AN OVERVIEW OF THE THEORY OF DRINFELD MODULES

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Abstract. These are the notes of an expository talk about Drinfeld modules I gave at the University of Michigan on September 18, 2017. The talk was aimed at graduate students.

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1. Carlitz module

1.1. **Carlitz zeta function.** Let q be a power of a prime number p. The ring of integers $\mathbb{Z}$ has many similarities with the ring

$$A = \mathbb{F}_q[T]$$

of polynomials in indeterminate T with coefficients in the finite field $\mathbb{F}_q$ with q elements, e.g., both are Euclidean domains, have finite residue fields and finite groups of units. But there are also deeper arithmetic similarities. One of those similarities arises in the theory zeta functions.

A famous result of Euler says that for *even* $m \geq 2$, we have

$$\zeta(m) = \sum_{n=1}^{\infty} \frac{1}{n^m} = -B_m (2\pi i)^m/2, \tag{1.1}$$

where $i = \sqrt{-1}$ and B_m's are the coefficients of the expansion

$$\frac{x}{e^x - 1} = \sum_{m=0}^{\infty} B_m x^m$$

($B_m \cdot m!$ are the Bernoulli numbers). For example, $\zeta(2) = \pi^2/6$. The key to the proof of this formula is the product expansion of

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

or rather,

$$\frac{e^x - e^{-x}}{2} = \pi x \prod_{n=1}^{\infty} \left(1 + \frac{x^2}{n^2}\right).$$

In [Car35], Carlitz proved an analog of (1.1) for A. Let A_+ be the set of monic polynomials in A; this is the analog of the set of positive integers $\mathbb{Z}_+ = \{1, 2, \dots\}$. For $m \geq 1$, consider

$$\zeta_C(m) = \sum_{a \in A_+} \frac{1}{a^m}.$$

Notation 1.1. Let $F = \mathbb{F}_q(T)$ be the fraction field of A and $F_\infty = \mathbb{F}_q(\!(1/T)\!)$ be the completion of F with respect to the norm $|a/b| = q^{\deg(a)-\deg(b)}$. Note that for $a \in A$ we have $\#(A/aA) = |a|$, so this norm is the analog of the usual absolute value $|n| = \#(\mathbb{Z}/n\mathbb{Z})$ on $\mathbb{Z}$. Let $\mathbb{C}_\infty$ be the completion of an algebraic closure of F_∞. If A is the analog of $\mathbb{Z}$, then $F, F_\infty, \mathbb{C}_\infty$ are the analogs of $\mathbb{Q}, \mathbb{R}, \mathbb{C}$, respectively.

It is easy to see that the series $\zeta_C(m)$ converges in F_∞.

The product

$$\prod_{\substack{a \in A \\ \deg(a) < d}} (x - a) \tag{1.2}$$

has a nice sum expansion (thanks to the fact that the set $\{a \in A \mid \deg(a) < d\}$ is an $\mathbb{F}_q$-vector space). Put $[n]_C = T^{q^n} - T$,

$$D_0 = 1, \quad D_n = [n]_C[n-1]_C^q \cdots [1]_C^{q^{m-1}},$$

$$\boldsymbol{\pi} = \prod_{n=1}^{\infty} \left(1 - \frac{[n]_C}{[n+1]_C}\right), \quad \mathbf{i} = (-[1]_C)^{1/(q-1)}.$$

By taking $d \to \infty$ in the expansion of (1.2) one arrives at the following crucial formula of Carlitz:

$$e_C(x) := \sum_{n=0}^{\infty} \frac{x^{q^n}}{D_n} = x \prod_{0 \neq \alpha \in \boldsymbol{\pi}\mathbf{i}A} \left(1 - \frac{x}{\alpha}\right). \tag{1.3}$$

Exercise 1.2. Show that $e_C(x) = \sum_{n=0}^{\infty} x^{q^n}/D_n$ is an entire function on $\mathbb{C}_\infty$.

The *Carlitz exponential* $e_C(x)$, as the name suggests, is the analog of e^x. It is easy to see that the kernel of $e_C(x) : \mathbb{C}_\infty \to \mathbb{C}_\infty$ is the lattice $\boldsymbol{\pi}\mathbf{i}A \subset \mathbb{C}_\infty$, as $2\pi i\mathbb{Z}$ is the kernel of $e^x : \mathbb{C} \to \mathbb{C}^\times$. From this perspective, D_n is the analog of $n!$, $\boldsymbol{\pi}$ is the analog of π, $\mathbf{i}$ is the analog of $\sqrt{-1}$, and $q - 1 = \#A^\times$ is the analog of $2 = \#\mathbb{Z}^\times$. Define BC_m by

$$x/e_C(x) = \sum_{m=0}^{\infty} BC_m x^m = 1 - \frac{1}{D_1} x^{q-1} + \cdots$$

With this at hand, mimicking Euler's argument one obtains:

Theorem 1.3. *For $(q-1) \mid m$, we have*

$$\zeta_C(m) = -BC_m(\boldsymbol{\pi}\mathbf{i})^m/(q-1).$$

Remark 1.4. In 1941, Wade [Wad41] proved that $\boldsymbol{\pi}$ is transcendental over F. More recently, by considering "tensor powers of the Carlitz module" (which are Anderson modules), Anderson and Thakur [AT90] deduced a formula for $\zeta_C(m)$ for arbitrary $m \geq 1$, and Jing Yu [Yu91] used this result to prove that $\zeta_C(m)$ is transcendental over F for all $m \geq 1$. The transcendence of zeta-values $\zeta(n)$, for $n \geq 3$ odd, is a major open problem in number theory.

1.2. **Analytic continuation and Riemann hypothesis.** The Riemann zeta function ζ has a meromorphic continuation to $\mathbb{C}$ with a pole at 1. Moreover, $\zeta(x)$ is in $\mathbb{Q}$ on the negative integers and zero on the negative evens. In analogy with this, Goss [Gos79] extended the domain of ζ_C from $\mathbb{Z}_+$ to $\mathbb{C}_\infty^\times \times \mathbb{Z}_p$ as follows: For a monic polynomial $a \in A_+$ set $\langle a \rangle := aT^{-\deg(a)}$ and for $(x, y) \in \mathbb{C}_\infty^\times \times \mathbb{Z}_p$ define

$$a^{(x,y)} := x^{\deg(a)} \langle a \rangle^y.$$

The term $\langle a\rangle^y$ is well defined since $\langle a\rangle^y \equiv 1 \pmod{T^{-1}}$. Goss showed that by grouping together terms of the same degree, ζ_C becomes well defined over all $\mathbb{C}_\infty^\times \times \mathbb{Z}_p$:

$$\zeta_{CG}(x,y) := \sum_{m\geq 0} x^{-m}\left(\sum_{\substack{a\in A_+\\ \deg(a)=m}} \langle a\rangle^{-y}\right).$$

Note that $a^{(T^m,m)} = a^m$ for any integer m and $a\in A_+$. Define $\zeta_{CG}(m) := \zeta_{CG}(T^m,m)$ for $m\in\mathbb{Z}$. Thus when $m>0$ we have $\zeta_{GC}(m) = \sum_{a\in A_+} a^{-m} = \zeta_C(m)$. In [Gos79], Goss proved that for $m>0$ we have $\zeta_{CG}(-m)\in A$ and $\zeta_{CG}(-m)=0$ when $m\equiv 0 \pmod{q-1}$.

The following statement is the analog of the Riemann Hypothesis for the Carlitz-Goss zeta function:

Theorem 1.5. *Fix $y\in\mathbb{Z}_p$. As a function of x, the zeros of $\zeta_{CG}(x,-y)$ are simple and lie in F_∞ (i.e., all lie on the same "real line").*

This was proved for $q=p$ by Wan [Wan96] using difficult ad hoc calculations with the Newton polygon of ζ_{CG}. Soon after, Thakur realized that a combinatorial result stated, but not proved, in an old paper by Carlitz [Car48] on power sums of polynomials would allow one to give a simpler proof, and this was carried out in the case $q=p$ by Diaz-Vargas [DV96], and for general q by Sheats [She98].

There are generalizations of Carlitz-Goss zeta function analogous to the Dedekind and Artin zeta functions. The meromorphic continuation of these zeta functions to $\mathbb{C}_\infty^\times\times\mathbb{Z}_p$ was proved by Taguchi and Wan [TW96] using analytic techniques introduced by Dwork. More recently, Böckle and Pink [BP09] found a cohomological proof of this analytic behavior. Taelman [Tae12] proved a formula for a special value of some of these zeta functions which can be interpreted as the analog of the class number formula and of the Birch and Swinnerton-Dyer conjecture. Whether ζ_{CG} and its generalizations posses functional equations, and the shape of such functional equations, is a major open problem in this area.

1.3. **Carlitz exponential.** From the expression $\zeta_C(x) = \sum_{i\geq 0} x^{q^i}/D_i$, it is easy to see that $\zeta_C(x)$ has the following properties:

- $e_C(x+y) = e_C(x)+e_C(y)$;
- $e_C(\alpha x) = \alpha e_C(x)$ for any $\alpha\in\mathbb{F}_q$;
- $e_C(Tx) = Te_C(x) + e_C(x)^q$.

This implies that for any $a\in A$ there is a polynomial

$$\rho_a(x) = ax + c_1x^q + \cdots + c_dx^{q^d}, \quad d=\deg(a), \quad c_1,\dots,c_d\in\mathbb{C}_\infty, c_d\neq 0,$$

such that

$$e_C(a\cdot x) = \rho_a(e_C(x))$$

For example,

$$\rho_T(x) = Tx + x^q, \quad \rho_{T^2+1}(x) = (T^2+1)x + (T^q+T)x^q + x^{q^2}.$$

Let $\mathbb{C}_\infty\{x\}$ be the (non-commutative) ring of polynomials of the form $\sum_{i=0}^n s_i x^{q^i}$, $n \geq 0$, where the addition is the usual addition of polynomials but multiplication is given by composition $(f \circ g)(x) = f(g(x))$. Essentially from the definition of ρ_a, the map

$$\rho : A \to \mathbb{C}_\infty\{x\}, \quad a \mapsto \rho_a,$$

is an injective ring homomorphism, called the *Carlitz module*.

