TATE UNIFORMIZATION OF DRINFELD MODULES AND COMPACTIFICATIONS OF DRINFELD MODULAR VARIETIES

1. Review

Let us fix the usual notation

- C is a geometrically connected smooth projective curve over a finite field $\mathbf{F}_q$.
- $\infty \in C$ is a closed point.
- $F = \mathbf{F}_q(C)$ is the function field of C.
- $A = H^0(C - \{\infty\}, \mathcal{O}_C)$ is the ring of functions on C that are regular outside ∞.
- $\widehat{A} = \lim_{0 \neq I \subset A} A/I$ and $\mathbf{A}_f = \widehat{A} \otimes_A F$.
- F_∞ is the completion of F at ∞ with valuation ring $\mathcal{O}_\infty$. The ring A is discrete inside F_∞, and the absolute value of F_∞ is normalized via $|a|_\infty = \#|A/a|$ for any $0 \neq a \in A$.
- $\mathbf{C}_\infty$ is the completed algebraic closure of $\mathbf{F}_\infty$.
- For a ring R of characteristic p, $R\{\tau\} \simeq \mathrm{End}_G(\mathbf{G}_{a,R})$ is the Frobenius twisted polynomial arising.

The main objects under consideration are:

Definition 1.1 (Drinfeld modules)**.** Let S be a scheme of characteristic p. A *Drinfeld module* $X = (L, \phi)$ of rank d on S is a line bundle L on S equipped with a ring homomorphism $\phi : A \to \mathrm{End}_S(L)$ such that, locally over open subsets of S trivializing L, we can write

$$\phi(a) = \sum_{i=0}^{m} a_i \tau^i$$

where $p^m = |a|^d$ and the coefficient a_m is a unit. Taking the derivative (i.e., setting $\tau = 0$) gives a map $S \to \mathrm{Spec}(A)$ that is called the *characteristic* of S.

If $X = (L, \phi)$ is a Drinfeld module over S, then we can view S as an A-scheme, and L as an A-module valued functor on S-schemes. This permits us to define the notion of level structure:

Definition 1.2 (Level structure)**.** Let X/S be a Drinfeld module of rank d. Let $I \subset A$ be a nonzero ideal. A level I structure on X is a homomorphism

$$\psi : (I^{-1}/A)^d \to X(S)$$

of A-modules such that for any $m \in V(I)$, we have

$$X_m = \sum_{\alpha \in m^{-1}/A} \psi(\alpha)$$

as divisors on X. When $im(S) \cap \mathrm{Spec}(A/I) = \emptyset$, then this is the same as specifying as isomorphism

$$(I^{-1}/A)^d \simeq X[I]$$

of A/I-module schemes over S.

We have seen the following:

thm1 **Theorem 1.3** (Existence of moduli spaces)**.** *(1) Fix an ideal $0 \neq I \subset A$ with $\#V(I) \geq 2$. The functor that sends an A-scheme S to the set of Drinfeld A-modules X of rank d over S equipped with a level I-structure ψ (up to isomorphism) is represented by a flat affine A-scheme M_I^d of relative dimension $d-1$ that is smooth outside $V(I)$. Moreover, M_I^d is itself a smooth $\mathbf{F}_q$-scheme, and there is an obvious action of $\mathrm{GL}_d(A/I)$ on this A-scheme.*

(2) For $J \subset I$, there is a forgetful mad $M_J^d \to M_I^d$ that is finite, flat, étale outside $V(J)$, that is compatible with group actions.

(3) The inverse limit $M^d = \lim_{0 \neq I \subset A} M_I^d$ acquires an action of $\mathrm{GL}_d(\mathbf{A}_f)/F^*$.

2. Tate uniformization

Tate

In this section, V denotes a complete dvr over A, $\mathfrak{m} \subset V$ is the maximal ideal, $K = \mathrm{Frac}(V)$, and K^s is a separable closure. The norm $|\cdot|$ always refers to the norm on K.

Definition 2.1 (Good and stable reduction)**.** Let X be a Drinfeld A-module of rank $d > 0$ over K. We say that X has *stable* reduction of rank $d_1 > 0$ is it can be represented by a map $\phi : A \to \mathcal{O}\{\tau\}$ such that ϕ modulo $\mathfrak{m}$ gives a Drinfeld A-module of rank d_1 over $\mathcal{O}/\mathfrak{m}$; if we can take $d_1 = d$, we say that X has *good* reduction.

