{C}, \oplus)$ be a monoidal category. Suppose that $\mathcal{C}/d$ is equivalent to a poset.

$$\oplus : \mathcal{C} \times \mathcal{C}/d \longrightarrow \mathcal{C}$$

induces a functor

$$\mathrm{Res}^{\oplus} : \mathrm{Mod}\,\mathcal{C} \longrightarrow \mathrm{Mod}\,(\mathcal{C} \times \mathcal{C}/d)$$

Choose a functor

$$F : \mathrm{Mod}\,(\mathcal{C}/d) \longrightarrow \mathrm{Ch}\,(\mathrm{Mod}\,k)$$

Then "applying F in the second factor" we get

$$F \circ \mathrm{Res}^{\oplus} : \mathrm{Mod}\,\mathcal{C} \longrightarrow \mathrm{Ch}(\mathrm{Mod}\,\mathcal{C})$$

$$F \circ \mathrm{Res}^{\oplus}(M) = \left(c \mapsto F(M_{c \oplus -}) \right)$$

What is F?

Poset homology w/ coefficients (after Baclawski):

Let P be a poset with top element $\hat{1}$

Let M be a P-module

Def: $B_P(M)$ is the following Bar complex

- In homological degree s:

$$B_P(M)_s = \bigoplus_{\hat{1} > p_1 > \cdots > p_s \in P} M_{p_s}$$

$$\sum_{i=0} (-1)^i \partial_i : B_P(M) \longrightarrow B_P(M)_s$$

$$\partial_i(\hat{1} > \cdots > p_s, m) =$$

$$\begin{cases} 0 & \text{if } i = 0 \\ \hat{1} > \cdots p_{i-1} > p_{i+1} > \cdots > p_s, m & \text{if } i = 1, \ldots, s-1 \\ \hat{1} > \cdots > p_{s-1}, M_{p_s p_{s-1}}(m) & \text{if } i = s \end{cases}$$

$B_P(M)$ is closely related to the order simplicial complex NP, whose simplices are chains of elts of P.

Def: When P is graded, and the order complex $N(p, \hat{1})$ is $r(p)-3$-connected $\forall\ p \in P$,

then $K_P(M)$ is the subcomplex

$$K_P(M)_s \subseteq \bigoplus_{\hat{1} > p_1 > \cdots > p_s,\ r(p_s)=s} M_{p_s}$$

$$\ker\left(\sum_{i=0}^{s-1} (-1)^i \partial_i \right)$$

By definition:

$$K_P(M)_s \cong \bigoplus_{p \in P,\ r(p)=s} \tilde{H}_{s-2}\left(N(p, \hat{1}), M_p \right)$$

Hall's theorem

$$\cong \bigoplus_{p \in P,\ r(p)=s} M_p^{\oplus \mu(p)}$$

Prop: $K_p(M) \longrightarrow B_p(M)$ induces an isomorphism on homology.

Pf: Use the spectral sequence for the rank filtration $\boxtimes$

Example:

$$FS^{op}/_3 \simeq P(3) = \begin{tikzcd} & 1|2|3 & \\ 12|3 \arrow[ur] & 13|2 \arrow[u] & 23|1 \arrow[ul] \\ & 123 \arrow[ul] \arrow[u] \arrow[ur] & \end{tikzcd}$$

basis the three length 2 chains

$$B_{P(3)}(M) = M_{1|2|3} \leftarrow \begin{array}{c} M_{12|3} \oplus M_{13|2} \oplus M_{23|1} \\ \oplus \\ M_{123} \end{array} \longleftarrow M_{123}^{\oplus 3}$$

$$K_{P(3)}(M) = M_{1|2|3} \xleftarrow{\begin{pmatrix} 1|2|3 & 1|2|3 & 1|2|3 \end{pmatrix}} M_{12|3} \oplus M_{13|2} \oplus M_{23|1} \xleftarrow{\begin{pmatrix} 12|3 & 0 \\ -13|2 & 13|2 \\ 0 & -12|3 \end{pmatrix}} M_{123}^{\oplus 2}$$

Ways to think about B_P, K_P:

- $B_P(M)$ computes $\mathrm{Tor}^P_\bullet(S(\hat{1}), M)$

  where $S(\hat{1})$ is the P^{op} representation
$$\begin{tikzcd} & k \arrow[dl] \arrow[d] \arrow[dr] & \\ 0 \arrow[dr] & 0 \arrow[d] & 0 \arrow[dl] \\ & 0 & \end{tikzcd}$$

  (when k is a field, this is the same as $\mathrm{Ext}^\bullet_P(M, S(\hat{1})^*)^*$)

- $B_P(M) = \mathrm{cone}\Big(\underset{p \in P - \hat{1}}{\underrightarrow{\mathrm{hocolim}}} M_p \longrightarrow M_{\hat{1}} \Big)$

so it "measures the difference between $\{M_p\}_{p<\hat{1}}$ and $M_{\hat{1}}$."

- $K_p(M)$ is a "Koszul complex" in the sense of Koszul duality theory.

By taking $F = B_{e/d}$ or $K_{e/d}$,

we obtain functors

$$B_d, K_d : \mathrm{Mod}(\mathcal{C}) \longrightarrow \mathrm{Ch}(\mathrm{Mod}\, \mathcal{C})$$

K_d is defined for FI, VI_L, VIC_L, FS^{op}, FI_r, ...

Examples: For a monoidal cat $\mathcal{C}, \oplus$

let $\Sigma^x M$ denote $\mathrm{Res}^{x \oplus -}(M) = M_{x \oplus -}$

- For $\mathcal{C} = FI$: $K_d(M)$ is

$$\Sigma^d M \leftarrow \Sigma^{d-1} M^{\oplus d} \leftarrow \Sigma^{d-2} M^{\oplus \binom{d}{2}} \leftarrow \cdots \leftarrow M$$

the complex obtained by iterating

cone $(M \longrightarrow \Sigma M)$ d-times

- For $\mathcal{C} = VI_L$: $K_W(M)$ is

$$\Sigma^W M \leftarrow \bigoplus_{\substack{V \subseteq W \\ \text{codim } 1}} \Sigma^V M \leftarrow \bigoplus_{\substack{V \subseteq W \\ \text{Codim } 2}} \Sigma^V M \otimes \mathrm{Stein}(W/V) \leftarrow \cdots$$

- For $\mathcal{C} = FS^{op}$: $K_d(M)$ is

Stirling # of 1st kind

$$\Sigma^d M \leftarrow \Sigma^{d-1} M^{\oplus \binom{d}{2}} \leftarrow \cdots \leftarrow \Sigma^{d-i} M^{\oplus S(d,d-i)} \leftarrow \cdots$$

- For $\mathcal{C} = \cdots \bullet \underset{y}{\overset{x}{\rightrightarrows}} \bullet \underset{y}{\overset{x}{\rightrightarrows}} \bullet \underset{y}{\overset{x}{\rightrightarrows}} \bullet \cdots$

$$\mathrm{Mod}\,\mathcal{C} \simeq \mathrm{Gr\,Mod}\,(k[x,y])$$

and $\forall d$

$K_d(M)$ is the Koszul complex:

$$M(d) \xleftarrow{(x \;\; y)} M(d-1)^{\oplus 2} \xleftarrow{\begin{pmatrix} -x \\ -y \end{pmatrix}} M(d-2)$$

One interpretation of K_d / B_d:

when $\mathcal{C}$ is an EI category, we can define

$\mathcal{C}$-module homology $H_\bullet(\mathcal{C})_d$ $\forall\, d \in \mathcal{C}$

Then $H_i(B_d(-))$ equals

$$\operatorname{Mod}\mathcal{C} \xrightarrow{\operatorname{Res}^{\otimes}} \operatorname{Mod}(\mathcal{C}\times\mathcal{C}) \xrightarrow{H_i^{\mathcal{C}}(-)_d} \operatorname{Mod}\mathcal{C}$$

"in second factor"

Case of FS^{op} modules

Background on FS^{op}

Proved Noetherian by Sam-Snowden, and used to prove the Lannes-Schwarz Artinian conjecture.

Applications to:

- Kazhdan-Lusztig polynomials of braid matroids (Proudfoot-Young)
- Homology of the moduli space of stable curves $\overline{M}_{g,n}$ (T.) and Kontsevich mapping spaces (T.-Petersen)
- Resonance arrangements (Proudfoot-Ramos)

Analogy / Connection (Joint w/ Sam-Snowden)

FI modules : $\mathbb{C}[x]$ modules

FS^{op} modules : $\mathfrak{g}$-modules for $\mathfrak{g}$ the Witt lie algebra of formal vector fields on unit disc.