One might wonder why ρ, which is a ring homomorphism, is called "module". To see the module, let $\mathbb{C}_\infty\{\tau\}$ be the twisted polynomial ring whose elements are polynomials $\sum_{i=0}^n s_i \tau^i$, $n \geq 0$, with coefficients $s_0, \dots, s_n \in \mathbb{C}_\infty$, and τ and $b \in \mathbb{C}_\infty$ do not commute but rather satisfy the commutation rule $\tau b = b^q \tau$. For example,

$$(a + b\tau)(c + d\tau) = ac + (bc^q + ad)\tau + bd^q\tau^2.$$

It is easy to check that the map $\mathbb{C}_\infty\{\tau\} \to \mathbb{C}_\infty\{x\}$, $\sum_{i=0}^n s_i\tau^i \mapsto \sum_{i=0}^n s_i x^{q^i}$, is a ring isomorphism. Moreover, we can identify $\mathbb{C}_\infty\{\tau\}$ with the $\mathbb{F}_q$-linear endomorphisms of the additive group $\mathbb{C}_\infty$ (or rather, the additive group-scheme $\mathbb{G}_{a,\mathbb{C}_\infty}$) if we treat τ as the q-power Frobenius endomorphism $\tau(s) = s^q$. Note that $\mathbb{C}_\infty$ is naturally an A-module, with A acting by multiplication $a \circ s = as$; via ρ we obtain a new A-module structure on $\mathbb{C}_\infty$, where a now acts as $a \circ s = \rho_a(s)$, so the "module" is $\mathbb{C}_\infty$ equipped with ρ-action of A.

2. Drinfeld modules

2.1. **Definition.** It is clear that the argument at the end of the previous section works in much larger generality. Let K be a field equipped with a homomorphism $\gamma : A \to K$; such fields are called A-fields and $\ker(\gamma)$ is called the A-characteristic of K. (Of course, K has characteristic p in the usual sense.) Note that $\mathbb{F}_q$ is a subfield of K, so we can define $K\{\tau\}$ as earlier, i.e., as the twisted polynomial ring with the commutation rule $\tau b = b^q\tau$.

Drinfeld module of rank r over K is a homomorphism

$$\phi : A \to K\{\tau\}, \quad a \mapsto \phi_a,$$

such that $\phi_T = \gamma(T) + c_1\tau + \cdots + c_r\tau^r$, $c_r \neq 0$. Note that specifying ϕ_T uniquely determines ϕ, since T generates A over $\mathbb{F}_q$. For example, the Carlitz module ρ is a Drinfeld module of rank 1 over $\mathbb{C}_\infty$.

Let $\partial : K\{\tau\} \to K$ be the homomorphism mapping a polynomial $\sum_{i=0}^n s_i\tau^i$ to its constant term s_0. Note that the differential of $\sum_{i=0}^n s_i x^{q^i}$ with respect to x is s_0, and $\gamma = \partial \circ \phi : A \to K$. Hence, through ϕ, K acquires a new A-module structure such that $a \in A$ acts on the "tangent space" by the usual multiplication by $\gamma(a)$. This is similar to considering an elliptic curve E over a field K as a $\mathbb{Z}$-module, where $n \in \mathbb{Z}$ acts on points of E through the group structure of E, but acts on the tangent space of E by usual multiplication by n on K.

A *morphism* $u : \phi \to \psi$ between two Drinfeld modules over K is $u \in K\{\tau\}$ such $u\phi_a = \psi_a u$ for all $a \in A$. In other words, u makes the following diagram commutative

$$\begin{tikzcd}
\mathbb{G}_{a,K} \arrow[r, "u"] \arrow[d, "\phi_a"'] & \mathbb{G}_{a,K} \arrow[d, "\psi_a"'] \\
\mathbb{G}_{a,K} \arrow[r, "u"] & \mathbb{G}_{a,K}
\end{tikzcd}$$

for all $a \in A$, so u is a homomorphism of underlying modules. We say that u is an *isomorphism* if $u \in K^\times$. By considering $u\phi_T = \psi_T u$, it is easy to see that non-zero morphisms exist only between Drinfeld modules of the same rank, and since the kernel of any non-zero u is finite, any such morphism is an "isogeny".

Remark 2.1. The choice of $A = \mathbb{F}_q[T]$ is made only for expository purposes. Given a smooth projective curve X over $\mathbb{F}_q$, fix a closed point ∞ on X and let A to be the subring of the field of rational functions on X which are regular away from ∞. Drinfeld A-module of rank r over K is an embedding $\phi : A \to K\{\tau\}$ such that $\partial\phi = \gamma$ and $\#\ker(\phi_a) = \#(A/aA)^r$ for all $a \in A$ (here $\ker(\phi_a)$ is considered as a group-scheme). Of course, it becomes quite difficult to write down equations for such Drinfeld modules even for $r = 1$ and X of genus 1; cf. [Hay91]. One can even define Drinfeld modules over an arbitrary A-scheme, instead of $\mathrm{Spec}(K)$; this is the set-up in Drinfeld's original paper [Dri74].

2.2. **Analytic uniformization.** A *lattice* $\Lambda \subset \mathbb{C}_\infty$ is a finitely generated A-submodule such that the intersection of Λ with any ball in $\mathbb{C}_\infty$ of finite radius is finite. (This is slightly stronger than requiring Λ to be discrete in $\mathbb{C}_\infty$ with respect to $|\cdot|$, e.g., the algebraic closure $\overline{\mathbb{F}}_q$ of $\mathbb{F}_q$ is discrete in $\mathbb{C}_\infty$ but is completely contained in the ball of radius 1.) The *rank* of Λ is its rank as a free A-module; for example, $\Lambda = A + \boldsymbol{\pi}\mathbf{i}A$ is lattice of rank 2. Note that unlike $\mathbb{Z}$-lattices in $\mathbb{C}$, there are lattices of arbitrary rank in $\mathbb{C}_\infty$, since the extension $\mathbb{C}_\infty/F_\infty$ has infinite degree.

Exercise 2.2. Let $\Lambda = Av_1 + \cdots + Av_r \subset \mathbb{C}_\infty$ be a free A-module of rank r. Show that if $v_1, \dots, v_r$ are linearly dependent over F_∞, then Λ is not a lattice.

For a lattice $\Lambda \subset \mathbb{C}_\infty$ of rank r define

$$e_\Lambda(x) = x \prod_{0 \neq \lambda \in \Lambda} \left(1 - \frac{x}{\lambda}\right).$$

Theorem 2.3 (Drinfeld)**.**

(1) *$e_\Lambda(x)$ is entire on $\mathbb{C}_\infty$ with simple zeros at the elements of Λ, and no other zeros.*
(2) *$e_\Lambda(x)$ is $\mathbb{F}_q$-linear.*
(3) *For each $a \in A$*

$$e_\Lambda(ax) = \phi_a^\Lambda(e_\Lambda(x)),$$

where

$$\phi_a^\Lambda(x) = \sum_{i=0}^{r\cdot\deg(a)} c_i x^{q^i}, \quad c_0 = a, \quad c_{r\cdot\deg(a)} \neq 0.$$

(4) *The map* $a \mapsto \phi_a^\Lambda$ *defines a Drinfeld module of rank* r *over* $\mathbb{C}_\infty$.

Proof. The proof of (1) and (2) is similar to the argument one uses to prove the corresponding statements for the Carlitz exponential e_C. Choose an A-basis $v_1, \dots, v_r$ of Λ, and consider $\Lambda_d := \{s_1v_1 + \cdots s_rv_r \mid \deg(s_i) \leq d, i = 1, \dots, r\}$. This is an $\mathbb{F}_q$-vector space so the polynomial $\prod_{\lambda\in\Lambda_d}(x - \lambda)$ is in $\mathbb{C}_\infty\{x\}$; thus, e_{Λ_d} is $\mathbb{F}_q$-linear; thus, $e_\Lambda = \lim_{d\to\infty} e_{\Lambda_d}$ is $\mathbb{F}_q$-linear. The discreteness of Λ in our stronger sense is essentially equivalent to e_Λ being entire. Let Λ' be the lattice generated by $v_1/a, \dots, v_r/a$. Let $z_1, \dots, z_n$ be coset representatives of Λ in Λ', so $n = q^{r\cdot\deg(a)}$. Let $P(x) = x\prod_{z_i\neq 0}(1 - x/e_\Lambda(z_i))$. It is not hard to show that $P(x)$ is independent of the choice of coset representatives $\{z_i\}$ and $P(x) \in \mathbb{C}_\infty\{x\}$. Since $e_\Lambda(ax)$ and $P(e_\Lambda(x))$ are entire functions with the same set of zeros Λ' and the derivative of both is a, one concludes that $e_\Lambda(ax)/P(e_\Lambda(x))$ is an entire function on $\mathbb{C}_\infty$ whose value at $x = 0$ is 1 and which has no zeros. By a fact from non-archimedean analysis, such a function must be constant. (Note that the analogous statement is false over $\mathbb{C}$, as e^x is entire and has no zeros.) Thus, $e_\Lambda(ax) = P(e_\Lambda(x))$. Finally, (4) easily follows from (2) and (3). □

Since an entire function on $\mathbb{C}_\infty$ is surjective, we have a commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \Lambda \arrow[r] \arrow[d, "a"] & \mathbb{C}_\infty \arrow[r, "e_\Lambda"] \arrow[d, "a"] & \mathbb{C}_\infty \arrow[r] \arrow[d, "\phi_a^\Lambda"] & 0 \\
0 \arrow[r] & \Lambda \arrow[r] & \mathbb{C}_\infty \arrow[r, "e_\Lambda"] & \mathbb{C}_\infty \arrow[r] & 0
\end{tikzcd}$$

Theorem 2.4 (Drinfeld)**.** *Let* ϕ *be a Drinfeld module of rank* r *over* $\mathbb{C}_\infty$. *There exists a lattice* $\Lambda \subset \mathbb{C}_\infty$ *of rank* r *such that* $\phi = \phi^\Lambda$.

