Connections between commutative algebra and representations of categories

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Representations of categories

Definition

A **representation** of a category $\mathcal{C}$ over a ring $\mathbf{k}$ is a functor $\mathcal{C} \to \mathbf{Mod}_{\mathbf{k}}$.

We also use the terminology $\mathbf{k}[\mathcal{C}]$-module.

Representations of categories

Two observations from recent years:

- Representations of categories are a useful tool to study a wide range of stability problems in algebra, topology, representation theory, statistics, and more.
- In many cases, $\mathbf{k}[\mathcal{C}]$-modules behave like modules over a commutative ring, and so ideas from commutative algbera and algebraic geometry are useful in studying them.

Examples

Basic finiteness properties

$M = \mathbf{k}[\mathcal{C}]$-module, $m \in M(x)$

$\to \langle m \rangle =$ smallest submodule containing m

Definition

M is **finitely generated** if $M = \langle m_1 \rangle + \cdots + \langle m_n \rangle$.

Definition

$\mathbf{k}[\mathcal{C}]$ is **noetherian** if any submodule of a finitely generated module is finitely generated.

Example 1: $\underline{\mathbf{N}}$

$\underline{\mathbf{N}}$ = category associated to the poset $\mathbf{N}$.

Objects are the non-negative integers and there is exactly one morphism $n \to m$ when $n \leq m$.

$\mathbf{k}[\underline{\mathbf{N}}]$-module	$\iff$	$\mathbf{k}$-module V_n for each $n \in \mathbf{N}$ with linear maps $V_n \to V_{n+1}$
	$\iff$	$\mathbf{N}$-graded $\mathbf{k}[t]$-module

Theorem (Hilbert)

$\mathbf{k}[\underline{\mathbf{N}}]$ *is noetherian if* $\mathbf{k}$ *is left-noetherian.*

Example 2: **FI**

FI = category of finite sets with injections.

$$\mathbf{k}[\mathbf{FI}]\text{-module} \iff \begin{cases} \text{representation } V_n \text{ of } S_n \text{ for each } n \in \mathbf{N} \text{ with} \\ S_n\text{-equivariant maps } V_n \to V_{n+1} \text{ satisfying } \ldots \end{cases}$$

Remark

$\mathbf{k}[\mathbf{FI}]$-modules are like $\mathbf{k}[t]$-modules, but much more complicated.

Example 2: **FI**

Theorem

$\mathbf{k}[\mathbf{FI}]$ *is noetherian if* $\mathbf{k}$ *is left-noetherian.*

Proved by:

- Snowden (char. 0)
- Church–Ellenberg–Farb (char. 0)
- Church–Ellenberg–Farb–Nagpal
- Sam–Snowden

Example 2: **FI**

X = topological space

$\mathrm{Conf}_S(X)$ = space of injections $S \to X$

$S \mapsto \mathrm{Conf}_S(X)$ is a contravariant functor $\mathbf{FI} \to \mathbf{Top}$

$\implies S \mapsto \mathrm{H}^i(\mathrm{Conf}_S(X), \mathbf{k})$ is a $\mathbf{k}[\mathbf{FI}]$-module (i fixed)

Theorem (Church–Ellenberg–Farb–(Nagpal))

Under suitable hypotheses, this $\mathbf{k}[\mathbf{FI}]$*-module is finitely generated.*

Remark

The noetherianity result for $\mathbf{k}[\mathbf{FI}]$ is crucial for this application.

Example 3: $\mathbf{Vec}^\Delta$

$\mathbf{Vec}^\Delta$ is the following category ($\mathbf{k}$ a field):

- An object is a finite family of $\mathbf{k}$-vector spaces $\{V_i\}_{i \in I}$.
- A morphism $\{V_i\}_{i \in I} \to \{W_j\}_{j \in J}$ consists of a surjection $f \colon J \to I$ and for each $i \in I$ a linear map $V_i \to \bigotimes_{f(j)=i} W_j$.

A Δ-**module** is a $\mathbf{k}[\mathbf{Vec}^\Delta]$-module satisfying a technical condition.

Example 3: $\mathbf{Vec}^\Delta$

Associating to $\{V_i\}_{i \in I}$ the (affine cone on the) Segre embedding

$$\prod_{i \in I} \mathbf{P}(V_i^*) \to \mathbf{P}\left(\bigotimes_{i \in I} V_i^*\right)$$

defines a contravariant functor

$$\mathbf{Vec}^\Delta \to \{\text{closed embeddings of varieties}\}.$$

$\implies$ associating to $\{V_i\}_{i \in I}$ the space of p-syzygies of the Segre embedding (for p fixed) defines a Δ-module.

Example 3: $\mathbf{Vec}^{\Delta}$

Theorem (Snowden, Sam–Snowden)

Noetherianity holds for Δ-modules.

