Drinfel′d Modules: Deformations of formal modules

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In addition to [Dri75, §4], we also list [Mat12i; Mat12ii] and [HG94] as references.

Throughout the talk, we fix the following:

Notation 0.1. Let $\mathcal{O}$ be the ring of integers of a local non-Archimedean field with uniformizer π. Let $p = \operatorname{char} \mathcal{O}/(\pi)$, and let $q = p^k = \#\mathcal{O}/(\pi)$. We denote by $\mathcal{O}^{\mathrm{nr}}$ a maximal unramified extension of $\mathcal{O}$, and by $\widehat{\mathcal{O}}^{\mathrm{nr}}$ the completion of $\mathcal{O}^{\mathrm{nr}}$. We denote by κ the field $\widehat{\mathcal{O}}^{\mathrm{nr}}/(\pi) = \overline{\mathcal{O}/(\pi)}$.

We also denote by (G, g) a formal $\mathcal{O}$-module over κ, i.e., $G \in \kappa[\![x, y]\!]$ is a formal group, and $g \colon \mathcal{O} \to \operatorname{End}(G)$ is a homomorphism such that $g_a = g(a) = \overline{a}x + \cdots$. We denote by $\mathscr{C}$ the category of complete, local $\widehat{\mathcal{O}}^{\mathrm{nr}}$-algebras with residue field κ.

1 Motivation

We start with some motivation from the theory of elliptic curves over a ground field k of characteristic $p > 0$. Let $\mathcal{M}_{1,1}$ be the moduli space of elliptic curves, so that

$$\mathcal{M}_{1,1}(k) = \bigl\{\text{elliptic curves } E/k\bigr\}.$$

One philosophy of moduli theory is that subspaces of a moduli space should also have geometric meaning. In our case, if T is any k-scheme, then

$$\mathcal{M}_{1,1}(T) = \operatorname{Hom}_k(T, \mathcal{M}_{1,1}) = \left\{ \text{families of elliptic curves } \begin{matrix} \mathcal{E} \\ \downarrow \\ T \end{matrix} \right\}.$$

One can then ask the following:

Question 1.1. What is the local structure of $\mathcal{M}_{1,1}$? E.g., if $[E] \in \mathcal{M}_{1,1}(k)$, then what is "$\mathcal{O}^\wedge_{\mathcal{M}_{1,1},[E]}$"?

Recall that if $X = \operatorname{Spec} R$ is an affine scheme, and $x = \mathfrak{m} \in \operatorname{Spec} R$ is a closed point, then

$$\mathcal{O}^\wedge_{X,x} = \varprojlim_n R/\mathfrak{m}^n.$$

In our case, the relevant ring is

$$(\mathcal{O}_{\mathcal{M}_{1,1}})^\wedge_{[E]} = \varprojlim_{\substack{\operatorname{Spec} A \to \mathcal{M}_{1,1} \\ \text{extending } [E], \\ A \text{ Artin local } k\text{-alg}, \\ A/\mathfrak{m}_A = k}} \mathcal{M}_{1,1}(A),$$

where

$$\mathcal{M}_{1,1}(A) = \left\{ \text{elliptic curves } \begin{matrix} \mathcal{E} \\ \downarrow \\ \operatorname{Spec} A \end{matrix} \;\middle|\; \mathcal{E}|_{\operatorname{Spec} k} = E \right\}$$

is the set of deformations of E over A. In this way, the local structure of $\mathcal{M}_{1,1}$ at $[E]$ is determined by infinitesimal deformations of E, and so we are led to ask the following:

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Question 1.2. What are the infinitesimal deformations of E?

One can determine them directly using methods we've already seen in Bhargav's course, but here we'll take an approach that we can adapt to Drinfel′d modules.