Proof. First, one shows that there is a unique power series $e(x) = \sum_{i\geq 0} b_i x^{q^i}$ with $b_0 = 1$, such that $e(Tx) = \phi_T(e(x))$. If $\phi_T(x) = Tx + c_1x^q + \cdots + c_rx^{q^r}$, then the relation $e(Tx) = \phi_T(e(x))$ leads to the equations $b_i(T^{q^i} - T) = \sum_{n=1}^r b_{i-n}^{q^n} c_n$, which can be solved recursively for b_i's. Next, one shows that $e(x)$ is entire. Finally, the set of zeros of an entire function is discrete in $\mathbb{C}_\infty$, so we obtain the lattice Λ as the set of zeros of $e(x)$. □

2.3. **Moduli space.** Once we have the correspondence of Theorem 2.4 between lattices and Drinfeld modules, we can try to classify all Drinfeld modules up to isomorphism.

Exercise 2.5. Let ϕ and ψ be Drinfeld modules over $\mathbb{C}_\infty$ with corresponding lattices Λ_ϕ and Λ_ψ. Show that ϕ is isomorphic to ψ if and only if there is $c \in \mathbb{C}_\infty^\times$ such that $\Lambda_\psi = c\Lambda_\phi$.

Hence classifying Drinfeld modules of rank r over $\mathbb{C}_\infty$ is equivalent to classifying lattices of rank r in $\mathbb{C}_\infty$ up to scaling (i.e. homothety). To construct a lattice in $\mathbb{C}_\infty$ we can start by choosing a vector $(s_1, \dots, s_r) \in \mathbb{C}_\infty^r$ and taking the A-span $As_1 + \cdots + As_r$. Since homothetic lattices are equivalent for our purposes, we can in fact take a point in the projective space $\mathbb{P}^{r-1}(\mathbb{C}_\infty)$. The corresponding A-span will give a lattice if and only if the coordinates of our point are linearly independent over F_∞, equiv. do not lie on an F_∞-rational hyperplane in $\mathbb{P}^{r-1}(\mathbb{C}_\infty)$. Let

$$\Omega^r = \mathbb{P}^{r-1}(\mathbb{C}_\infty) - \bigcup H,$$

where the union is over all F_∞-rational hyperplanes in $\mathbb{P}^{r-1}(\mathbb{C}_\infty)$. Finally, to get rid of the choice of basis of the lattice in our construction, we mod out by the action of $\mathrm{GL}_r(A)$, where the action of $\mathrm{GL}_r(A)$ on Ω^r is induced from its natural (left) action on $\mathbb{C}_\infty^r$. It is easy to see that $\mathrm{GL}_r(A)$ preserves $\Omega^r \subset \mathbb{P}^{r-1}(\mathbb{C}_\infty)$, since it preserves the union of F_∞-rational hyperplanes. Overall, the set of isomorphism classes of rank r Drinfeld modules over $\mathbb{C}_\infty$ is in natural bijection with the set of orbits

$$\mathrm{GL}_r(A) \setminus \Omega^r.$$

Example 2.6. $\Omega^2 = \mathbb{C}_\infty - F_\infty$, so rank 2 Drinfeld modules are classified by

$$\mathrm{GL}_2(A) \setminus (\mathbb{C}_\infty - F_\infty),$$

which is similar to elliptic curves over $\mathbb{C}$ being classified by $\mathrm{GL}_2(\mathbb{Z}) \setminus (\mathbb{C} - \mathbb{R})$. In this case, $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(A)$ acts on $z \in \mathbb{C}_\infty - F_\infty$ as $\gamma z = (az+b)/(cz+d)$.

Ω^r is more than just a set. Nowadays, it is called the *Drinfeld symmetric space*, and, as the name suggests, Ω^r has a structure of a non-archimedean analytic space. To give Ω^r a structure of an analytic space, Drinfeld [Dri74] constructs a map $\lambda : \Omega^r \to \mathcal{B}^r$ to a simplicial complex $\mathcal{B}^r$, called the *Bruhat-Tits building of* $\mathrm{PGL}_r(F_\infty)$. The vertices of $\mathcal{B}^r$ correspond to left cosets of $\mathrm{GL}_r(\mathcal{O}_\infty)F_\infty^\times$ in $\mathrm{GL}_r(F_\infty)$, equiv. the similarity classes of $\mathcal{O}_\infty$-submodules of rank r in F_∞^r; simplices correspond to flags of these submodules. (Here $\mathcal{O}_\infty$ denotes the ring of integers of F_∞.) For example, $\mathcal{B}^2$ turns out to be an infinite tree in which every vertex is adjacent to eactly $q+1$ other vertices. The map λ is $\mathrm{GL}_2(F_\infty)$-equivariant, so the complex $\mathcal{B}^r$ can be used to visualize Ω^r and its quotients.

For a congruence subgroup Γ of $\mathrm{GL}_r(A)$, the quotient $\Gamma \setminus \Omega^r$ classifies isomorphism classes of Drinfeld modules equipped with some "level structure". These quotients are the analogs of various modular curves classifying elliptic curves. Drinfeld proved in [Dri74] that $\Gamma \setminus \Omega^r$ is the analytification of an algebraic affine variety of dimension $r-1$ defined over a finite extension of F. Instead of discussing these modular varieties formally, we give one example:

Example 2.7. Let $M^r(T)$ be the modular variety classifying Drinfeld modules of rank r with level-T structure, where by level-T structure we mean a choice of ordered basis of $\phi[T] := \ker(\phi_T)$, or rather an A-linear isomorphism $\iota : \phi[T] \xrightarrow{\sim} (A/TA)^r = \mathbb{F}_q^r$. Let

$$V_r = \mathbb{P}_F^{r-1} \setminus \bigcup H,$$

where the union is over $\mathbb{F}_q$-rational hyperplanes, i.e., hyperplanes whose equation has coefficients in $\mathbb{F}_q$. Since the number of these hyperplanes in finite, V_r is an affine subvariety of

$\mathbb{P}_F^{r-1}$. A Drinfeld module is uniquely determined by its T-torsion, since

$$\phi_T(x) = Tx \prod_{0 \neq \alpha \in \phi[T]} \left(1 - \frac{x}{\alpha}\right).$$

Now let $(\alpha_1 : \cdots : \alpha_r) \in V_r$, and consider the $\mathbb{F}_q$-vector space $W = \sum \mathbb{F}_q \alpha_i$ spanned by α_i's. Since, by assumption, the α_i's are linearly independent over $\mathbb{F}_q$, we have $\dim_{\mathbb{F}_q} W = r$. Hence

$$\phi_T(x) = Tx \prod_{0 \neq \alpha \in W} \left(1 - \frac{x}{\alpha}\right)$$

defines a Drinfeld module of rank r, and ι is the choice of $(\alpha_1, \dots, \alpha_r)$ as an ordered basis of $\phi[T]$. Note that $(\alpha_1 : \cdots : \alpha_r)$ is only well-defined up to scaling. If we repeat the previous construction with $(\beta\alpha_1 : \cdots : \beta\alpha_r)$ then we get a Drinfeld module ψ such that $\beta^{-1}\psi_T(\beta x) = \phi_T(x)$. Thus, $\beta^{-1}\psi\beta = \phi$, i.e., these Drinfeld modules are isomorphic, and the corresponding level structures are related by $\beta\iota = \iota'$. Hence $(\phi, \iota) \cong (\psi, \iota')$. This construction gives a well-defined morphism $V_r \to M^r(T)$ over F, which is not hard to show to be an isomorphism; see [Pin13, p. 358]. For $r = 2$, we get $M^2(T) = \mathbb{P}_F^1 - \mathbb{P}_F^1(\mathbb{F}_q)$.

The congruence group in this example is

$$\Gamma(T) = \{\gamma \in \mathrm{GL}_r(A) \mid \gamma \equiv 1 \bmod T\}$$

and $M^r(T) \otimes \mathbb{C}_\infty \cong \Gamma(T) \setminus \Omega^r$.

Definition 2.8. Another important example of a congruence group is

$$\Gamma_0(\mathfrak{n}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(A) \Big| \; c \equiv 0 \pmod{\mathfrak{n}} \right\}.$$

where $\mathfrak{n} \in A$ be a non-zero monic polynomial. The curve $Y_0(\mathfrak{n}) = \Gamma_0(\mathfrak{n}) \setminus \Omega^2$ is an affine curve defined over F. The unique smooth projective connected curve that contains $Y_0(\mathfrak{n})$ is usually denoted $X_0(\mathfrak{n})$. There are known formulas for genus of $X_0(\mathfrak{n})$ and the number of "cusps" $X_0(\mathfrak{n}) - Y_0(\mathfrak{n})$. For example, if $\mathfrak{n} = \mathfrak{p}$ is irreducible, then genus of $X_0(\mathfrak{p})$ is $(q^{\deg \mathfrak{p}} - q^2)/(q^2 - 1)$ or $(q^{\deg \mathfrak{p}} - q)/(q^2 - 1)$ depending on whether $\deg \mathfrak{p}$ is even or odd, and there are exactly two cusps (just as for classical modular curves $X_0(p)$).