Example 3: $\mathbf{Vec}^{\Delta}$

Theorem (Snowden, Sam–Snowden)

The Δ-module of p-syzygies of the Segre embedding is finitely generated.

Remark

Explicit generators for this Δ-module are known only for $p \leq 3$.

General results

Noetherianity

Problem

Find a general condition on $\mathcal{C}$ that ensures $\mathbf{k}[\mathcal{C}]$ is noetherian, for any left-noetherian ring $\mathbf{k}$.

Solution

Adapt the theory of Gröbner bases to $\mathbf{k}[\mathcal{C}]$-modules.

Monomials

For $x \in \mathcal{C}$ define a $\mathbf{k}[\mathcal{C}]$-module P_x by $P_x(y) = \mathbf{k}[\mathrm{Hom}(x, y)]$.

$\rightarrow$ The modules P_x take the role of free modules.

For $f \in \mathrm{Hom}(x, y)$, write e_f for the corresponding element of $P_x(y)$.

$\rightarrow$ The elements e_f take the role of monomials.

Monomials

Define $e_f \mid e_{f'}$ if $f' = gf$ for some g.

Define $e_f \sim e_{f'}$ if $e_f \mid e_{f'}$ and $e_{f'} \mid e_f$.

$\mathcal{M}_x$ = set of monomials in P_x up to $\sim$. Partially ordered by divisibility.

An **admissible order** on $\mathcal{M}_x$ is a well-order $<$ such that $e_f < e_{f'}$ implies $e_{gf} < e_{gf'}$, whenever this makes sense.

Gröbner bases

Theorem (Richter, Sam–Snowden)

Suppose that:

1. $\mathbf{k}$ *is left-noetherian.*
2. $\mathcal{C}$ *is* **directed** *(any self-map is the identity).*
3. $\mathcal{M}_x$ *is well partially ordered under divisibility.*
4. $\mathcal{M}_x$ *admits an admissible order.*

Then any submodule N of P_x is finitely generated.

Idea of proof:

- Admissible order $\rightarrow$ initial submodule $\mathrm{in}(N)$.
- Well partial order $\implies$ $\mathrm{in}(N)$ finitely generated.
- Standard Gröbner basis argument $\implies$ N finitely generated.

Gröbner categories

Definition

A category $\mathcal{C}$ is **Gröbner** if it is directed and $\mathcal{M}_x$ is well partially ordered and admits an admissible order for all $x \in \mathcal{C}$.

Theorem

If $\mathcal{C}$ is Gröbner then $\mathbf{k}[\mathcal{C}]$ is noetherian, for any left-noetherian ring $\mathbf{k}$.

Quasi-Gröbner categories

Definition

A category $\mathcal{C}$ is **quasi-Gröbner** if there exists a Gröbner category $\mathcal{C}'$ and a functor $\mathcal{C}' \to \mathcal{C}$ that is essentially surjective and satisfies a certain technical finiteness condition (property F).

Theorem

If $\mathcal{C}$ is quasi-Gröbner then $\mathbf{k}[\mathcal{C}]$ is noetherian, for any left-noetherian ring $\mathbf{k}$.

Example: **FI**

OI = category of totally ordered finite sets with order-preserving injections.

Theorem

OI *is Gröbner.*

Theorem

FI *is quasi-Gröbner, via the natural functor* **OI** $\to$ **FI**.

Corollary

k[**FI**] *is noetherian for any left-noetherian ring* **k**.

Example: **FS**

FS = category of finite sets with surjections.

Theorem

$\mathbf{FS}^{\mathrm{op}}$ *is quasi-Gröbner, via a functor* $\mathbf{OS}^{\mathrm{op}} \to \mathbf{FS}^{\mathrm{op}}$.

Remark

The noetherianity of Δ-modules is deduced from this.

Example: $\mathbf{VA}_q$

$\mathbf{VA}_q$ = category of finite dimensional vector spaces over $\mathbf{F}_q$.

Theorem

$\mathbf{VA}_q$ *is quasi-Gröbner, via a functor* $\mathbf{OS}^{\mathrm{op}} \to \mathbf{VA}_q$.

Corollary (Lannes–Schwartz artinian conjecture)

$\mathbf{k}[\mathbf{VA}_q]$ *is noetherian.*

Hilbert series

Can often define a "Hilbert series" H_M of a $\mathbf{k}[\mathcal{C}]$-module M.

Problem

Find a general condition on $\mathcal{C}$ that ensures H_M is "rational" for any finitely generated $\mathbf{k}[\mathcal{C}]$-module M.

Solution

Connect to the theory of formal languages.

Lingual categories

We define a condition on $\mathcal{C}$ called **lingual**.