Suppose that E is supersingular, and let $\widehat{E}$ be the completion of E at the identity, which is a formal group as Andy described last time. In this case, we can answer Question 1.2 with the following:

Theorem 1.3 (Serre–Tate)**.** *Let A be an Artin local k-algebra with residue field k. Then, there is an equivalence of categories*

$$\begin{Bmatrix}\text{deformations of } E\\ \text{over } A\end{Bmatrix} \simeq \begin{Bmatrix}\text{deformations of } \widehat{E}\\ \text{over } A\end{Bmatrix}.$$

Theorem 1.4 (Lubin–Tate)**.** *The functor*

$$A \longmapsto \begin{Bmatrix}\text{deformations of } \widehat{E}\\ \text{over } A\end{Bmatrix}$$

is pro-representable by $k[\![t]\!]$.

We will see a version of the Lubin–Tate theorem for formal modules soon. Although $k[\![t]\!]$ is a limit of Artin rings, it is not itself Artin. Thus, the functor is not representable, but only "pro-representable". For us, it will be a bit more convenient to work in a slightly larger category of complete local rings.

These results combined show, roughly speaking, that $\mathcal{M}_{1,1}$ is smooth of dimension 1 at $[E]$.

2 Deformations of formal modules [Dri75, §4A]

With notation as in Notation 0.1, we have the following:

Goal 2.1. Describe deformations of (G, g) over all $R \in \mathscr{C}$, i.e., formal $\mathcal{O}$-modules (F, f) over R which reduce to (G, g) modulo $\mathfrak{m}_R$.

We can break this down into three parts:

(*i*) Determine all automorphisms of a given deformation,
(*ii*) Parametrize all possible deformations, and
(*iii*) Find obstructions to the existence of deformations.

The solutions to all three problems are usually found in certain cohomology groups. In our case, we can also deal with (i) explicitly. Moreover, there are no obstructions in (*iii*) by Cor. 2.9 in Andy's notes, which can be interpreted as a statement about formal smoothness of the moduli space.

We will make the following:

Additional Assumption 2.2. (G, g) is of finite height h, i.e.,

$$g_\pi(x) = ux^{q^h} + \cdots \quad \text{for some} \quad 0 \neq u \in \kappa.$$

2.1 Automorphisms of deformations

We start by answering (*i*):

Proposition 2.3 [Dri75, Prop. 4.1]**.** *Let $R \in \mathscr{C}$, and suppose (F, f) and (F', f') are deformations of (G, g) over R. If*

$$\phi\colon (F, f) \longrightarrow (F', f')$$

is a homomorphism such that $\phi \equiv 0 \bmod \mathfrak{m}_R$, then $\phi = 0$.

Proof. Since $R = \varprojlim R/\mathfrak{m}_R^n$, it suffices to show inductively that $\phi \equiv 0 \bmod \mathfrak{m}_R^n$ for all n. The case $n = 1$ follows by assumption, so consider the inductive case. Assume ϕ is a homomorphism that reduces to 0 modulo $\mathfrak{m}_R^n$. Then,

$$\phi \circ f_\pi = \phi(ux^{q^h} + \cdots)$$

while

$$f'_\pi \circ \phi \equiv 0 \mod \mathfrak{m}_R^{n+1}$$

since $f'_\pi = \pi x + \cdots$ where the ellipses consist of degree ≥ 2 terms in X, and so all coefficients of $f'_\pi \circ \phi$ are in $\mathfrak{m}_R^{n+1}$. Thus, $\phi \equiv 0 \bmod \mathfrak{m}_R^{n+1}$. □

Corollary 2.4. *If (F, f) is as in Proposition 2.3, and $\phi\colon (F, f) \to (F, f)$ is a homomorphism such that $\phi \equiv \mathrm{id} \bmod \mathfrak{m}_R$, then $\phi = \mathrm{id}$.*

Proof. Apply Proposition 2.3 to $\phi - \mathrm{id}$. □

In other words, a deformation (F, f) of (G, g) has in fact no non-trivial automorphisms.