Exercise 2.9. Over an algebraically closed field K, the Carlitz module $\rho_T = \gamma(T) + \tau$, up to isomorphism, is the unique Drinfeld module of rank 1.

Exercise 2.10. For a Drinfeld module ϕ of rank 2 over K given by $\phi_T = \gamma(T) + g\tau + \Delta\tau^2$, define $j(\phi) = g^{q+1}/\Delta$. This is the analog of the j-invariant of elliptic curves.

(i) Prove that for any $j \in K$, there is some ϕ over K with $j(\phi) = j$.
(ii) Prove that two rank-2 Drinfeld modules ϕ and ψ are isomorphic over the algebraic closure K^{alg} of K if and only if $j(\phi) = j(\psi)$.

Exercise 2.11. The previous exercise implies that $\mathrm{GL}_2(A) \setminus \Omega^2$ is the affine line $\mathbb{A}^1_{\mathbb{C}_\infty}$. Show by direct elementary calculations that $\mathrm{GL}_2(A) \setminus \mathcal{B}^2$ is an infinite half-line (or see [Ser03, p. 111])

These two facts are intimately related.

Remark 2.12. Drinfeld was not aware of Carlitz's work at the time of writing [Dri74]. Although Carlitz's paper [Car35] provides a beautiful analog of Euler's formula for $\zeta(2m)$, and the module ρ gives the correct analog of cyclotomic polynomials over F (as Carlitz shows in [Car38]), these papers did not receive the attention they deserved from the larger mathematical community. This partly might be the result of Carlitz's over-productive output[1], the bland titles of his papers[2], and the relative lack of good exposition/motivation. It seems [Car35], [Car38] were mostly forgotten[3] until 1974, when Hayes [Hay74] (who was a student of Carlitz) used ρ to give an explicit description of the maximal abelian extension of F, similar to the Kronecker-Weber theorem for $\mathbb{Q}$. Note that ρ is similar to a Lubin-Tate formal group law, which can be used to explicitly construct the totally ramified abelian extensions of a local field, although [Car35], [Car38] precede the work of Lubin and Tate by about 30 years.

Remark 2.13. For a more in-depth discussion of the results in Drinfeld's paper [Dri74] the survey article by Deligne and Husemöller [DH87] and the expository articles in the conference proceedings [GvdPRVG97] are highly recommended .

3. Endomorphisms and Galois representations

3.1. **Endomorphism rings of Drinfeld modules.** By definition of morphisms between Drinfeld modules, the ring of endomorphisms $\mathrm{End}(\phi)$ of a Drinfeld module ϕ over K is the centralizer of $\phi(A)$ in $K\{\tau\}$:

$$\mathrm{End}(\phi) = \{u \in K\{\tau\} \mid u\phi_a = \phi_a u \text{ for all } a \in A\}.$$

Obviously $\phi(A) \subset \mathrm{End}(\phi)$, so $\mathrm{End}(\phi)$ is an A-algebra, where A acts via $\phi(A)$: for $a \in A$, $u \in \mathrm{End}(\phi)$, we have $a \circ u := \phi_a u = u\phi_a$. (Keep in mind that $\phi : A \to K\{\tau\}$ is always injective, even if $\gamma : A \to K$ is not injective, so $A \cong \phi(A)$.) Because $\mathrm{End}(\phi)$ is torsion free as an A-module, we have embeddings

$$\mathrm{End}(\phi) \hookrightarrow \mathrm{End}(\phi) \otimes_A F \hookrightarrow \mathrm{End}(\phi) \otimes_A F_\infty.$$

Theorem 3.1 (Drinfeld)**.** *Let ϕ be a Drinfeld module of rank r over K.*

- (i) $\mathrm{End}(\phi)$ *is a free A-module of rank $\leq r^2$.*
- (ii) *If the A-characteristic of K is* 0 *(i.e.,* $\ker(\gamma) = 0$*), then* $\mathrm{End}(\phi)$ *is a commutative ring and its rank as an A-module is $\leq r$.*
- (iii) $\mathrm{End}(\phi) \otimes_A F_\infty$ *is a division ring.*
- (iv) *If $K = \mathbb{C}_\infty$ and Λ_ϕ is the lattice uniformizing ϕ, then* $\mathrm{End}(\phi) \cong \{c \in \mathbb{C}_\infty \mid c\Lambda_\phi \subset \Lambda_\phi\}$.

[1]More than 750 papers listed in MathSciNet.

[2]The titles of [Car35] and [Car38] are examples of this.

[3]According to MathSciNet, [Car35] and [Car38] are not cited anywhere before 1974, except in a paper by Carlitz himself in 1952.

Proof. (i) can be proved by an argument similar to the proof of the corresponding statement for elliptic curves: endomorphisms of ϕ act faithfully on the Tate modules of ϕ, from which one obtains an embedding $\mathrm{End}(\phi) \hookrightarrow M_r(A_\mathfrak{p})$ for any prime ideal $\mathfrak{p} \lhd A$ not equal to the A-characteristic of K. The problem then reduces to showing that the previous map remains injective after we tensor $\mathrm{End}(\phi)$ with $A_\mathfrak{p}$; cf. Theorem III.7.4 in [Sil86]. (ii) can be reduced to (iv) by a "Lefschetz principle" type argument. (iii) follows from observing that the map $\delta : \mathrm{End}(\phi) \to \mathbb{Z}$, $u \mapsto \deg_\tau(u)$, extends to a norm on the finite dimensional F_∞-algebra $\mathrm{End}(\phi) \otimes_A F_\infty$ such that $\delta(su) = |s|^r\delta(u)$ for all $s \in F_\infty$ and $u \in \mathrm{End}(\phi)$. (iv) The map

$$\mathrm{End}(\phi) \to \{c \in \mathbb{C}_\infty \mid c\Lambda_\phi \subset \Lambda_\phi\}$$

is given by $u \mapsto \partial u$. If $\partial u = 0$, then $u = s_m\tau^m + s_{m+1}\tau^{m+1} + \cdots$, $m \geq 1$, $s_m \neq 0$. On the other hand, $u\phi_T = \phi_T u$ gives $s_m\gamma(T)^{q^m} = \gamma(T)s_m$. Hence $\gamma(T)^{q^m-1} = 1$, which leads to a contradiction if $\ker(\gamma) = 0$. Hence the above map is injective. The bijectivity requires a little bit of diagram chasing; cf. [Gek83, Prop. 2.4] or [Dri74, §3]. □

The previous theorem implies that if K has A-characteristic 0, then $\mathrm{End}(\phi) \otimes_A F$ is a field extension of F of degree $\leq r$ in which ∞ does not split. Field extensions of F of finite degree in which ∞ does not split are called *imaginary* extensions, in analogy with imaginary extensions of $\mathbb{Q}$.

Exercise 3.2. Assume the characteristic of F is odd. Let K be a quadratic extension of F. Show that K is the splitting field of $x^2 = a$ for some $a \in A$, and describe those a for which the extension K/F is imaginary.

Example 3.3. Let $\phi_T = T + (h + h^q)\tau + \tau^2 = (h + \tau)(h + \tau)$ where $h^2 = T$. Consider ϕ over $K = F(h)$. It is clear from the construction that $\tilde{h} := (h + \tau) \in \mathrm{End}(\phi)$, and $\tilde{h}^2 = \phi_T$. Hence $A[\sqrt{T}] \subset \mathrm{End}(\phi)$, and since $A[\sqrt{T}]$ is the maximal A-order in K, we conclude from Theorem 3.1 that $A[\sqrt{T}] = \mathrm{End}(\phi)$, so ϕ has "complex multiplication". Also note that

$$j := j(\phi) = h^{q+1}(1 + h^{q-1})^{q+1} = T^{\frac{q+1}{2}}(1 + T^{\frac{q-1}{2}})^{q+1}.$$

If q is odd, then $j \in F$. The Drinfeld module $\psi_T = T + j\tau + j^q\tau^2$ is defined over F. Since $j(\psi) = j(\phi)$, we have $\psi \cong \phi$ over K, so ϕ can be defined over F. This is the analog of the fact that an elliptic curve with CM by the ring of integers of a quadratic imaginary extension with class number one can be defined over $\mathbb{Q}$.

In general, if ϕ has CM by the integral closure of A in a quadratic imaginary extension K of F, then j is integral over A and $K(j)$ is the Hilbert class field of K (the maximal abelian extension of K in which the place of K over ∞ splits completely); see [Gek83, §3]. In particular, ϕ cannot be defined over K itself, but only over its Hilbert class field. Again, these facts are the analogs of classical results from the theory of CM elliptic curves.

There is a trick which can be used in many problems dealing with CM Drinfeld modules to reduce the problem to a (simpler) question about rank-1 Drinfeld modules: If ϕ is a Drinfeld A-module of rank r over K with CM by the ring of integers $\mathcal{O}_L$ of an imaginary extension L/F of degree r, then there is a Drinfeld $\mathcal{O}_L$-module ψ of rank 1 over K whose restriction to

A is ϕ. The module ψ is the embedding $\mathcal{O}_L \to K\{\tau\}$ we get from the CM action. That ψ is a Drinfeld module and the claim about its rank are easy to check. The reader will notice that this trick is used in the previous example, where ψ is defined by $\psi_h = h + \tau$.

Remark 3.4. Drinfeld modules with CM give "Heegner cycles" on Drinfeld modular varieties. An analog of the Gross-Zagier formula with Heegner points on Drinfeld modular curves $X_0(\mathfrak{n})$, $\mathfrak{n} \lhd A$, is proved in [RT00] assuming q is odd. The proof in [RT00] closely follows the strategy in the original paper by Gross and Zagier. (For a very detailed calculation of Heegner points and a verification of Gross-Zagier formula for $X_0(T^3)$ over $\mathbb{F}_2(T)$, see my write-up [Pap00] of an Arizona Winter School project from 2000.) Quite recently, Yun and Zhang [YZ17] proved a vast generalization of the Gross-Zagier formula over function fields (so far, only for everywhere unramified cuspidal automorphic representation of PGL_2). Their proof is very different from the proof of Gross and Zagier.