In other words, if we represent X by an action of A on $\mathbf{G}_{a,\mathcal{O}}$, then having stable reduction means that at least some $a \in A$ acts by a non-scalar $\mathbf{G}_{a,\mathcal{O}/\mathfrak{m}}$.

Example 2.2 (Non-stable reduction)**.** Taking $A = \mathbf{F}_p[T]$, $\mathcal{O} = \mathbf{F}_p[\![T]\!]$, and $\phi : A \to \mathcal{O}\{\tau\}$ given by $T \mapsto T + T \cdot \tau$ gives a Drinfeld module of rank 1 over $\mathcal{O}[\frac{1}{T}]$, but does not give a Drinfeld module over $\mathbf{F}_p = \mathcal{O}/(T)$. From the proof of the proposition below, it follows that this Drinfeld module acquires stable reduction after extracting a $(p-1)$-st root of T.

The following is an analog of the stable reduction theorem for abelian varieties.

propGR

Proposition 2.3. *Every Drinfeld A-module X over K has potentially stable reduction.*

Proof. Choose a representative $\phi : A \to \mathcal{O}\{\tau\}$ for X; one can show that this is always possible (and is clear from the formula for the conjugation by c action given below). Choose some non-constant $a \in A$ and write $\phi_a = \sum_i a_i \tau^i$. It will be enough to show that, after replacing ϕ with $\phi' : A \to \mathcal{O}\{\tau\}$ obtained by a change of variables on $\mathbf{G}_a$ given by some scalar $c \in K$, the polynomial ϕ'_a is non-constant modulo $\mathfrak{m}$; equivalently, we want to ensure $\phi'(y_i)$ has some non-constant coefficient being a unit.

Unwinding definitions, one finds that ϕ'_a is given by

$$c \cdot (\sum_i a_i \tau^i) \cdot c^{-1} = \sum_i c^{1-p^i} a_i \tau^i.$$

The constant coefficient has not changed, so it lies in $\mathcal{O}$. We need to ensure that all coefficients appearing above lie in $\mathcal{O}$, and that some non-constant coefficient is a unit. Now if we had some $c \in K$ with

$$\mathrm{val}(c) = \min_{i>0} \frac{\mathrm{val}(a_i)}{p^i - 1},$$

then we would be done: all coefficients would lie in $\mathcal{O}$ as the RHS is the minimum, and the index i attaining the minimum would give the unit coefficient. We might not be able to find such a c in K, but we can always find it after extending scalars. This proves the potential stable reduction. $\square$

There is also a classification of of Drinfeld modules with arbitrary reduction types in terms of Drinfeld modules with good reduction equipped with certain lattices (*Tate data*), akin to p-adic uniformization of abelian varieties. We need the following definition:

Definition 2.4 (Lattices)**.** Let X be a Drinfeld module over K. A lattice in X is a Galois invariant A-submodule $\Gamma \subset X(K^s)$ that is finite projective as an A-module and discrete as a subspace of $X(K^s)$, i.e., its intersection with every residue disc is finite. Each lattice determines a representation $G_K \to \mathrm{Aut}(\Gamma)$.

Proposition 2.5 (Tate uniformization)**.** *There is an equivalence of categories between Drinfeld modules X of rank d over K and the category of pairs (Y, Γ) where Y is a Drinfeld module of rank d_1 ($d_1 \le d$) over K with potentially good reduction, and $\Gamma \subset Y(K^s)$ is a lattice of rank $d - d_1$.*

Note that Γ need not be a subset of $Y(K^s)$. We shall see later that in rank 1, the level structure forces Γ to lie in $Y(K)$.

Proof sketch. We sketch how to construct the correspondence between Drinfeld modules X of rank d over K with stable reduction and the category of pairs (Y, Γ) where Y is a Drinfeld module of rank d_1 ($d_1 \le d$) over K with good reduction, and $\Gamma \subset Y(K^s)$ is a lattice of rank $d - d_1$. The general case is deduced by Galois descent.

Given (Y, Γ), one can construct X by passage to a quotient, as in the classification of Drinfeld modules over F_∞ seen earlier.

