Drinfel′d Modules: Overview

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The topic this semester is Drinfel′d modules. Today, we will give an overview of what will be doing this semester.

1 The Langlands program for GL_2

The motivation for Drinfel′d modules comes from the Langlands program, so we will start by discussing the Langlands program. Note that the material we will cover this semester is mostly independent from the Langlands program; the connection with the Langlands program will only show up at the end.

Fix a global field K, e.g., the function field of a curve over a finite field, and fix a prime $\ell \neq \operatorname{char}(k)$. The Langlands program (for GL_2) conjectures a bijection

$$\left\{ \begin{array}{c} \text{certain automorphic} \\ \text{representations of } \mathrm{GL}_2(\mathbf{A}_K) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{certain 2-dimensional} \\ \ell\text{-adic representations of } G_K \end{array} \right\}. \tag{1.1}$$

The restriction on representations on both sides is weak enough that the sets on either side contain most things you would be interested in. There is also a more general version of the Langlands program which considers GL_n for all n.

It is very hard to construct the maps in either direction. In a series of papers starting with [Dri75], Drinfel′d showed the bijection in (1.1) when K is a function field. Our goal for this seminar will be the following part of Drinfel′d's work:

Goal 1.1. Describe the map $\to$ in (1.1) when K is a function field.

In this seminar, we will study *Drinfel′d modules*, which are the key ingredient in showing (1.1).

Today, we will describe the big picture for how Drinfel′d modules work. Their construction is motivated by an analogous construction used in showing (1.1) when $K = \mathbf{Q}$, so we will review that first.

1.1 The case $K = \mathbf{Q}$

When $K = \mathbf{Q}$, the bijection in (1.1) is

$$\left\{ \begin{array}{c} \text{certain automorphic} \\ \text{representations of } \mathrm{GL}_2(\mathbf{A}_{\mathbf{Q}}) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{certain 2-dimensional} \\ \ell\text{-adic representations of } G_{\mathbf{Q}} \end{array} \right\}. \tag{1.2}$$

A basic fact is:

$$K \cdot O_2(\mathbf{R}) \cdot \mathbf{R}^\times \backslash^{\mathrm{GL}_2(\mathbf{A})} /_{\mathrm{GL}_2(\mathbf{Q})} \xrightarrow{\sim} \mathbf{H}/\Gamma,$$

where K is a compact open subgroup of $\mathrm{GL}_2(\mathbf{A}_{\mathrm{fin}})$, the general linear group of degree 2 over the ring of finite adeles, and where Γ is a subgroup of $\mathrm{SL}_2(\mathbf{Z})$, which acts on the upper-half plane $\mathbf{H}$. Thus, the map $\to$ boils down to associating a 2-dimensional ℓ-adic representation of $G_{\mathbf{Q}}$ to a modular form.

To produce such an ℓ-adic representation, we will consider the ℓ-adic cohomology of a variety. There is a nice candidate for such a variety, that is,

$$Y_\Gamma \coloneqq \mathbf{H}/\Gamma,$$

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which is a moduli space of elliptic curves with level structure as long as Γ is a finite index subgroup of $\mathrm{SL}_2(\mathbf{Z})$ in a nice way.

Example 1.2. When $\Gamma = \mathrm{SL}_2(\mathbf{Z})$, we have a bijection

$$\begin{aligned} \mathbf{H}/\operatorname{SL}_2(\mathbf{Z}) &\xrightarrow{\sim} \left\{\begin{matrix}\text{isomorphism classes of} \\ \text{elliptic curves over } \mathbf{C}\end{matrix}\right\} \\ \tau &\longmapsto \mathbf{C}/\langle 1, \tau\rangle \end{aligned}$$

Example 1.3. Let Γ be the following subgroup of $\mathrm{SL}_2(\mathbf{Z})$:

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, c \equiv 0 \bmod N \right\} \subset \mathrm{SL}_2(\mathbf{Z}).$$

The quotient space $\mathbf{H}/\Gamma_0(N)$ is in bijection with isomorphic classes of pairs (E, G), where $G \subset E$ is cyclic of order N. This is what we mean by *level structure:* the moduli space keeps track of some torsion points.

Although these constructions are purely analytic, it is important to note that the resulting moduli spaces make sense in greater generality, e.g., over $\operatorname{Spec} \mathbf{Q}$ or $\operatorname{Spec} \mathbf{Z}$. Thus, these spaces can actually be thought of as varieties over $\mathbf{Q}$.