Example 3.5. Let $K = A/(T) \cong \mathbb{F}_q$, so $\ker(\gamma) = (T)$. Let $\phi_T = \gamma(T) + \tau^2 = \tau^2$. Clearly, $\tau \in \mathrm{End}_K(\phi)$. Since $\tau^2 = \phi_T$, we see that $A[\sqrt{T}] \subset \mathrm{End}_K(\phi)$. In fact, it is not hard to show that $A[\sqrt{T}] = \mathrm{End}_K(\phi)$. Now consider ϕ as a Drinfeld module over the quadratic extension $L = \mathbb{F}_{q^2}$ of K. Then $\mathbb{F}_{q^2} \subset \mathrm{End}_L(\phi)$, since for any $\alpha \in \mathbb{F}_{q^2}$ we have $\tau^2\alpha = \alpha^{q^2}\tau^2 = \alpha\tau^2$. Hence $\mathbb{F}_{q^2}\{\tau\} \subset \mathrm{End}_L(\phi)$. Thus, $\mathrm{End}_L(\phi)$ is strictly larger than $\mathrm{End}_K(\phi)$. Moreover, $\mathbb{F}_{q^2}\{\tau\}$ is not commutative since $\tau\alpha = \alpha^q\tau$ and $\alpha^q = \alpha$ if and only if $\alpha \in \mathbb{F}_q$. In fact, $\mathbb{F}_{q^2}\{\tau\}$ is isomorphic to a maximal A-order in a quaternion division algebra over F ramified at T and ∞. To see that there is a quaternion algebra lurking around, assume for simplicity that q is odd. Fix a non-square α in $\mathbb{F}_q^\times$ and let $j \in \mathbb{F}_{q^2}$ be such that $j^2 = \alpha$. Then $\mathbb{F}_{q^2} = \mathbb{F}_q(j)$, and $\tau j = j^q\tau = -j\tau$. If we denote $i = \tau$, then $i^2 = \phi_T$, and we see that $\mathbb{F}_{q^2}\{\tau\} \cong A[i,j]$ where $i^2 = T$, $j^2 = \alpha$, $ij = -ji$. Finally note that $\phi[T] := \ker(\phi_T)$ is the set of zeros of x^{q^2}, so $\phi[T]$ is a connected group-scheme. Thus, ϕ is the analog of supersingular elliptic curve over $\overline{\mathbb{F}}_p$.

More generally, let $\mathfrak{p}$ be a monic irreducible polynomial in A. Let $\mathbb{F}_\mathfrak{p} := A/(\mathfrak{p})$. Let ϕ be a rank r Drinfeld module over $\overline{\mathbb{F}}_\mathfrak{p}$. Then the following are equivalent (see [Gek91]):

- Some power of τ lies in $\phi(A)$.
- $\mathrm{End}(\phi)$ is a maximal A-order in the central division r^2-dimensional algebra over F ramified only at $\mathfrak{p}$ and ∞, with invariants $1/r$ and $-1/r$, respectively.
- $\phi[\mathfrak{p}]$ is connected.

There is also an analog of Honda-Tate theory in this setting; see [Dri77b] or [Yu95].

Exercise 3.6. Let ϕ be a Drinfeld module over $\overline{\mathbb{F}}_\mathfrak{p}$ of rank 2 with $j(\phi) = 0$. Prove that ϕ is supersingular if and only if $\deg(\mathfrak{p})$ is odd. (See [Gek83, §5] for the solution.)

Exercise 3.7. Let ϕ be a Drinfeld module over $K = K^{\mathrm{alg}}$ of rank r. Prove that $\mathrm{End}(\phi)^\times =: \mathrm{Aut}(\phi) \cong \mathbb{F}_{q^s}^\times$ for some s dividing r.

3.2. **Galois representations arising from Drinfeld modules.** Let $\mathfrak{a} \lhd A$ be an ideal of A; by abuse of notation, we denote by $\mathfrak{a}$ also the monic polynomial that generates this ideal. Let ϕ be a Drinfeld module of rank r over K. Let

$$\phi[\mathfrak{a}] = \ker(\phi_\mathfrak{a}) = \{\alpha \in K^{\mathrm{alg}} \mid \phi_\mathfrak{a}(\alpha) = 0\}.$$

It is clear that $\phi[\mathfrak{a}]$ is a finite A-module (via $\phi(A)$). If $\ker(\gamma) \nmid \mathfrak{a}$, then

$$\phi_{\mathfrak{a}}(x) = \gamma(\mathfrak{a})x + c_1 x^q + \cdots + c_n x^{q^n}, \quad n = r \cdot \log_q \#(A/\mathfrak{a}),$$

is a separable polynomial since $\phi_{\mathfrak{a}}(x)' = \gamma(\mathfrak{a}) \neq 0$. Thus, $\#\phi[\mathfrak{a}] = \#(A/\mathfrak{a})^r$. It is not hard to show that as an A-module we have

$$\phi[\mathfrak{a}] \cong (A/\mathfrak{a})^{\oplus r}.$$

Since $\phi_{\mathfrak{a}}(x)$ is separable, $\phi[\mathfrak{a}]$ is equipped with an action of $G_K := \mathrm{Gal}(K^{\mathrm{sep}}/K)$, which commutes with the action of A. Hence we get a continuous representation

$$\pi_{\mathfrak{a}} : G_K \to \mathrm{GL}_r(A/\mathfrak{a}).$$

Example 3.8. Consider the Carlitz module $\rho_T = T + \tau$ over F. Then for any $\mathfrak{a}$

$$\mathrm{Gal}(F(\phi[\mathfrak{a}])/F) \cong (A/\mathfrak{a})^\times,$$

and $F(\phi[\mathfrak{a}])/F$ is ramified only at the primes dividing $\mathfrak{a}$ and at ∞. More precisely, if $\mathfrak{a} = \mathfrak{p}^s$ is a power of prime, then $F(\phi[\mathfrak{p}^s])/F$ is totally ramified at $\mathfrak{p}$ and is unramified at all other primes of A. The place ∞ splits into $\#(A/\mathfrak{p}^s)^\times/(q-1)$ primes in $F(\phi[\mathfrak{p}^s])$, and the ramification index at each of these primes is $q-1$. These facts are proved in Hayes' paper [Hay74]. Hence the Carlitz module gives the correct analog of cyclotomic theory over $\mathbb{Q}$.

Exercise 3.9. What can you say about the image of $\pi_{\mathfrak{a}}$ if ϕ has CM?

Theorem 3.10 (Boston, Ose). *Let K be an A-field of infinite order. Let*

$$\pi : G_K \to \mathrm{GL}_r(\mathbb{F}_q)$$

be a continuous representation. There is a Drinfeld module of rank r over K such that π is equivalent to π_T, i.e., π arises from the action of G_K on $\phi[T]$.

Proof. See Theorem 6.1 in [BO00]. □

The previous theorem is somewhat analogous to the fact that continuous representations of $G_{\mathbb{Q}}$ into $\mathrm{GL}_2(\mathbb{F}_3)$ or $\mathrm{GL}_2(\mathbb{F}_5)$, whose determinant is the cyclotomic character, arise from 3, resp. 5-torsion, of an elliptic curve over $\mathbb{Q}$; see [SBT97]. Somewhat surprisingly, the proof of the corresponding fact for Drinfeld modules in [BO00] is comparatively elementary, and does not involve any algebraic geometry, although from the geometric perspective it is related to the fact that a compactification of $M^r(T)_K$ is $\mathbb{P}^{r-1}_K$, which has (infinitely many) K-rational "non-cuspidal" points.

Exercise 3.11. Prove that the image of π_T arising from the given Drinfeld module over F is isomorphic to the indicated subgroup of $\mathrm{GL}_2(\mathbb{F}_q)$:

- $\phi_T = T + \tau + \tau^2$ gives $\mathrm{GL}_2(\mathbb{F}_q)$.
- $\phi_T = T + \tau + T^q\tau^2$ gives $\mathrm{SL}_2(\mathbb{F}_q)$.
- $\phi_T = T + \tau + (1-T)\tau^2$ gives the Borel subgroup of upper triangular matrices.
- $\phi_T = T + \tau^2$ gives the normalizer of split Cartan subgroup.

Exercise 3.12. Let ϕ be a Drinfeld module of rank r over K. Note that we can write $\phi_T(x) = xf(x^{q-1})$, where $f(x) \in K[x]$ is a polynomial of degree $(q^r - 1)/(q - 1)$. Show that the splitting field of $f(x)$ is the subfield of $K(\phi[T])$ fixed by $\pi_T(G_K) \cap Z(\mathbb{F}_q)$, where $Z(\mathbb{F}_q)$ denotes the center of $\mathrm{GL}_r(\mathbb{F}_q)$. Hence the Galois group of f is a subgroup of $\mathrm{PGL}_r(\mathbb{F}_q)$.

Remark 3.13. One can use Drinfeld modules to prove reciprocity results for non-solvable extensions of F, i.e., statements about the set of primes that split completely in such extensions; see [CP15].

Let $\mathfrak{p} \lhd A$ be a prime distinct from the A-characteristic of K. Let ϕ be Drinfeld module of rank r over K. Define the *Tate module* of ϕ

$$\mathrm{Ta}_\mathfrak{p}(\phi) = \varprojlim \phi[\mathfrak{p}^n] \cong A_\mathfrak{p}^{\oplus r},$$

where $A_\mathfrak{p}$ is the completion of A at $\mathfrak{p}$.