2.2 Parametrization of possible deformations via symmetric cohomology

We now answer (*ii*). Let M be a vector space over κ. We can then consider the following chain complex:

$$\begin{array}{lcl}
C^1(G,g;M) = M[\![x]\!] & & \psi \\
\quad\Big\downarrow d^1 & & \quad\Big\downarrow \\
C^2(G,g;M) = \left\{ \begin{array}{l|l} (\Delta, \{\delta_a\}_{a\in\mathcal{O}}) & \Delta(x,y) = \Delta(y,x) \\ \Delta \in M[\![x,y]\!] & \delta_{a+b}(x) = \delta_a(x) + \delta_b(x) + \Delta\big(g_a(x), g_b(x)\big) \\ \delta_a \in M[\![x]\!] & \delta_{ab}(x) = a\delta_b(x) + \delta_a\big(g_b(x)\big) \end{array} \right\} & & \begin{pmatrix} \Delta(x,y) = \psi\big(G(x,y)\big) - \psi(x) - \psi(y) \\ \delta_a(x) = \psi\big(g_a(x)\big) - a\psi(x) \end{pmatrix} \\
 & & (\Delta, \{\delta_a\}) \\
\quad\Big\downarrow d^2 & & \quad\Big\downarrow \\
C^3(G,g;M) = \left\{ \begin{array}{l|l} (\Gamma, \{\gamma_a\}_{a\in\mathcal{O}}) & \\ \Gamma \in M[\![x,y,z]\!] & \text{certain symmetry conditions} \\ \gamma_a \in M[\![x,y]\!] & \end{array} \right\} & & \begin{pmatrix} \Gamma(x,y,z) = \Delta(y,z) - \Delta\big(G(x,y),z\big) + \Delta\big(x, G(y,z)\big) - \Delta(x,y) \\ \gamma_a(x,y) = \delta_a(x) - \delta_a\big(G(x,y)\big) + \delta_a(y) - a\Delta(x,y) + \delta\big(g_a(x), g_a(y)\big) \end{pmatrix}
\end{array}$$

We then define the second symmetric cohomology group

$$H^2(G, g; M)_s := \frac{\ker d^2}{\operatorname{im} d^1},$$

which inherits an $\mathcal{O}$-module structure.

Remark 2.5. If we define the chain complex more generally for a deformation (F, f) of (G, g) over some $R \in \mathscr{C}$ and $M = R$, then the H^1-group parametrizes automorphisms of (F, f). However, to our knowledge, there is no full theory of symmetric cohomology—this would presumably require derived methods, reflecting the fact that deforming commutative groups is hard!

We can now state the following:

Lemma 2.6 [Dri75, p. 571, Lem.]**.** *Let $R \in \mathscr{C}$ such that $\mathfrak{m}_R^{k+1} = 0$ for some $k > 1$, and let (F, f) be a deformation of (G, g) over R. Then, there is a one-to-one correspondence*

$$H^2(G, g; \mathfrak{m}^k) \xrightarrow{\text{1-1}} \left\{ \begin{array}{c} \textit{deformations } (F', f') \textit{ of } (G, g) \textit{ over } R \\ \textit{such that } (F', f') \equiv (F, f) \bmod \mathfrak{m}^k \end{array} \right\} \Big/ \textit{isomorphism}$$

$$(\Delta, \delta) \longmapsto \begin{pmatrix} F'(x,y) = F\big(F(x,y), \Delta(x,y)\big) \\ f'_a(x) = F\big(f_a(x), \delta_a(x)\big) \end{pmatrix}$$

Proof. Tedious exercise. The conditions on (Δ, δ) give exactly that F' is commutative and associative, that f'_a is an endomorphism of F' for all $a \in \mathcal{O}$, and that f' is a ring homomorphism. $\square$

Computation of $H^2(G, g; \kappa)_s$ In [Dri75, §1], we saw that we can always reduce to the case of normal formal $\mathcal{O}$-modules, in which case

$$\begin{aligned} G(x,y) &\equiv x + y + u \cdot \frac{p}{\pi} C_{q^h}(x,y) \quad \bmod \deg q^{h+1} \\ g_a(x) &\equiv ax + u \cdot \frac{a^{q^h} - a}{\pi} \cdot x^{q^h} \quad \bmod \deg q^{h+1} \end{aligned} \tag{2.1}$$

for some $0 \neq u \in \kappa$, where we recall $C_{q^h} = \frac{1}{p}\big((x+y)^{q^h} - x^{q^h} - y^{q^h}\big)$.