Remark 1.4. There are some details you have to be careful with; for example, if N is small in Example 1.3, there can be lots of issues.

We now have a way to produce varieties, which in our case are curves. Furthermore, we can compactify these curves Y_Γ by adding some cusp points to form a complete curve X_Γ. We then look at

$$H^1_{\text{ét}}\big((X_\Gamma)_{\overline{\mathbf{Q}}}, \mathbf{Q}_\ell\big)$$

which is an ℓ-adic representation of $G_{\mathbf{Q}}$. This representation is not 2-dimensional, and not associated to any modular form; in fact, it has dimension $2g$ where g is the genus of X_Γ, and g increases as Γ gets smaller.

The solution is to then break up this ℓ-adic representation into 2-dimensional pieces, and the way to do that is to use the action of the Hecke algebra $\mathbf{T}$ on this cohomology space. This Hecke algebra $\mathbf{T}$ acts on the modular curve via correspondences, which can also be thought of as endomorphisms on the Jacobian of the curve; either way, we have an action of $\mathbf{T}$ on H^1. In the case when $\Gamma = \Gamma_0(N)$, where N is a prime, we can decompose H^1 with respect to different characters of the Hecke algebra, which index modular forms, to obtain

$$H^1_{\text{ét}}\big((X_\Gamma)_{\overline{\mathbf{Q}}}, \mathbf{Q}_\ell\big) = \bigoplus_{\substack{\text{weight 2 cuspidal} \\ \text{modular forms } f \\ \text{of level } N}} V_f,$$

where each summand V_f is 2-dimensional. The assignment $f \mapsto V_f$ realizes the correspondence we want.

Remark 1.5. This construction is known as the Eichler–Shimura construction, and was discovered in the 1950s. The map in the other direction in (1.2) is due to work of Wiles and others, which occurred much more recently.

1.2 The function field case

We now consider the function field case, where we will try to carry out a similar construction as for $K = \mathbf{Q}$.

Let K be a function field, and choose a place ∞ of K.

Definition 1.6. We denote by K_∞ the completion of K at ∞, and $\mathbf{C}_K$ the completion of an algebraic closure $\overline{K_\infty}$ of K_∞. If C is the complete curve with function field K, then we denote by A the affine coordinate ring of $C \smallsetminus \{\infty\}$.

Example 1.7. If $K = \mathbf{F}_p(T)$, then $K_\infty = \mathbf{F}_p((\frac{1}{T}))$, and $A = \mathbf{F}_p[T]$.

The inclusion $A \subset K_\infty$ is an analogue of the inclusion $\mathbf{Z} \subset \mathbf{R}$. We will take this analogy further by having the ring A play the role of $\mathbf{Z}$. In particular, abelian groups will be replaced by A-modules, and elliptic curves with something involving A^2. We will then construct a moduli space of these things, which will realize the correspondence (1.1).

1.2.1 Carlitz modules

We start with the A-version of $\mathbf{G}_m$, which are called *Carlitz modules*.

Let $\Gamma \subset A$ be a finite subgroup under addition. We can then consider the polynomial

$$f(x) = \prod_{\gamma \in \Gamma} (x - \gamma).$$

The fact that Γ is a subgroup under addition implies f is an *additive polynomial*, that is, a polynomial satisfying $f(x+y) = f(x) + f(y)$. Note that as a polynomial, this implies that the only powers of x with nonzero coefficients are the p-powers of x. We can rewrite the polynomial above as

$$f(x) = x \prod_{\gamma \neq 0 \in \Gamma} \left(1 - \frac{x}{\gamma}\right),$$

at least up to scalars. Writing it in this way, it suggests that we can maybe allow infinite subgroups $\Gamma \subset A$. The biggest thing you can do is $\Gamma = A$, in which case

$$f(x) = x \prod_{\alpha \neq 0 \in A} \left(1 - \frac{x}{\alpha}\right).$$

Fact 1.8. This is an additive power series over K, and converges to an element of $\mathbf{C}_K$ whenever you plug in an element of $\mathbf{C}_K$. The coefficients of x involve a q-analogue of factorials.

Writing

$$x \prod_{\alpha \neq 0 \in A} \left(1 - \frac{x}{\alpha}\right) =: \frac{1}{\xi} e(\xi x),$$

for some element $\xi \in \mathbf{C}_K$ (that is analogous to $2\pi i$), we obtain the *Carlitz exponential* $e(x)$. There will be an explicit formula for ξ.