In general, they are not well understood.

Example of an FS^{op} module

Consider

$$M_X = \mathbb{Q}\{ \text{set partitions of } X \text{ into 2 blocks} \}$$

unordered

Then M is an FS^{op} module, via pull-back of set partitions:

$$f: Y \longrightarrow X.$$

p

$\{a\}\{b\}$

$\{a\}$

In fact M is generated in degree ≤ 2

$Tr(\sigma)$ = # of partitions fixed by σ

$\sigma : X \to X$

- If σ has a cycle of odd length then σ must fix each block

$$Tr(\sigma) = \frac{2^{X_1(\sigma) + X_2(\sigma) + \cdots}}{2}$$

where

$$\chi_i(\sigma) := \# \text{ of } i\text{-cycles of } \sigma$$

- If all of the cycles of σ have even length, then

$$\mathrm{Tr}(\sigma) = \underbrace{\frac{2^{\chi_2(\sigma)+\chi_4(\sigma)+\cdots}}{2}}_{\text{preserving blocks}} + \underbrace{\frac{2^{\chi_2(\sigma)+\chi_4(\sigma)+\cdots}}{2}}_{\text{swapping blocks}}$$

Can iterate $K_\ell(M)$ by taking total complex.

Thm: (T.) Let M be an FS^{op} module, which is a submodule of one generated in degree d. Then $\exists s \in \mathbb{N}$ such that

$$\forall\ \ell_1, \dots, \ell_s \in \mathbb{N} \qquad \ell_i > d$$

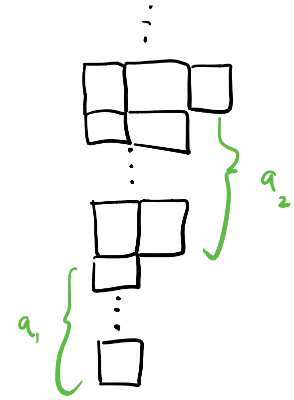
$K_{\ell_1} \circ K_{\ell_2} \circ \cdots \circ K_{\ell_r}(M)$ is exact.

- Key ingredient is Sam-Snowden Gröbner theory
- Taking Frobenius characters, this translates into a system of linear differential equations for $ch(M)$.

Def: An FS^{op} module is of class (d,s) if the hypothesis + conclusion of the previous theorem apply to it.

To describe the character of a class (d,s) module we need the following functions.

Character Exponentials



Def Let $A = 1^{a_1} 2^{a_2} \cdots$ be an integer partition.

Symmetric function

$$A^{\chi} := \prod_n \Big(\sum_{d|n} d\, a_d\Big)^{\chi_n} \longleftrightarrow P_A\Big[\sum_{m\geq 0} h_m\Big]$$

Ex: • $A = 1^2$: $A^{\chi} = \prod_n 2^{\chi_n} = 2^{\sum_n \chi_n}$

• $B = 2^1$: $B^{\chi} = \prod_{n \text{ odd}} 0^{\chi_n} \prod_{n \text{ even}} 2^{\chi_n}$

0 unless all cycles of σ are even

The example above had character function

$$\frac{A^{\chi} + B^{\chi}}{2}$$

Def:

Let $\nu = 1^{m_1} 2^{m_2} \cdots$ be another integer partition.

$$\binom{\mathbb{X}}{\nu} A^{\mathbb{X}-\nu} := \prod_n \binom{\mathbb{X}_n}{m_n} \left(\sum_{d|n} d a_d \right)^{\mathbb{X}-m_n} \longleftrightarrow \frac{p_\nu}{z_\nu} \cdot \overbrace{\left(P_A \left[\sum_{m \geq 0} h_m \right] \right)}^{\text{Symmetric function}}$$

Ex: If $A = 0$, $\displaystyle\prod_n \binom{\mathbb{X}_n(\sigma)}{m_n} 0^{\mathbb{X}_n(\sigma) - m_n}$

is nonzero iff $\sigma \in S_n$ has cycle type ν.