Conversely, fix an X of rank d with stable reduction of rank d_1, represented by a map $\phi : A \to \mathcal{O}\{\tau\}$. The hypothesis ensures that $\phi \mod \mathfrak{m}$ is a Drinfeld module of rank d_1. This means that for any $0 \neq a \in A$, the coefficient $a_{d_1 \cdot \log_p |a|^d}$ of $\phi(a) = \sum_i a_i \tau^i$ is a unit in $\mathcal{O}$, and all higher coefficients are topologically nilpotent (i.e., lie in $\mathfrak{m}$). By the lemma on bringing Drinfeld modules to "standard form" from Haoyang's talk, modulo each $\mathfrak{m}^n$, we can find a unique pair (ψ_m, u_m) where $\psi : A \to \mathcal{O}/\mathfrak{m}^n\{\tau\}$ is a Drinfeld module of rank d_1 in standard form (i.e., the highest coefficient of $\psi(a)$ is in degree $d_1 \cdot \log_p |a|$ and is a unit) and u has constant coefficient 1 such that $u \cdot \psi(a) = \phi \cdot u(a)$ for all $a \in A$. By uniqueness, these patch to give similar (ψ, u) over $\mathcal{O}$. The Drinfeld module attached to ψ gives us our Y. One checks (by a difficult but elementary calculation) that u provides a homomorphism $Y \to X$ of Drinfeld modules, and that the kernel Γ of the induced map $Y(K^s) \to X(K^s)$ is a lattice of rank $d - d_1$. □

Cons1 **Construction 2.6** (Weak analog of j-invariants)**.** Let $\phi : A \to R\{\tau\}$ be a Drinfeld module of rank 2 over a characteristic p ring R. Fix some $0 \neq a \in A$ with $|a| > 0$, and let $s = \log_p |a|$. As in the definition of Drinfeld modules, locally on $\mathrm{Spec}(R)$, we can write

$$\phi(a) = \sum_{i=0}^{2d} a_i \tau^i$$

with highest coefficient a_{2d} being a unit. The ratio

$$t = \frac{a_s^{|a|+1}}{a_{2s}}$$

is a well-defined global section $t_i \in R$ (i.e., is independent of the choice of ϕ). Thus, we have constructed a map

$$M_I^2 \xrightarrow{f_a} \mathrm{Spec}(A[t])$$

depending on a. (There is a similar construction in any rank.)

We now explain how to detect reduction behaviour in terms of the classifying map to the moduli space using the map f_a constructed above. In the process, this gives a good way to probe the moduli space itself.

Cor1 **Corollary 2.7.** *Let $0 \neq I \subset A$ with $\#V(I) \geq 2$. Let X be a Drinfeld module of rank 2 over K with level I structure. Then:*

(1) If X has potentially good reduction, then it has good reduction.
(2) X has good reduction if and only if the classifying map $c_X : \mathrm{Spec}(K) \to M_I^2$ carries the section $t \in \mathcal{O}_{M_I^2}$ from the preceding construction into $\mathcal{O} \subset \mathcal{K}$.
(3) The map f_a constructed above is finite surjective and flat.

Proof. (1) is clear from the representability of M_I^2.

For (2), if X has good reduction, then c_X extends to a map $\mathrm{Spec}(\mathcal{O}) \to M_I^2$, so one direction is clear. Conversely, if X does not have good reduction, then it must have stable reduction of rank 1 over some extension L/K. But, from the definition (and well-definedness) of the function t, this means that the pullback of t to L does not lie in $\mathcal{O}_L$: the numerator is a unit, while the denominator is topologically nilpotent. As $\mathcal{O} = \mathcal{O}_L \cap K$, the claim follows.

For (3), as both sides are regular affine schemes of dimension 2, it is enough to check f_a is proper: this will imply that f_a is finite (Zariski's main theorem), thus flat (miracle flatness lemma), and thus surjective (it is open by flatness, and closed by properness). But (2) gives exactly the valuative criterion of properness, so we are done. □

3. Complex multiplication

In rank 1, Drinfeld's moduli space M^1 explicitly realizes an abelian extension of F totally split at ∞ whose Galois group is $\mathbf{A}_f^*/F^*$; the congruence relation and class field theory then show that this space "realizes" class field theory for F.

Theorem 3.1. *The A-scheme M^1 is the spectrum of the ring of integers in a maximal abelian extension of F that is totally split at ∞. The action of $\mathbf{A}_f^*/k^*$ constructed above coincides with the action from class field theory.*

Proof sketch. Let $0 \neq I \subset A$ with $\#V(I) \geq 2$. The $d = 1$ analog of Corollary 2.7 says that the map $M_I^1 \to \operatorname{Spec}(A)$ is finite; more simply, this map is proper as any rank 1 Drinfeld module has potentially good reduction by Proposition 2.3. Thus, by Theorem 1.3, M_I^1 is a smooth curve, and the map $M_I^1 \to \operatorname{Spec}(A)$ is a finite surjective map ramified only over $V(I)$.