Theorem 3.14. *Let K be a finite extension of F or $A/\mathfrak{q}$. Let ϕ and ψ be Drinfeld modules over K. Then*

$$\mathrm{End}_K(\phi, \psi) \otimes_A A_\mathfrak{p} \xrightarrow{\sim} \mathrm{Hom}_{A_\mathfrak{p}[G_K]}(\mathrm{Ta}_\mathfrak{p}(\phi), \mathrm{Ta}_\mathfrak{p}(\psi)).$$

Proof. When K is a finite extension of $A/\mathfrak{q}$, this theorem is due to Drinfeld [Dri77b], and can be proved by an argument similar to Tate's argument for abelian varieties over finite fields; see [Yu95, Thm. 2]. When K is a finite extension of F, this theorem is due to Taguchi and Tamagawa; see [Tag95]. The proof is quite different from Faltings proof of the corresponding statement for abelian varieties over number fields, and relies on an idea of Anderson related to t-motives. □

The Tate module $\mathrm{Ta}_\mathfrak{p}(\phi)$ gives rise to a representation $G_K \to \mathrm{GL}_r(A_\mathfrak{p})$. An analog of the Mumford-Tate conjecture for abelian varieties and a generalization of Serre's Open Image Theorem for elliptic curves are known in this context:

Theorem 3.15. *Assume K is a finitely generated field. Let $\mathfrak{p}_0$ be the A-characteristic of K and ϕ be a Drinfeld module of rank r over K such that $\mathrm{End}_{K^{\mathrm{alg}}}(\phi) \cong A$. Then the image of the adelic representation*

$$G_K \to \prod_{\mathfrak{p}_0 \neq \mathfrak{p} \lhd A} \mathrm{GL}_r(A_\mathfrak{p})$$

arising from ϕ is open.

Proof. This was proved by Pink and his students; see [PR09], [DP12]. □

Exercise 3.16. Let ϕ be a Drinfeld module of rank r over F. Define a reasonable notion of ϕ having "good reduction" at a prime $\mathfrak{q} \lhd A$. Your definition should be such that the following theorem is true: ϕ has good reduction at $\mathfrak{q}$ if and only if $\mathrm{Ta}_\mathfrak{p}(\phi)$ is unramified at $\mathfrak{q}$ for some $\mathfrak{p} \neq \mathfrak{q}$; see [Gos96, §4.10] for the proof. This is the analog of Ogg-Néron-Shafarevich criterion for abelian varieties.

Note that there are Drinfeld modules over F with good reduction at every prime of A, e.g., ϕ defined by $\phi_T = T + T\tau + \tau^r$ is such a Drinfeld module. This gives an interesting geometric construction: the splitting field of $f(x) = T + Tx + x^{(q^r-1)/(q-1)}$ corresponds to a Galois covering of $\mathbb{P}^1_{\mathbb{F}_q}$ with Galois group $\mathrm{PGL}_r(\mathbb{F}_q)$, which is unramified everywhere except at 0 and ∞. These type of extensions were extensively studied by Abhyankar, who for awhile was not aware of the connection of this problem with the theory of Drinfeld modules; cf. [Abh01].

Not to give the wrong impression that analogs of all theorems for elliptic curves are known for Drinfeld modules, here are two problems, still *open*, whose analogs for elliptic curves over $\mathbb{Q}$ are famous theorems:

(1) Let ϕ be a Drinfeld module of rank r over F. Denote by $({}^\phi F)_{\mathrm{tor}}$ all elements of F which are torsion for $\phi(A)$. Then the order $\#({}^\phi F)_{\mathrm{tor}}$ should be uniformly bounded in terms of r, i.e. does not depend on ϕ. For $r = 2$, there is a more precise conjecture by Schweizer [Sch03]:

$$({}^\phi F)_{\mathrm{tor}} \cong A/\mathfrak{m} \oplus A/\mathfrak{n}, \quad \text{where} \quad \mathfrak{m} \mid \mathfrak{n}, \quad \deg(\mathfrak{m}) + \deg(\mathfrak{n}) \leq 2.$$

This is the analog of Mazur's theorem classifying rational torsion of elliptic curves.

(2) It is not known, even conjecturally, if the Galois representations arising from Tate's modules of Drinfeld modules of rank ≥ 2 over F are related to modular forms in any way; cf. [Gos02]. For elliptic curves over $\mathbb{Q}$ such relation is a consequence of the modularity theorem of Wiles and others.

4. Generalizations of Drinfeld modules

4.1. **Anderson modules.** Drinfeld modules are one-dimensional objects in the sense that their underlying group-scheme is $\mathbb{G}_{a,K}$. This suggests an obvious generalization, where we replace $\mathbb{G}_{a,K}$ by $\mathbb{G}_{a,K}^d$. Let K be an A-field, $\gamma : A \to K$. It is not hard to show that

$$\mathrm{End}_{\mathbb{F}_q}(\mathbb{G}_{a,K}^d) \cong M_d(K\{\tau\}),$$

where $M_d(K\{\tau\})$ is the ring of $d \times d$ matrices with entries in $K\{\tau\}$. Note that we can write any element of $M_d(K\{\tau\})$ as a polynomial $\sum_{i=0}^n S_i\tau^i$ for some $n \geq 0$, where $S_0, \dots, S_n \in M_d(K)$, $S_n \neq 0$, and τ^i denotes the scalar matrix $\tau^i I_d$. An element $S = \sum_{i=0}^n S_i\tau^i$ acts on the tangent space of $\mathbb{G}_{a,K}^d$ via S_0. Let $\partial : M_d(K\{\tau\}) \to M_d(K)$ be the homomorphism which maps S to S_0.

Anderson module [And86] (also called *abelian t-module*) is an embedding

$$\phi : A \to M_d(K\{\tau\})$$

such that $\partial(\phi_T) = \gamma(T)I_d + N$, where N is a nilpotent matrix, and $\ker(\phi_T)$ is a non-trivial finite group scheme. The *dimension* of ϕ is d, and its *rank* is $\log_q \# \ker(\phi_T)$. Hence Drinfeld modules are Anderson modules of dimension 1.

Remark 4.1. It is clear that $\mathrm{GL}_d(K) \subset \mathrm{GL}_d(K\{\tau\})$, but this latter group is strictly larger. For example, it contains all upper-triangular unipotent matrices in $M_d(K\{\tau\})$. In particular, $\phi_T = \begin{pmatrix} \gamma(T) & \tau \\ 0 & \gamma(T) \end{pmatrix}$ is not an Anderson module.

Example 4.2. The reason for allowing the presence of a nilpotent matrix N, instead of just insisting that $\partial(\phi_T) = \gamma(T)I_d$, is that some natural constructions in the category of Anderson modules lead to Anderson modules with non-zero N. The "d-th tensor power of the Carlitz module" mentioned in Remark 1.4 is the following Anderson module of dimension d and rank 1:

$$\phi_T = (\gamma(T)I_d + N_d) + V_d\tau,$$

where

$$N_d = \begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix}, \qquad V_d = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & & & \vdots \\ 1 & 0 & \cdots & 0 \end{pmatrix}.$$

The endomorphism ring of a d-dimensional Anderson module ϕ, $\mathrm{End}(\phi)$, is the centralizer of $\phi(A)$ in $M_d(K\{\tau\})$. Anderson proved in [And86] that if ϕ has rank r then $\mathrm{End}(\phi)$ is a free A-module of rank $\leq r^2$, which is a generalization of Theorem 3.1 (i).

Next, we would like to classify Anderson modules by relating them to simpler objects, similar to the bijection between lattices in $\mathbb{C}_\infty$ and Drinfeld modules. In fact, in [And86], Anderson tries to follow the strategy of the proof of Theorem 2.4. Given a d-dimensional Anderson module ϕ over $\mathbb{C}_\infty$, one formally looks for an exponential function $e_\phi : \mathbb{C}_\infty^d \to \mathbb{C}_\infty^d$ of the form

$$e_\phi\left(\begin{pmatrix} x_1 \\ \vdots \\ x_d \end{pmatrix}\right) = \begin{pmatrix} x_1 \\ \vdots \\ x_d \end{pmatrix} + \sum_{i=i}^{\infty} S_i \begin{pmatrix} x_1^{q^i} \\ \vdots \\ x_d^{q^i} \end{pmatrix},$$

where $S_i \in M_d(\mathbb{C}_\infty)$ $(i \geq 1)$, such that

$$e_\phi\left(T \begin{pmatrix} x_1 \\ \vdots \\ x_d \end{pmatrix}\right) = \phi_T\left(e_\phi\left(\begin{pmatrix} x_1 \\ \vdots \\ x_d \end{pmatrix}\right)\right).$$

As in the case of Drinfeld modules, this equation leads to recursive formulas for $S_1, S_2, \dots$ which are possible to solve uniquely. Moreover, e_ϕ turns out to be entire on $\mathbb{C}_\infty^d$ (i.e., everywhere convergent), and the kernel of e_ϕ to be a discrete A-module in $\mathbb{C}_\infty^d$. However, as Anderson discovered, as soon as $d > 1$ a fundamental problem arises in that there exist Anderson modules for which e_ϕ is *not* surjective. In fact, Anderson proved that e_ϕ is surjective if and only if the A-rank of $\ker(e_\phi)$ is equal to the rank of ϕ, but also gave an example of ϕ for which e_ϕ is injective, so its kernel is trivial; see [And86]. We say that ϕ is *uniformizable* if e_ϕ is surjective.

Because not every Anderson module is uniformizable, their moduli spaces cannot be represented as quotients of Drinfeld symmetric domains. On the other hand, pieces of their moduli spaces can be parametrized using analogs of Rapoport-Zink spaces, introduced into function field arithmetic by Hartl, and Genestier and V. Lafforgue. The interested reader should consult the papers of Hartl, for example, [Har05], [Har11].