We first compute $\operatorname{im} d^1$, for which it suffices to compute $d^1(x^n)$ for all $n \in \mathbf{Z}_{>0}$. If $n \neq q^i$, then

$$\begin{aligned} \Delta(x,y) &= G(x,y)^n - x^n - y^n \\ &\equiv (x+y)^n - x^n - y^n \quad \bmod \deg n+1 \end{aligned}$$

since $G(x,y) = x + y + \cdots$ where the ellipses consist of degree ≥ 2 terms in x and y, and

$$\begin{aligned} \delta_a(x) &= \big(g_a(x)\big)^n - ax^n \\ &\equiv (a^n - a)x^n \quad \bmod \deg n+1 \end{aligned}$$

since $g_a(x) = ax + \cdots$ where the ellipses consist of degree ≥ 2 terms in x. If $n = q^i$, then

$$\begin{aligned} \Delta(x,y) &= G(x,y)^{q^i} - x^{q^i} - y^{q^i} \\ &\equiv u^{q^i} \cdot \frac{p}{\pi} \cdot C_{q^h}(x,y)^{q^i} \quad \bmod \deg q^{h+i} + 1 \\ &\equiv u^{q^i} \cdot \frac{p}{\pi} \cdot C_{q^{h+i}}(x,y) \quad \bmod \deg q^{h+i} + 1 \end{aligned}$$

using the form of $G(x,y)$ given in (2.1) and some binomial coefficient arithmetic [Haz12, §4.2]. Likewise,

$$\delta_a(x) \equiv u^{q^i} \frac{a^{q^{h+i}} - a}{\pi} \cdot x^{q^{h+i}} \quad \bmod \deg q^{h+i} + 1.$$

To compute $\ker d^2$, it suffices to describe, for all $n \in \mathbf{Z}_{>0}$, those cocycles (Δ, δ) for which $(\Delta, \delta) \equiv 0 \bmod \deg n$. This is essentially Prop. 2.10 from Andy's talk. If $n \neq q^i$, then

$$\begin{aligned} \Delta(x,y) &\equiv v\big((x+y)^n - x^n - y^n\big) \quad \bmod \deg n+1 \\ \delta_a(x) &\equiv v(a^n - a)x^n \quad \bmod \deg n+1 \end{aligned}$$

for some $0 \neq v \in \kappa$, and if $n = q^i$, then

$$\begin{aligned} \Delta(x,y) &\equiv v \cdot \frac{p}{\pi} C_{q^i}(x,y) \quad \bmod \deg q^i + 1 \\ \delta_a(x) &\equiv v \cdot \frac{a^{q^i} - a}{\pi} x^{q^i} \quad \bmod \deg q^i + 1 \end{aligned}$$

again for some $0 \neq v \in \kappa$. Thus,

$$H^2(G, g; \kappa) = \bigoplus_{1 \leq i \leq h-1} \kappa \cdot (\Delta_i, \delta_i)$$

where each (Δ_i, δ_i) is in degree q^i. We can then show the following:

Proposition 2.7 [Dri75, Prop. 4.2]**.** *Let (G, g) be a formal $\mathcal{O}$-module over κ of finite height h. Then, the functor*

$$\begin{aligned} \Psi \colon \mathscr{C} &\longrightarrow \mathsf{Set} \\ R &\longmapsto \left\{ \begin{matrix} \text{deformations of} \\ (G,g) \text{ over } R \end{matrix} \right\} \Big/ \text{isomorphism} \end{aligned}$$

is represented by $\widehat{\mathcal{O}}^{\mathrm{nr}}[\![t_1, \ldots, t_{h-1}]\!]$.