For a point $v \in \operatorname{Spec}(A) - V(I)$, pick a uniformizer $\pi \in A_v$. By the congruence relation from Haoyang's talk, the action of π on the fibre $M_{I,v}^1$ of M_I^1 over v coincides with the Frobenius over the field $A_v/(\pi)$. But then each connected component of M_I^1 is also left invariant by this action. As this is true for almost all v's, it follows that each connected component of M_I^1 is invariant under the $\mathbf{A}_f^*$-action, and thus the same holds true for the infinite level version M^1.

On the other hand, by the adelic description from Matt's talk, we have $M^1(F_\infty) = M^1(F_\infty^s) = \mathbf{A}_f^*/k^*$, compatibly with the group action. As the $\mathbf{A}_f^*/k^*$-action on this set is simply transitive, it follows from Galois theory that M^1 is a connected generically Galois cover of $\operatorname{Spec}(A)$ with group $\mathbf{A}_f^*/k^*$. The congruence relation and class field theory then gives the desired identification. $\square$

4. Compactification

The main theorem about compactifications is:

thmc **Theorem 4.1.** *Let $0 \neq I \subset A$ with $\#V(I) \geq 2$.*

1. *There exists a unique smooth surface $\overline{M}_I^2$ over A containing M_I^2 as a dense open such $\overline{M}_I^2 \to \operatorname{Spec}(A)$ is proper and $\Delta_I^2 := \overline{M}_I^2 - M_I^2 \to \operatorname{Spec}(A)$ is finite. The group action extends.*
2. *If $J \subset I$, then $M_J^2 \to M_I^2$ extends to a finite morphism $\overline{M}_I^2 \to \overline{M}_J^2$. The group action also extends.*
3. *The completion of $\overline{M}_I^2$ along the boundary Δ_I^2 can be explicitly described in terms of Tate data. In particular, the map $\overline{M}_I^2 \to \operatorname{Spec}(A)$ is smooth outside $V(I)$.*

The strategy of the proof is the following:

1. Build the formal completion $\widehat{\overline{M}}_I^2$ of $\overline{M}_I^2$ along the boundary Δ_I^2 (none of these exist yet) directly in terms of Tate data. This part is analogous to the construction of the Tate curve at the cusp in the modular curve $X(1)$, except that one must also use the group action to cover all the cusps. The output is a formal affine scheme; the affineness is specific to $d = 2$, as that ensures Δ_I^2 is finite over $\operatorname{Spec}(A)$, and thus affine. We shall view it as an actual affine scheme.
2. Define the "punctured formal neighbourhood" $\widehat{\overline{M}}_I^2 - \Delta_I^2 =: M_I^{2,\circ}$. If the theorem were true, then we would have a structure map $M_I^{2,\circ} \to M_I^2$; one builds this map by hand.
3. Glue $\widehat{\overline{M}}_I^2$ with M_I^2 along $M_I^{2,\circ}$. This is analogous to constructing $\mathbf{P}_A^1$ by glueing $\operatorname{Spec}(A[t])$ with $\operatorname{Spec}(A[\![\frac{1}{t}]\!])$ along $\operatorname{Spec}(A((\frac{1}{t})))$. In fact, there is a very general glueing theorem of Artin that says that
$$\operatorname{Coh}(\mathbf{P}_A^1) \simeq \operatorname{Coh}(A[t]) \times_{\operatorname{Coh}(A((\frac{1}{t})))} \operatorname{Coh}(A[\![\frac{1}{t}]\!]).$$
In other words, specifying a coherent sheaf F on $\mathbf{P}_A^1$ is the same as specifying a triple (F_1, F_2, ϕ), where F_1 is a finitely generated $A[t]$-module, F_2 is a finitely generated $A[\![\frac{1}{t}]\!]$-module, and ϕ is an identification of their pullbacks to $A((\frac{1}{t}))$. This equivalence is symmetric monoidal, so it implies a similar equivalence for categories of commutative algebras in coherent sheaves. Applying $\underline{\operatorname{Spec}}$ gives a similar glueing statement for finite morphisms:
$$\operatorname{FinSch}_{\mathbf{P}_A^1} \simeq \operatorname{FinSch}_{A[t]} \times_{\operatorname{FinSch}_{A((\frac{1}{t}))}} \operatorname{FinSch}_{A[\![\frac{1}{t}]\!]}.$$
The scheme $\overline{M}_I^2$ will be constructed as a finite cover of $\mathbf{P}_A^1$ using this statement: using the weak analog of the j invariant from Construction 2.6, one shows that the constructions from steps (1) and

(2) fit together to give a triple $(M^2_I, \widehat{\overline{M}}^2_I, \phi)$ on the right hand side above, and the corresponding finite scheme over $\mathbf{P}^1_A$ is defined to $\overline{M}^2_I$.