4.2. **Drinfeld-Stuhler modules.** Another generalization of Drinfeld modules arises when one equips $\mathbb{G}_{a,K}^d$ with an action of not just A but of an A-order in a division algebra. These objects are analogs in Drinfeld's theory of abelian surfaces equipped with an action of an order in an indefinite quaternion algebra over $\mathbb{Q}$.

Let D be a central division algebra over F of dimension d^2. Assume $D \otimes_F F_\infty \cong M_d(F_\infty)$. Fix a maximal A-order O_D in D. (All such orders are conjugate to each other in D.)

A *Drinfeld-Stuhler O_D-module* defined over K is an embedding

$$\phi : O_D \to M_d(K\{\tau\}), \quad b \mapsto \phi_b,$$

such that the composition

$$A \to O_D \xrightarrow{\phi} M_d(K\{\tau\}) \xrightarrow{\partial} M_d(K)$$

maps $a \in A$ to $\gamma(a)I_d$, and for any non-zero $b \in O_D$ the kernel of the endomorphism ϕ_b of $\mathbb{G}_{a,K}^d$ is a finite group scheme over K of order $\#(O_D/O_D \cdot b)$.

A Drinfeld-Stuhler module can also be considered as an Anderson module of dimension d, rank d^2, with O_D in its ring of endomorphisms. When $d = 1$, a Drinfeld-Stuhler module is simply a Carlitz module.

Exercise 4.3. Let ϕ be a Drinfeld-Stuhler O_D-module defined over K. Assume A-characteristic of K is 0. Prove that $D \otimes_F K \cong M_d(K)$, i.e., K splits D. This implies that ϕ cannot be defined over F itself. (Hint: Consider the map $\partial \circ \phi : O_D \to M_d(K)$.)

Example 4.4. As a consequence of the Grunwald-Wang theorem, every central simple F-algebra is cyclic. We first give a specific example of a cyclic algebra.

Let $\mathbb{F}_{q^d}$ denote the degree d extension of $\mathbb{F}_q$. Let $F' = \mathbb{F}_{q^d}(T)$, $A' = \mathbb{F}_{q^d}[T]$. The Galois group $\mathrm{Gal}(F'/F) \cong \mathrm{Gal}(\mathbb{F}_{q^d}/\mathbb{F}_q)$ has a canonical generator σ given by the Frobenius automorphism (i.e., σ induces the qth power morphism on $\mathbb{F}_{q^d}$). Let $\mathfrak{r} \in A$ be a monic square-free polynomial with prime decomposition $\mathfrak{r} = \mathfrak{p}_1 \cdots \mathfrak{p}_m$. Assume the degree of each prime $\mathfrak{p}_i$ is coprime to d. Let D be the cyclic algebra

$$D = \bigoplus_{i=0}^{d-1} F' z^i, \quad z^d = \mathfrak{r}, \quad zy = \sigma(y)z, \quad y \in F'.$$

D is a division algebra, as can be seen by computing its invariants:

$$\mathrm{inv}_\mathfrak{p}(D) = \frac{\mathrm{ord}_\mathfrak{p}(\mathfrak{r})\deg(\mathfrak{p})}{d} \in \mathbb{Q}/\mathbb{Z}, \quad \mathfrak{p} \lhd A \text{ is prime}.$$

Since the sum of the invariants of D over all places of F must be 0, if we assume that $\sum_{i=1}^m \deg(\mathfrak{p}_i)$ is divisible by d, then D will be split at ∞ and will ramify only at the primes of A dividing $\mathfrak{r}$. The order

$$O_D = \bigoplus_{i=0}^{d-1} A' z^i$$

is maximal in D, since its discriminant is equal to $\mathfrak{r}^{d(d-1)}$.

Let K be an A'-field $\gamma : A' \to K$. Let $\varphi : A' \to K\{\tau\}$ be defined by $\varphi_T = \gamma(T) + \tau^d$; this is a rank-1 Drinfeld A'-module and a rank-d Drinfeld A-module. Then

$$\phi : O_D \to M_d(K\{\tau\})$$

given by

$$\begin{aligned}
\phi_T &= \mathrm{diag}(\varphi_T, \dots, \varphi_T),\\
\phi_h &= \mathrm{diag}(h, h^q, \dots, h^{q^{d-1}}), \quad h \in \mathbb{F}_{q^d},\\
\phi_z &= \begin{pmatrix} 0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 1 & \cdots & 0\\ & & & \ddots & \\ 0 & 0 & 0 & \cdots & 1\\ \varphi_{\mathfrak{r}} & 0 & 0 & \cdots & 0 \end{pmatrix},
\end{aligned}$$

is a Drinfeld-Stuhler module.

The endomorphism ring $\mathrm{End}(\phi)$ of a Drinfeld-Stuhler module ϕ is defined to be

$$\mathrm{End}(\phi) = \{u \in M_d(K\{\tau\}) \mid u\phi_b = \phi_b u \text{ for all } b \in O_D\}.$$

Obviously, $\phi(A)$ is in the center of $\mathrm{End}(\phi)$, so $\mathrm{End}(\phi)$ is an A-module. In [Pap17], it is proved that:

- $\mathrm{End}(\phi)$ is a free A-module of rank $\le d^2$.
- $\mathrm{End}(\phi) \otimes_A F_\infty$ is isomorphic to a subalgebra of the central division algebra over F_∞ with invariant $-1/d$.
- If K has A-characteristic 0, then $\mathrm{End}(\phi)$ is an A-order in an imaginary field extension of F which embeds into D. In particular, $\mathrm{End}(\phi)$ is commutative and its rank over A divides d.

Exercise 4.5. Assume K has A-characteristic 0, and ϕ is the Drinfeld-Stuhler module from Example 4.4. Show that $\mathrm{End}(\phi) \cong A'$. (Hint: Note that $\mathrm{diag}(h, h, \dots, h) \in \mathrm{End}(\phi)$ for all $h \in \mathbb{F}_{q^d}$.)

Exercise 4.6. Assume ϕ is the Drinfeld-Stuhler module from Example 4.4 and $K = A'/(T) \cong \mathbb{F}_{q^d}$, so K has A-characteristic T. Show that $\kappa = \phi_z \tau^{d-1} \in \mathrm{End}(\phi)$ and $A' \subset \mathrm{End}(\phi)$. Show that $\mathrm{End}(\phi)\otimes F = F'[\kappa]$ is the central division algebra $\overline{D}$ over F with invariants

$$\mathrm{inv}_v(\overline{D}) = \begin{cases} 1/d & \text{if } v=(T),\\ -1/d & \text{if } v=\infty,\\ -\mathrm{inv}_v(D) & \text{otherwise.}\end{cases}$$

Hence $\mathrm{End}(\phi)$ is an order in $\overline{D}$. (See Example 5.2 in [Pap17] for the solution.) Note that $\ker\phi_T$ is connected, so ϕ is "supersingular".

Drinfeld-Stuhler modules are uniformizable (as Anderson modules). One can use this to show that the set of isomorphism classes of Drinfeld-Stuhler modules over $\mathbb{C}_\infty$ is in natural bijection with the set of orbits

$$O_D^\times \setminus \Omega^d,$$

where $O_D^\times$ acts on Ω^d through the embedding $O_D^\times \hookrightarrow (D\otimes F_\infty)^\times \cong \mathrm{GL}_d(F_\infty)$. Unlike the case of Drinfeld modular varieties $\mathrm{GL}_d(A)\setminus\Omega^r$, which are affine, the quotient $O_D^\times\setminus\Omega^r$ is projective.

Realizing an idea of U. Stuhler, the shtuka version of Drinfeld-Stuhler modules was introduced by Laumon, Rapoport and Stuhler in [LRS93] under the name of *$\mathscr{D}$-elliptic sheaves*, where their modular varieties play an important role in the proof of the Langlands correspondence for GL_d over local fields of positive characteristic. The more elementary module version of these objects, as given above, is introduced and studied in [Pap17].

Besides applications to the Langlands conjectures, modular varieties of Drinfeld-Stuhler modules have many other interesting applications. We mention two of those:

Let $\Gamma_\mathfrak{n}$ be the principal conguence subgroup of $O_D^\times$ and let $M^D(\mathfrak{n})$ be the modular variety corresponding to $\Gamma_\mathfrak{n}\setminus\Omega^d$; this variety parametrizes Drinfeld-Stuhler modules with certain level structures. Fix a prime $\mathfrak{p}\lhd A$ which does not divide $\mathfrak{n}$ or the discriminant of D. Then $M^D(\mathfrak{n})$ has good reduction $M^D(\mathfrak{n})_{/\mathbb{F}_\mathfrak{p}}$ at $\mathfrak{p}$, which is a smooth projective variety of dimension $d-1$ over $\mathbb{F}_\mathfrak{p}$. Let $\mathbb{F}_\mathfrak{p}^{(d)}$ be the degree-d extension of $\mathbb{F}_\mathfrak{p}$. One can show (see [Pap09]) that

$$\lim_{\deg(\mathfrak{n})\to\infty} \frac{\#M^D(\mathfrak{n})\left(\mathbb{F}_\mathfrak{p}^{(d)}\right)}{\chi(M^D(\mathfrak{n}))} = \frac{(-1)^{d-1}}{d}\prod_{i=1}^{d-1}(|\mathfrak{p}|^i-1),$$

where $\chi(M^D(\mathfrak{n}))$ is the ℓ-adic Euler-Poincaré characteristic of $M^D(\mathfrak{n})$. This result says that the number of $\mathbb{F}_\mathfrak{p}^{(d)}$-rational points on $M^D(\mathfrak{n})_{/\mathbb{F}_\mathfrak{p}}$ asymptotically comes close to the Weil-Deligne bound. This is a generalization to the higher dimensional modular varieties of a well-known result of Ihara, Drinfeld and Vladut for modular curves; cf. [VD83].