Sketch of Proof. The "tangent space"

$$\Psi\big(\kappa[\varepsilon]/(\varepsilon^2)\big) = H^2(G, g; \kappa)$$

has basis (Δ_i, δ_i) for $1 \le i \le h-1$. We can then "glue" the (Δ_i, δ_i) to form an element in

$$\Psi\big(\kappa[\![t_1, \dots, t_{h-1}]\!]/(t_1, \dots, t_{h-1})^2\big);$$

this element maps to (Δ_i, δ_i) under the morphism $\Psi\big(\kappa[\![t_1, \dots, t_{h-1}]\!]/(t_1, \dots, t_{h-1})^2\big) \to \Psi\big(\kappa[\![\varepsilon]\!]/(\varepsilon^2)\big)$ induced by the projection $\kappa[\![t_1, \dots, t_{h-1}]\!]/(t_1, \dots, t_{h-1})^2 \twoheadrightarrow \kappa[\![t_i]\!]/(t_i)^2$. By formal smoothness, it lifts to an element

$$(F^0, f^0) \in \Psi\big(\widehat{\mathcal{O}}^{\mathrm{nr}}[\![t_1, \dots, t_{h-1}]\!]\big).$$

By the Yoneda lemma, this corresponds to the natural transformation

$$\begin{aligned} \mathrm{Hom}\big(\widehat{\mathcal{O}}^{\mathrm{nr}}[\![t_1, \dots, t_{h-1}]\!], -\big) &\longrightarrow \Psi(-) \\ \big(\varphi\colon \widehat{\mathcal{O}}^{\mathrm{nr}}[\![t_1, \dots, t_{h-1}]\!] \to R\big) &\longmapsto \varphi(F^0, f^0). \end{aligned}$$

We want this to be a bijection for every R. Since every (F, f) over R is the inverse limit of its reductions modulo $\mathfrak{m}_R^k$, it suffices to show that

$$\mathrm{Hom}\big(\widehat{\mathcal{O}}^{\mathrm{nr}}[\![t_1, \dots, t_{h-1}]\!], R\big) \longrightarrow \Psi(R)$$

is a bijection for every Artin algebra $R \in \mathscr{C}$.

First, consider the case when $\ell(R) = 1$, i.e., when $R = \kappa[\varepsilon]/(\varepsilon^2)$. In this case, we have

$$\begin{array}{ccc} \mathrm{Hom}\big(\widehat{\mathcal{O}}^{\mathrm{nr}}[\![t_1, \dots, t_{h-1}]\!], R\big) & \longrightarrow & \Psi(R) \\ \| & & \| \\ \mathrm{Hom}\big(\kappa[\![t_1, \dots, t_{h-1}]\!]/(t_1, \dots, t_{h-1})^2, R\big) & & \| \\ \| & & \| \\ \bigoplus_i \mathrm{Hom}\big(\kappa[t_i]/(t_i)^2, R\big) & \longrightarrow & \bigoplus_i \kappa \cdot (\Delta_i, \delta_i) \\ \left(\varphi_i\colon \begin{matrix} t_i \mapsto \varepsilon \\ t_j \mapsto 0 \end{matrix}\right) & \longmapsto & (\Delta_i, \delta_i) \end{array}$$

which is a bijection by construction.