The above strategy outlines the proofs of Theorem 4.1 (1) and (3). To see (2) (which is crucial for applications), one uses the following general lemma in algebraic geometry:

Lemma 4.2. *Let X_1 and X_2 be two normal surfaces that are proper over* $\operatorname{Spec}(A)$*. Assume we are given closed subschemes $D_1 \subset X_1$ and $D_2 \subset X_2$ that are fine over* $\operatorname{Spec}(A)$ *and an A-morphism $f : X_1 - D_1 \to X_2 - D_2$. Then f extends to a finite A-morphism $X_1 \to X_2$.*

Proof. By taking closures, there is some proper birational map $\pi : Y \to X_1$ that is an isomorphism outside a closed subscheme supported in D_1 and a morphism $\overline{f} : Y \to X_2$ extending f. Moreover, the map $X_1 - D_1 \xrightarrow{\pi^{-1}} Y \to X_2$ agrees with f, and thus has image in $X_2 - D_2$. As this map is finite and $X_1 - D_1$ is dense in Y, it follows that $\pi^{-1}(X_2 - D_2) = X_1 - D_1$.

On the other hand, by normality of X_1, if π were not an isomorphism, there would be a π-exceptional irreducible curve $C \subset X_1$ that is not collapsed by f. As C is π-exceptional, it lies in a fibre over a point $a \in \operatorname{Spec}(A)$. Then $\pi(C) \subset X_2$ is a non-constant curve in the fibre over a. As D_2 is finite over A, we have $\pi(C) \not\subset D_2$. But then $C \cap \pi^{-1}(X_2 - D_2)$ would be non-empty. Now we know that $\pi^{-1}(X_2 - D_2) = X_1 - D_1$, so we get $C \cap (X_1 - D_1) \neq \emptyset$. But this means $\pi : Y \to X$ does not kill C, which contradicts the π-exceptionality of C. $\square$

4.1. **An idea of the construction of $\overline{M}^2_I$.** Let us assume for a moment that theorem has been proven, and study $\overline{M}^2_I$ via valuations. Fix $(K, \mathcal{O})$ as in §2. Fix an A-algebra structure on $\mathcal{O}$, and an A-map $\operatorname{Spec}(K) \to M^2_I$ corresponding to a Drinfeld module X over K with level I structure γ. By properness of $\overline{M}^2_I$ over A, the classifying map $c_X : \operatorname{Spec}(K) \to M^2_I$ extends to a map $\overline{c_X} : \operatorname{Spec}(\mathcal{O}) \to \overline{M}^2_I$. Then there are exactly two possibilities:

1. (1) (X, γ) has good reduction. By definition of the moduli problem, this happens exactly when the map $\overline{c_X}$ factors through $M^2_I \subset \overline{M}^2_I$. In this case, the compactification plays no role.
2. (2) (X, γ) does not have good reduction. This happens exactly when the map $\overline{c_X}$ factors through $\widehat{\overline{M}}^2_I$ as there are no other possibilities. By Tate uniformization, there is a corresponding rank 1 Drinfeld module Y over $\mathcal{O}$ (i.e., with good reduction) and a rank 1 lattice $\Gamma \subset Y(K^s)$ with a K-isomorphism $Y/\Gamma \simeq X$. In particular, Γ is Galois stable, and thus corresponds to a character[1] $G_K \to \operatorname{Aut}(\Gamma) \simeq A^* \simeq \mathbf{F}^*_q$. But the level structure on X then tells us that the Galois action on $I^{-1}\Gamma/\Gamma$, which means that the character has image in $(1 + I) \subset A^*$; as $(1 + I) \cap \mathbf{F}^*_q = \{1\} \subset A$, this character is trivial. Thus, $\Gamma \subset Y(K)$ is actually K-rational.