The Carlitz exponential defines an isomorphism of abelian groups

$$e(\xi \cdot -) \colon \mathbf{C}_K/A \xrightarrow{\sim} \mathbf{C}_K,$$

which is analogous to

$$\exp \colon \mathbf{C}/\mathbf{Z} \xrightarrow{\sim} \mathbf{C}^\times.$$

Although we have a group isomorphism, the group structure is not what should be considered. Instead, just as the group isomorphism $\exp \colon \mathbf{C}/\mathbf{Z} \overset{\sim}{\to} \mathbf{C}^\times$ can be used to give $\mathbf{C}^\times$ the structure of an abelian group, we can think of $\mathbf{C}_K$ as having an A-module structure via the group isomorphism $e(\xi \cdot -)$.

Definition 1.9. The group $\mathbf{C}_K$ together with its A-module structure inherited from the group isomorphism $e(\xi \cdot -)$ is called the *Carlitz module*, and is denoted $\mathbf{C}_K^{\mathrm{Car}}$.

As groups, $\mathbf{C}_K$ and $\mathbf{C}_K^{\mathrm{Car}}$ are isomorphic, but their A-module structures are different. For example, $\mathbf{C}_K/A$ contains K/A, which has torsion elements; we see that $\mathbf{C}_K^{\mathrm{Car}}$ therefore has torsion elements, even though $\mathbf{C}_K$ does not.

The use of the group isomorphism $e(\xi \cdot -)$ makes the discussion above somewhat analytic in flavor. One can also study Carlitz modules from a more algebraic point of view, as follows. Given $a \in A$, write C_a for the endomorphism of $\mathbf{C}_K^{\mathrm{Car}}$ it defines.

Fact 1.10. C_a is an additive polynomial.

Example 1.11. Let $A = \mathbf{F}_q[T]$. Then,

$$e(Tx) = Te(x) + e(x)^q,$$

and so letting $a = T$, we have $C_a(x) = x^q + Tx$.

Writing $K = \mathrm{Frac}(A)$, we will consider the ring $K\{\tau\}$. As an additive group, this ring consists of polynomials in τ over K, but the multiplication has some twisting: for $a \in K$, we have $\tau a = a^p \tau$. The ring $K\{\tau\}$ can be used to represent additive polynomials, which form a ring under composition (this is why we have the twist). Each $a \in A$ gives an additive polynomial C_a, and so we get a map

$$C\colon A \longrightarrow K\{\tau\},$$

which is a ring homomorphism. One can then study Carlitz modules by studying this homomorphism C.

We therefore have two points of view of the Carlitz module:

(1) Analytic: $A \subset \mathbf{C}_K$ is a lattice with exponential function e.
(2) Algebraic: the ring homomorphism $C\colon A \to K\{\tau\}$.

There will be a talk about Carlitz modules.

1.2.2 Drinfel′d modules

Drinfel′d modules are a generalization of Carlitz modules, where A is replaced by a bigger lattice; interestingly, there are lattices of arbitrary rank in K, unlike in the case for $\mathbf{C}$. We will mostly consider rank 2 lattices, for which there will be two equivalent points of view as above. The story for Drinfel′d modules then follows that for elliptic curves in the case $K = \mathbf{Q}$, as follows:

1. There is a moduli space of rank 2 Drinfel′d modules, which can have level structure.
2. This moduli space is a smooth curve over K, which is an analogue of Y_Γ.
3. This curve can be compactified by adding cusps, using some representability techniques from algebraic geometry, specifically deformation theory. There is also an analytic description of this via uniformization: it is a quotient of the *Drinfel′d upper half plane*

$$\mathbf{P}^1(\mathbf{C}_K) \smallsetminus \mathbf{P}^1(K_\infty)$$

by some group.
4. The ℓ-adic cohomology of this complete curve realizes our correspondence (1.1).

We will not get to the automorphic stuff until the end. Most of the talks will just be talking about elliptic modules from an elementary point of view.

References

[Dri75] V. G. Drinfel′d. "Elliptic modules." Trans. by M. B. Nathanson. *Math. USSR-Sb.* 23.4 (1974), pp. 561–592. DOI: 10.1070/SM1974v023n04ABEH001731. MR: 0384707.

[Gos96] D. Goss. *Basic structures of function field arithmetic.* Ergeb. Math. Grenzgeb. (3), Vol. 35. Berlin: Springer-Verlag, 1996. DOI: 10.1007/978-3-642-61480-4. MR: 1423131.