We now consider the inductive case when $\ell(R) = n+1$. The key observation, which we have already used, is that we can glue deformations. More precisely, since deformations have no nontrivial automorphisms by Corollary 2.4, Ψ is product-preserving: for every cartesian diagram

$$\begin{array}{ccc} R \times_T S & \longrightarrow & S \\ \downarrow & & \downarrow \\ R & \longrightarrow & T \end{array}$$

of local Artin algebras, we have

$$\Psi(R \times_S T) = \Psi(R) \times_{\Psi(S)} \Psi(T).$$

Choose $x \in R$ such that $\mathfrak{m}_R \cdot x = 0$, and consider the commutative diagram

$$\begin{array}{ccc} R \times \kappa[\varepsilon]/(\varepsilon^2) \simeq R \times_{R/x} R & \longrightarrow & R \\ \downarrow & & \downarrow \\ R & \longrightarrow & R/x \end{array}$$

where the top-left isomorphism maps (r,ε) to $(r, r+x)$. Applying Ψ, we obtain the following cartesian diagram:

$$\begin{array}{ccc} \Psi(R) \times \Psi\big(\kappa[\varepsilon]/(\varepsilon^2)\big) & \longrightarrow & \Psi(R) \\ \downarrow & & \downarrow \\ \Psi(R) & \longrightarrow & \Psi(R/x) \end{array}$$

We get a similar cartesian diagram for $\mathrm{Hom}(\widehat{\mathcal{O}}^{\mathrm{nr}}[\![t_1,\ldots,t_{h-1}]\!], -)$ and a map between these two diagrams. Then a diagram chase shows that $\mathrm{Hom}\big(\widehat{\mathcal{O}}^{\mathrm{nr}}[\![t_1,\ldots,t_{h-1}]\!], R\big) \longrightarrow \Psi(R)$ is bijective. $\square$

The cartesian-ness of the last diagram implies that $\Psi(R)$ is naturally a torsor over $\Psi(R/x)$ for $\Psi\big(\kappa[\varepsilon]/(\varepsilon^2)\big)$.

3 Level structures [Dri75, §4B]

We now discuss level structures, which are a way to remove automorphisms in moduli problems so that the resulting moduli space is more manageable.

We first recall the classical case. Let $E/\mathbf{C}$ be an elliptic curve, so that $E = \mathbf{C}/(\mathbf{Z} + \mathbf{Z}\cdot\tau)$. Adding level structure remembers the lattice $\mathbf{Z} + \mathbf{Z}\cdot\tau$ modulo n, that is, it is an isomorphism

$$(\mathbf{Z}/n\mathbf{Z})^2 \simeq E[n].$$

Definition 3.1. Let $R \in \mathscr{C}$, and let (F,f) be a formal $\mathcal{O}$-module over R. In particular, (F,f) imposes an $\mathcal{O}$-module structure on $\mathfrak{m}$. Assume (F,f) modulo $\mathfrak{m}$ has finite length h. Then, for $n \in \mathbf{Z}_{\geq 0}$, a *level-$n$ structure on* (F,f) is an $\mathcal{O}$-module homomorphism

$$\psi\colon \big(\tfrac{1}{\pi^n}\mathcal{O}/\mathcal{O}\big)^h \longrightarrow \mathfrak{m}$$

such that $f_\pi(x)$ is divisible by

$$\prod_{a\in(\frac{1}{\pi}\mathcal{O}/\mathcal{O})^h} \big(x - \phi(a)\big) \tag{3.1}$$

when $n > 0$.

In fact, $f_\pi(x)$ and (3.1) will divide each other. The domain of ψ is the analogue of $(\mathbf{Z}/n\mathbf{Z})^2$, and the condition on divisibility is a condition similar to being n-torsion.

Example 3.2. If $R = \kappa$ and $f_\pi(x) \equiv u \cdot X^{q^h} \bmod \deg q^h + 1$, then ϕ must be trivial.

Definition 3.3. Let (G,g) be a formal $\mathcal{O}$-module over κ of finite height h. A *deformation of* (G,g) *of level* n is a deformation of (G,g) with level-n structure.