Characters

Thm: (T.) Let M be an FS^{op} module of class (d,s). Then the character of M is of the form:

$$\sum_{\nu, A} c_{\nu,A} \binom{X}{\nu} A^{X-\nu}$$

for $|A| \leq d$, $\operatorname{rank}(\nu) \leq d$, $c_{\nu,A} \in \mathbb{Q}$.

\# of rows in Young diagram of ν.

Finite dimensionality

Thm (T.) There is a finite dimensional space of class functions $U_{d,s}$ containing the character functions of FS^{op} modules of class (d,s)

$$\dim U_{d,s} = \binom{d+s-1}{s-1} \sum_{i \leq d} \#\text{ int partitions of } i.$$

Multiplicity polynomiality

Thm (T.) Let $(n,\lambda) = n \geqslant \lambda_1 \geqslant \lambda_2 \geqslant \cdots$

Then for M an FS^{op} module of class (d,s)

$$\mathrm{mult}_{(n,\lambda)}\left(M_{n+|\lambda|}\right)$$ is eventually quasi-polynomial

of period $\leqslant d$ and degree $\leqslant ds$.

Questions:

- For which categories does M finitely generated $\Rightarrow K_d^{or}(M)$ (mod torsion) exact?

  e.g. VI_q in describing characteristic

- How does the class of an FS^{op} module behave in short exact sequences?

- What is the representation theory of FS^{op}?
Can the complexes $K_d(M)$ be used analogously to $M \longrightarrow \Sigma M$ for FI modules?

Thanks!

Thanks Andrew + Jenny!

Idea of proof for FS^{op} modules.

1 Use $OS^{op} \longrightarrow FS^{op}$ where OS^{op} is the category of totally ordered sets and "ordered surjections"

$f: X \twoheadrightarrow Y$ s.t. $i \leq j \Rightarrow \min(f^{-1}(i)) \leq \min(f^{-1}(j))$

Reduce to OS^{op} submodules of $k\,OS(-, n)$

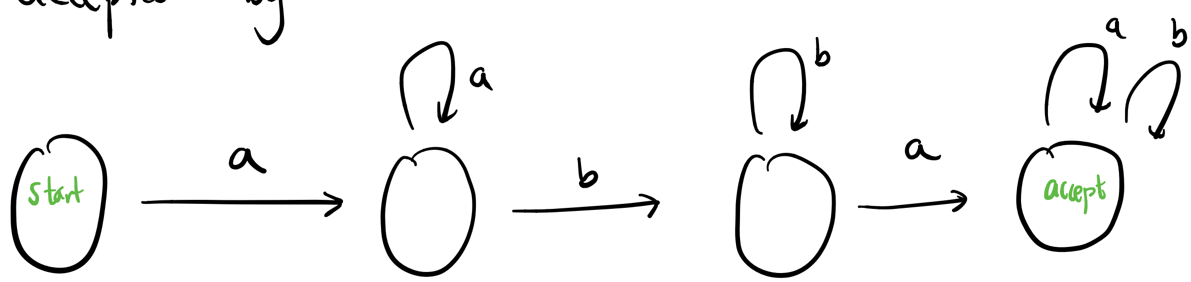
2 Gröbner theory: reduce to OS^{op} subsets of $OS(-, T)$

Sam-Snowden showed using Higman's lemma that these are classified by certain DFAs

Ex: If $T = \{a, b\}$ and aba describes the surjection $[3] \twoheadrightarrow T$

$$1 \mapsto a$$
$$2 \mapsto b$$
$$3 \mapsto a$$

then aba generates the OS^{op} subset accepted by



3 Associate $P(d)^{\times r}$-sets to DFA's

and prove that they yield exact complexes

$$I(w, A) \subseteq P(d)^{\times n}$$

$$= \left\{ \text{partitions } p \;\middle|\; \begin{array}{l} w \text{ factors through } p, \\ w/p \text{ accepted by } A \end{array} \right\}$$

Prove that the $P(d)^{\frown}$ set associated to $I(w, A)$ is exact.

Arnold:

$$\sum_i \dim H_i(\mathrm{Conf}_n \mathbb{C})\, t^i = \prod_{j=0}^{n-1} (1 + j$$

Goresky - Macpherson:

$\mathrm{Conf}_n \mathbb{C} \subseteq \mathbb{C}^n$ is a hyperplane complement, w/ associated poset the lattice of set partitions.