$\mathcal{C}$ lingual and M finitely generated $\implies$ $\exists$ formal languages $\mathcal{L}$ and $\mathcal{L}'$ s.t.

$$\mathrm{H}_M(t) = \mathrm{H}_{\mathcal{L}}(t) - \mathrm{H}_{\mathcal{L}'}(t),$$

where $\mathrm{H}_{\mathcal{L}}(t)$ is the generating function for $\mathcal{L}$.

Can then appeal to results about generating functions of formal languages to obtain results about H_M.

Example: **FI**

For an $\mathbf{k}[\mathbf{FI}]$-module M, with $\mathbf{k}$ a field, define the Hilbert series by

$$\mathrm{H}_M(t) = \sum_{n \geq 0} \dim M([n]) \cdot t^n.$$

Theorem

Let M be a finitely generated $\mathbf{k}[\mathbf{FI}]$-module. Then $\mathrm{H}_M(t) = \frac{f(t)}{(1-t)^n}$ for some polynomial f and some $n \geq 0$. Equivalently, $n \mapsto \dim M([n])$ is eventually a polynomial in n.

Corollary (Church–Ellenberg–Farb)

Fix a topological space X and an index i. Then $n \mapsto \beta_i(\mathrm{Conf}_n(X))$ is eventually a polynomial of n. ($\beta_i = i$th Betti number)

Example: $\mathbf{FS}^{\mathrm{op}}$

Define Hilbert series for $\mathbf{FS}^{\mathrm{op}}$ modules as for $\mathbf{FI}$-modules.

Theorem

Let M be a finitely generated $\mathbf{k}[\mathbf{FS}^{\mathrm{op}}]$-module. Then $\mathrm{H}_M(t)$ is a rational function whose poles have the form $1/n$ with $n \in \mathbf{N}$.

Example

Let $M = P_2$. Then

$$\mathrm{H}_M(t) = \frac{1}{1-2t} - \frac{2}{1-t} + 1.$$

Specific results: **FI**

From now on: $\mathbf{k}$ is a field of characteristic 0.

Structure theorem

Structure theorem for $\mathbf{k}[t]$: every finitely generated module M is a sum of a torsion module T and free module F.

Analogous result for $\mathbf{k}[\mathbf{FI}]$:

Theorem

Let M be a finite length complex of finitely generated $\mathbf{k}[\mathbf{FI}]$-modules. Then there is an exact triangle

$$T \to M \to F \to$$

where T is a finite length complex of torsion $\mathbf{k}[\mathbf{FI}]$-modules and F is a finite length complex of projective $\mathbf{k}[\mathbf{FI}]$-modules.

Projective resolutions

Consider the projective resolution of a finitely generated $\mathbf{k}[\mathbf{FI}]$-module M:

$$\cdots \to P_2 \to P_1 \to P_0 \to M \to 0$$

If M is not projective, the resolution is necessarily infinite.

Define the ith **linear strand** $L_i(M)$ by $L_i(M)([n]) = \mathrm{Tor}_{n-i}(M, \mathbf{k})_n$.

$\to$ $L_i(M)$ has the structure of a $\mathbf{k}[\mathbf{FI}]$-module.

Theorem (Hilbert syzygy theorem)

Each linear strand is finitely generated as a $\mathbf{k}[\mathbf{FI}]$*-module. Only finitely many linear strands are non-zero (i.e., regularity is finite).*

Proof: structure theorem.

Injective resolutions

Theorem

The category of finitely generated $\mathbf{k}[\mathbf{FI}]$*-modules has enough injectives, and every object has finite injective dimension.*

Remark

In fact, all projective modules are injective.

Local cohomology

$\mathrm{H}^0_{\mathfrak{m}}(M) =$ torsion submodule of M.

$\mathrm{H}^i_{\mathfrak{m}}(-) =$ ith right derived functor of $\mathrm{H}^0_{\mathfrak{m}}(-)$.

$\rightarrow$ $\mathrm{H}^i_{\mathfrak{m}}(-)$ is called **local cohomology**.

Theorem

If M is finitely generated then $\mathrm{H}^i_{\mathfrak{m}}(M)$ is finite dimensional for all i, and vanishes for i sufficiently large.

Proof: theorem on injective resolutions.

Remark

There is a vanishing theorem for local cohomology similar to the one in commutative algbera.

Local cohomology

$P =$ Hilbert polynomial of M

$\rightarrow \dim M([n]) = P(n)$ for $n \gg 0$.

Theorem

$$\dim M([n]) - P(n) = \sum_{i \geq 0} (-1)^i \dim \mathrm{H}^i_{\mathfrak{m}}(M)_n$$

Corollary

$\dim M([n]) = P(n)$ *if n is large enough so that* $\mathrm{H}^i_{\mathfrak{m}}(M)_n = 0$ *for all i.*

Specific results: $\mathbf{FI}_d$

The category $\mathbf{FI}_d$

$\mathbf{FI}_d$ is the category of finite sets where morphisms are injections together with a d-coloring of the complement of the image.