Proposition 3.4 [Dri75, Prop. 4.3]**.**

(1) *The functor*

$$\begin{aligned} \Psi_n\colon \mathscr{C} &\longrightarrow \mathsf{Set} \\ R &\longmapsto \left\{ \begin{matrix} \text{deformations of level } n \\ \text{of } (G,g) \text{ over } R \end{matrix} \right\} \Big/ \text{isomorphism} \end{aligned}$$

is represented by a ring D_n.

(2) *D_n is regular. Moreover, suppose $n \geq 1$ and e_i, for $i = 1,\ldots,h$, is a basis for $(\frac{1}{\pi^n}\mathcal{O}/\mathcal{O})^h$. Then, the universal deformation of level n induces a map*

$$\phi_n\colon \big(\tfrac{1}{\pi^n}\mathcal{O}/\mathcal{O}\big)^h \longrightarrow \mathfrak{m}_{D_n}$$

under which the e_i map to a regular system of parameters.

(3) *If $m \leq n$, the natural map $D_m \to D_n$ is finite and flat.*

We note that the map in (3) makes sense since every level-n structure induces a level-m structure.

Proof. We proceed by induction on n. If $n = 0$, then $D_0 = \widehat{\mathcal{O}}^{\mathrm{nr}}[\![t_1, \dots, t_{h-1}]\!]$ and we denote by (F^0, f^0) the universal deformation over D_0.

We now assume we have constructed D_1, and prove inductively that D_{n+1} exists satisfying the statement of the Proposition assuming the existence of D_n. Let $b_1, \dots, b_n \in D_n$ be the regular system of parameters that is induced by the basis for $(\frac{1}{\pi^n}\mathcal{O}/\mathcal{O})^h$. Then, given the universal level-n structure ϕ_n we want to be able to construct $\phi_{n+1} \colon (\frac{1}{\pi^{n+1}}\mathcal{O}/\mathcal{O})^h \to \mathfrak{m}_{D_{n+1}}$ such that $\phi_n = \phi_{n+1} \circ \pi = f_\pi \circ \phi_n$. We therefore set

$$D_{n+1} = \frac{D_n[\![y_1, \dots, y_h]\!]}{\bigl(f_\pi(y_1) - b_1, \dots, f_\pi(y_h) - b_h\bigr)},$$

which is regular with regular system of parameters $y_1, \dots, y_h$, and such that $D_n \to D_{n+1}$ is finite flat.

It remains to construct D_1, for which we need another inductive argument, in the form of the following:

Lemma 3.5 [Dri75, p. 572, Lem.]**.**

(i) *For $0 \le r \le h$, the functor*

$$\Phi_r \colon R \longmapsto \left\{ \phi \colon \bigl(\tfrac{1}{\pi}\mathcal{O}/\mathcal{O}\bigr)^r \to \mathfrak{m}_R \;\middle|\; \begin{array}{c} f_\pi(x) \text{ divisible by} \\ \displaystyle\prod_{a \in (\frac{1}{\pi}\mathcal{O}/\mathcal{O})^r} \bigl(x - \phi(a)\bigr) \end{array} \right\}$$

for $R \in \mathscr{C}$ a D_0-algebra, is representable by a ring L_r.

(ii) *L_r is regular with a regular system of parameters given by the images of a basis e_i, $i = 1, \dots, r$ of $(\frac{1}{\pi}\mathcal{O}/\mathcal{O})^r$.*

(iii) *$L_{r-1} \to L_r$ is finite and flat.*

We omit the proof of this Lemma, noting that the case $r = 0$ is easy, and the case $r = h$ gives the desired construction of D_1. □