Summarizing, we have shown:

Lemma 4.3 (Valuative characterisation of $\overline{M}^2_I$)**.** *Let $\overline{c} : \operatorname{Spec}(\mathcal{O}) \to \overline{M}^2_I$ be a map that restricts to a map $c : \operatorname{Spec}(K) \to M^2_I$ carrying classifying a Drinfeld module X with level I-structure over K. Then the map $\overline{c_X}$ factors through the formal boundary $\widehat{\overline{M}}^2_I$ exactly when X has the form Y/Γ for a rank 1 Drinfeld module Y with good reduction and a K-rational lattice $\Gamma \subset X(K)$.*

We now construct another space which has a similar universal property with respect to valuations as the one formulated in the lemma above. This space will eventually be the formal completion of the boundary at one point, and the entire formal boundary will be built by glueing multiple copies of this space using the group action.

Construction 4.4 (Local model at the cusp)**.** In addition to the ideal I, fix an ideal $J \subset A$. Let X^{univ} be the universal rank 1 Drinfeld module over M^1_I. One can form the quotient $X_J := X^{univ}/X[J]$ to a new Drinfeld module on M^1_1, isogenous to the original one. Giving a map $S \to X_J$ is the same as specifying a Drinfeld module (Y, γ) of rank 1 with level I structure over S (corresponding to the map down to M^1_I)

[1] Here we use that $\operatorname{Spec}(A)$ is the complement of a single point on a complete curve, so there are no non-constant invertible functions on it: any finite map $\pi : C \to \mathbf{P}^1$ is surjective, so the preimage of $\mathbf{G}_m$ is an open subset whose complement has at least two points.

together with a point of $Y_J(S)$. Interpreting this point as a map $A \to Y_J(S)$ of A-modules, it follows that we can lift this to a map $\tau : J \to Y(S)$ compatible with the isogeny $Y \to Y_J$. Moreover, this process is reversible: there is a bijective correspondence between $X_J(S)$ and triples (Y, γ, τ) where $(Y, \gamma) \in M^1_I(S)$ and τ is an A-module map $J \to Y(S)$.

As a scheme, X_J is a line bundle over M^1_J. Write $\overline{X}_J$ for its 1-point compactification[2]. There is a section $s_\infty : M^1_J \to \overline{X}_J$ at infinity, write $\widehat{\overline{X}_J}$ for the formal completion at the boundary; again, this is affine, and we view it as an affine scheme instead of formal scheme.

Lemma 4.5. *Let $\bar{c} : \mathrm{Spec}(\mathcal{O}) \to \overline{X}_J$ be a morphism inducing a map $c : \mathrm{Spec}(K) \to X_J$ that corresponds to a triple (Y, γ, τ) as above. Then $\tau : J \to Y(S)$ gives a lattice (i.e., a discrete A-submodule of rank 1) if and only if $\bar{c}$ factors through the boundary $\hat{\overline{X}}_J$*

Proof. Note that $\bar{c}$ factors through the boundary $\hat{\overline{X}}_J$ exactly when it does not factor through $X_J \subset \overline{X}_J$.

Assume first that $\bar{c}$ factors over X_J. Then we have a map $\mathrm{Spec}(\mathcal{O}) \to X_J$ lifting c. This means that the homomorphism $\tau : J \to Y(K)$ classified by c actually has image in $Y(\mathcal{O})$. But $Y(\mathcal{O})$ is compact, so τ cannot give a lattice: compactness forces any discrete subset to be finite.

Conversely, suppose $\bar{c}$ factors over the boundary $\hat{\overline{X}}_J$. As $\bar{c}$ does not have image in X_J, the map $\tau : J \to Y(K)$ does not have image $Y(\mathcal{O})$. Thus, there must be some $a \in A$ such that $\tau(a) \notin Y(\mathcal{O})$, so $|\tau(a)| > 1$ (where the norm on $Y(K)$ is defined by making $Y(\mathcal{O})$ the unit ball). As τ is A-linear, we have $\tau(A \cdot a) = A \cdot \tau(a)$; as $|\tau(a)| > 1$, one easily checks that $A \cdot \tau(a)$ is a rank 1 lattice in $Y(K)$. But the index of $A \cdot a$ in J is finite, so $\tau(J)$ is also a lattice of rank 1. $\square$

Thus, the triple $X_J \subset \overline{X}_J \supset \widehat{\overline{X}_J}$ has strong formal resemblance to $M^2_I \subset \overline{M}^2_I \supset \widehat{\overline{M}}^2_J$. To actually construct the latter, we choose an ideal J which is A-isomorphic to I^{-1}, glue together many copies of $\overline{X}_J \supset \widehat{\overline{X}_J}$ to get $\overline{M}^2_I \supset \widehat{\overline{M}}^2_J$.

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[2] For any line bundle bundle L, this $\mathbf{P}(\mathcal{L}^{-1} \oplus \mathcal{O}_S)$.