$\mathbf{k}[\mathbf{FI}]$ modules are analogous to $\mathbf{k}[t]$-modules; $\mathbf{k}[\mathbf{FI}_d]$-modules are analogous to $\mathbf{k}[t_1, \ldots, t_d]$-modules.

Theorem

$\mathbf{FI}_d$ *is quasi-Gröbner, and so* $\mathbf{k}[\mathbf{FI}_d]$ *is noetherian.*

Many results about $\mathbf{FI}$ (e.g., the Hilbert syzygy theorem) carry over to $\mathbf{FI}_d$.

Interesting new behavior: continuous families of modules.

Tensor products

$M, N = \mathbf{k}[\mathbf{FI}_d]$-modules $\implies$ can define a $\mathbf{k}[\mathbf{FI}_d]$-module $M \otimes N$:

$(M \otimes N)(S) =$ quotient of $\bigoplus_{S = A \amalg B} M(A) \otimes N(B)$ by relations $f_*(x) \otimes y = x \otimes f_*(y)$, where f is a morphism in $\mathbf{FI}_d$.

The unit object for $\otimes$ is the principal projective $P_\emptyset$.

Remark

This construction applies to $\mathbf{k}[\mathcal{C}]$-modules whenever $\mathcal{C}$ has a symmetric monoidal structure.

Tannakian algebraic geometry

$\mathcal{A}$ = abelian category with tensor product $\otimes$

R = the unit object of $\otimes$

Ideal = subobject of R

IJ = image of $I \otimes J$ under $R \otimes R \to R$

P is **prime** if $IJ \subset P$ implies $I \subset P$ or $J \subset P$

$\mathrm{Spec}(\mathcal{A})$ = set of prime ideals with Zariski topology

The spectrum of $\mathbf{FI}_d$

Theorem

$$\mathrm{Spec}(\mathbf{k}[\mathbf{FI}_d]) = \coprod_{i=0}^{d} \mathbf{Gr}(i, d).$$

Even better: the category of $\mathbf{k}[\mathbf{FI}_d]$-modules is filtered by Serre subcategory $\mathcal{A}_0 \subset \cdots \subset \mathcal{A}_d$ and $\mathcal{A}_i/\mathcal{A}_{i-1}$ can be described as a category of sheaves on $\mathbf{Gr}(i, d)$.

Corollary

$$\mathrm{K}_0(\mathbf{Mod}^{\mathrm{fg}}_{\mathbf{k}[\mathbf{FI}_d]}) = \bigoplus_{i=0}^{d} \Lambda \otimes \mathrm{K}_0(\mathbf{Gr}(i, d))$$

Here Λ is the ring of symmetric functions. In particular, the left side is free of rank 2^d over Λ.

Specific results: **UB**

The category **UB**

UB is the category of finite sets where morphisms are injections together with a perfect matching on the complement of the image.

Theorem (Nagpal–Sam–Snowden)

$\mathbf{k}[\mathbf{UB}]$ *is noetherian (if* $\mathbf{k}$ *is a field of characteristic 0).*

Remark

UB is not quasi-Gröbner. There is a combinatorial approach to noetherianity, but the combinatorics problem is unsolved!

Applications of **UB**

Theorem (Dan-Cohen–Penkov–Verganova, Sam–Snowden)

Finite length $\mathbf{k}[\mathbf{UB}]$*-modules are equivalent to "algebraic" representations of the infinite orthogonal group.*

The spectrum of **UB**

$\mathbf{k}[\mathbf{UB}]$-modules are equivalent to $\mathbf{GL}_\infty$-equivariant modules over $\mathrm{Sym}(\mathrm{Sym}^2(\mathbf{k}^\infty)) = \mathbf{k}[x_{i,j}]$ (via Schur–Weyl duality).

nth determinantal ideal in $\mathbf{k}[x_{i,j}] \implies$ a prime ideal P_n in $\mathbf{k}[\mathbf{UB}]$.

Surprise: $\mathrm{Spec}(\mathbf{k}[\mathbf{UB}]) = \mathbf{N}^2 \cup \{\infty\}$

$\rightarrow$ $\mathbf{N}^2$ is the set of **super ranks**.

The category of $\mathbf{k}[\mathbf{UB}]$-modules has a filtration indexed by $\mathbf{N}^2$ so that the (n, m) graded piece is (closely related to) the category of representations of the orthosymplectic group $\mathbf{O}(n \mid m)$. (There is also a "generic piece" that is related to representations of $\mathbf{O}(\infty)$.)