4 Divisible $\mathcal{O}$-modules [Dri75, §4C]

Before, we were in the situation of E a supersingular elliptic curve, for which the completion $\widehat{E}$ of E at 0 is a formal group of height 2. If E is ordinary instead, then $\widehat{E}$ does not give the full picture, and one has to consider the p-divisible groups

$$E[p^\infty] = \varinjlim_n E[p^n] \simeq \varinjlim_n (\mathbf{Z}/p^n \times \mu_{p^n}) = \mathbf{Q}_p/\mathbf{Z}_p \times \mu_{p^\infty},$$

where the first factor is the divisible part, and the second factor is the completion of E at 0. This is still a formal group, i.e., a group object in the category of formal schemes. For all formal groups G, we still have the connected-étale sequence

$$0 \longrightarrow G^\circ \longrightarrow G \longrightarrow G^{\text{ét}} \longrightarrow 0,$$

where G° is the connected part and $G^{\text{ét}}$ is the étale part. For example, for the elliptic curve E,

$$0 \longrightarrow \mu_{p^\infty} \longrightarrow E[p^\infty] \longrightarrow \mathbf{Q}_p/\mathbf{Z}_p \longrightarrow 0.$$

In the setting of formal modules, we make the following:

Definition 4.1. Let $R \in \mathscr{C}$. A *divisible $\mathcal{O}$-module over R* is a formal group F over R together with a homomorphism $f \colon \mathcal{O} \to \operatorname{End}(F)$ such that F° is a formal $\mathcal{O}$-module and $F^{\text{ét}} \simeq \operatorname{Spf} R \times (K/\mathcal{O})^j$ for some $j < \infty$,

Now assume F° mod $\mathfrak{m}_R$ has finite height. A *level-n structure on* (F, f) is a homomorphism

$$\bigl(\tfrac{1}{\pi^n}\mathcal{O}/\mathcal{O}\bigr)^{j+h} \longrightarrow \operatorname{Hom}(\operatorname{Spf} R, F)$$

which induces a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \left(\frac{1}{\pi^n}\mathcal{O}/\mathcal{O}\right)^h & \xrightarrow[\text{first } h \text{ factors}]{\text{inclusion}} & \left(\frac{1}{\pi^n}\mathcal{O}/\mathcal{O}\right)^{j+h} & \xrightarrow[\text{last } j \text{ factors}]{\text{projection}} & \left(\frac{1}{\pi^n}\mathcal{O}/\mathcal{O}\right)^j & \longrightarrow & 0 \\
 & & {\scriptstyle \text{level-}n \text{ structure for } (F^\circ, f^\circ)}\big\downarrow & & \big\downarrow & & \big\downarrow & & \\
0 & \longrightarrow & \operatorname{Hom}(\operatorname{Spf} R, F^\circ) & \longrightarrow & \operatorname{Hom}(\operatorname{Spf} R, F) & \longrightarrow & \operatorname{Hom}(\operatorname{Spf} R, F^{\text{ét}}) & & \\
 & & \wr\Vert & & & & \wr\Vert & & \\
 & & \mathfrak{m} & & & & (K/\mathcal{O})^j & &
\end{array}$$

where the isomorphism on the bottom-left follows from the fact that $F^\circ \simeq \operatorname{Spf} R[\![x]\!]$.

We can generalize Proposition 2.7 in the following manner:

Proposition 4.2 [Dri75, Prop. 4.5]**.** *Let (G, g) be a divisible $\mathcal{O}$-module over κ with level-n structure such that G° has height h, and $G/G^\circ \simeq (K/\mathcal{O})^j$. Let $n \in \mathbf{Z}_{\geq 0}$. Then, the functor*

$$\begin{aligned}
\Psi_n \colon \mathscr{C} &\longrightarrow \mathsf{Set} \\
R &\longmapsto \left\{ \begin{matrix} \textit{deformations of level } n \\ \textit{of } (G,g) \textit{ over } R \end{matrix} \right\} \Big/ \textit{isomorphism}
\end{aligned}$$

is represented by the regular ring $E_n \simeq D[\![d_1, \ldots, d_j]\!]$. In particular, E_n is regular of dimension $j + h$, and E_0 is smooth over $\widehat{\mathcal{O}}^{\mathrm{nr}}$. If $m \leq n$, then the homomorphism $E_m \to E_n$ is finite flat.

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