Let $\mathscr{B}^d$ be the Bruhat-Tits building of $\mathrm{PGL}_d(F_\infty)$. The quotient $\Gamma_\mathfrak{n}\setminus\mathscr{B}^d$ is a finite simplicial complex which describes the structure of the reduction $M^D(\mathfrak{n})$ of ∞. It turns out that $\Gamma_\mathfrak{n}\setminus\mathscr{B}^d$ is a Ramanujan hypergraph (see [Li04]), which is a higher dimensional generalization of Ramanujan graphs. Li's proof uses the results in [LRS93]. The Ramanujan graphs are of great importance in combinatorics and computer science.

4.3. **Anderson motives and shtukas.** Let K be an A-field with a fixed homomorphism $\gamma : A \to K$. Denote by $K[T,\tau]$ the ring generated by K and the elements T, τ satisfying

$$\begin{aligned} \tau T &= T\tau \\ T\alpha &= \alpha T, \quad \text{for } \alpha \in K \\ \tau\alpha &= \alpha^q\tau, \quad \text{for } \alpha \in K. \end{aligned}$$

Note that $K[T,\tau]$ contains $K[T]$ and $K\{\tau\}$ as subrings.

An *Anderson motive* (also called *t-motive*) is a left $K[T,\tau]$-module such that:

(i) M is free of finite rank over $K[T]$.
(ii) M is free of finite rank over $K\{\tau\}$.
(iii) $(T-\gamma(T))^N(M/\tau M) = 0$ for some integer $N > 0$.

A *morphism* of Anderson motives $f : M \to M'$ is simply a $K[T,\tau]$-linear map.

Given an Anderson module ϕ, let $M(\phi)$ be the group $\mathrm{Hom}_{\mathbb{F}_q}(\mathbb{G}_{a,K}^d, \mathbb{G}_{a,K})$ equipped with the structure of left $K[T,\tau]$-module given by

$$\begin{aligned} (\alpha m)(e) &= \alpha(m(e)) \\ (\tau m)(e) &= m(e)^q \\ (Tm)(e) &= m(\phi_T(e)) \end{aligned}$$

for all $e \in \mathbb{G}_{a,K}^d$, $\alpha \in K$, and $m \in \mathrm{Hom}_{\mathbb{F}_q}(\mathbb{G}_{a,K}^d, \mathbb{G}_{a,K})$.

Anderson proved in [And86] that $M(\phi)$ is an Anderson motive, and the categories of Anderson modules and motives are anti-equivalent under the functor $\phi \to M(\phi)$. Moreover, the dimension and the rank of ϕ are equal to $\mathrm{rank}_{K\{\tau\}} M(\phi)$ and $\mathrm{rank}_{K[T]} M(\phi)$, respectively. The advantage of motives is that one has the module-theoretic operations of tensor products and exterior powers on them. This leads to some important constructions in the theory, such as the tensor products of the Carlitz module.

Remark 4.7. It is somewhat ironic that an Anderson module, which is not quite a module but a homomorphism, is called "module", whereas an Anderson motive, which is actually a module, is called "motive". I think, the term "motive" was chosen by Anderson due to a result in [And86] which implies that $\mathrm{Ta}_\mathfrak{p}(\phi)$ can be recover from $M(\phi) \otimes A_\mathfrak{p}$ by taking τ-invariants.

An Anderson motive M can be considered as a coherent sheaf $\mathcal{F}$ on

$$\mathrm{Spec}(K[T]) = \mathrm{Spec}(A) \times_{\mathbb{F}_q} \mathrm{Spec}(K)$$

equipped with a map $\tau : \mathcal{F} \to \mathcal{F}$ which is A-linear and K-semi-linear. Extending this sheaf to $\mathbb{P}^1_{\mathbb{F}_q} \times \mathrm{Spec}(K)$ in a specific way gives a Drinfeld shtuka. The observation that the category of Drinfeld's modules is equivalent to the category of certain shtukas (called *elliptic sheaves*) was made by Drinfeld in a short but very important paper [Dri77a]. This observation lead to the construction of more general modular varieties which played a crucial role in the proof of the Langlands conjecture for GL_d over function fields by Drinfeld (for $d = 2$) and L. Lafforgue (for $d \geq 2$) [Laf02]. A nice exposition of the dictionary "Drinfeld modules"$\longleftrightarrow$"Elliptic sheaves" can be found in [BS97].

5. Modular forms

In the Drinfeld modular context, there are two different concepts that generalize classical modular forms, namely, *Drinfeld automorphic forms*, which are $\mathbb{C}$-valued functions on some adele groups, and *Drinfeld modular forms*, which are rigid-analytic functions on Drinfeld symmetric spaces with values in extensions of the function field. Automorphic forms are the objects relevant to the Langlands program, and their theory can be developed in a unified manner with automorphic forms over number fields. On the other hand, Drinfeld modular forms exhibit many strange and poorly understood phenomena, so their theory is still in the early stages of development compared to the classical theory of modular forms. To motivate the discussion, we first recall some facts from the classical theory of modular forms.

5.1. **Classical modular forms.** Let $\mathcal{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ denote the upper half-plane in $\mathbb{C}$. The group $\mathrm{SL}_2(\mathbb{R})$ acts on $\mathcal{H}$ via linear fractional transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}. \tag{5.1}$$

Note that the element $-1 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ acts trivially on $\mathcal{H}$. One can show that the group $\mathrm{PSL}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})/\{\pm 1\}$, which acts faithfully on $\mathcal{H}$, is the full group of complex analytic automorphisms of $\mathcal{H}$; see [Miy06, p. 3]. The set

$$\mathcal{F} = \{z \in \mathcal{H} \mid |z| \geq 1, |\mathrm{Re}(z)| \leq 1/2\}$$

is a fundamental domain for the action of the modular group $\mathrm{SL}_2(\mathbb{Z})$; cf. [Ser73, p. 78].

A *modular form of weight* $k \in \mathbb{Z}_+$ *with respect to* $\mathrm{SL}_2(\mathbb{Z})$ is a function $f : \mathcal{H} \to \mathbb{C}$ satisfying the following conditions:

(i) f is holomorphic on $\mathcal{H}$;
(ii) $f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$;
(iii) f is holomorphic at the cusp ∞.

Note that (ii) implies $f(z+1) = f(z)$; thus $f(z)$ has a Fourier expansion

$$f(z) = \sum_{n \in \mathbb{Z}} a_n u^n, \quad u = e^{2\pi i z}$$

and (iii) means that $a_n = 0$ for $n < 0$. A modular form is called a *cusp form* if $a_0 = 0$.

Let M_k denote the $\mathbb{C}$-vector space of modular forms of weight k, and $S_k \subset M_k$ denote the subspace of cusp forms. Note that for $-1 \in \mathrm{SL}_2(\mathbb{Z})$ condition (ii) gives $f(z) = (-1)^k f(z)$. Hence, for odd k we have $M_k = 0$. For even k, it is known that M_k is a finite dimensional vector space and

$$\dim M_k = \begin{cases} \left[\frac{k}{12}\right] & \text{if } k \equiv 2 \pmod{12} \\ \left[\frac{k}{12}\right] + 1 & \text{if } k \not\equiv 2 \pmod{12} \end{cases}$$

The direct sum $M = \bigoplus_{i\geq 0} M_{2i}$ is an algebra (since the product of two modular forms of weights k' and k'' is a modular form of weight $k'+k''$). It is not hard to show that for even $k\geq 4$, the Eisenstein series

$$G_k(z) = \sum_{(0,0)\neq(m,n)\in\mathbb{Z}^2} \frac{1}{(mz+n)^k}$$

is a modular form of weight k. It is a classical fact that

$$M = \mathbb{C}[G_4, G_6],$$

in other words, G_4 and G_6 generate M as an algebra over $\mathbb{C}$. Moreover, these modular forms appear as the coefficient forms of the universal elliptic curve over $\mathcal{H}$: the elliptic curve $E_z := \mathbb{C}/(\mathbb{Z}+z\mathbb{Z})$ is given by the Weierstrass equation

$$y^2 = 4x^3 - 60G_4(z)x - 140G_6(z).$$

The discriminant of E_z is

$$\Delta(z) = (60G_4(z))^3 - 27(140G_6(z))^2 = (2\pi)^{12}u\prod_{n=1}^{\infty}(1-u^n)^{24}, \tag{5.2}$$

which is a cusp form of weight 12 (see [Ser73, p. 95]); in fact, 12 is the lowest weight where non-zero cusp forms exist, and $\Delta(z)$ spans S_{12}. From the product decomposition of Δ, or from the fact that $\Delta(z)$ is the discriminant of E_z, we see that $\Delta(z)$ has no zeros on $\mathcal{H}$.

For an integer $n\geq 1$, the Hecke operator T_n is defined as a correspondence on the set of lattices of $\mathbb{C}$ which transforms a lattice to the sum of its sublattices of index n, i.e., $T_n(\Lambda) = \sum_{(\Lambda:\Lambda')=n}\Lambda'$. (The sum on the right hand-side is an element of the free abelian group generated by the set of lattices, i.e., T_n is a homomorphism from this free abelian group to itself.) Since any lattice in $\mathbb{C}$ is homothetic to a lattice $\mathbb{Z}+z\mathbb{Z}$, $z\in\mathcal{H}$, a modular form can be interpreted as a function of the set of lattices. Using this observation, one defines an action of T_n on modular forms, which for prime p and $f\in M_k$ is given by

$$(T_pf)(z) := p^{k-1}f(pz) + \frac{1}{p}\sum_{0\leq b<p} f((z+b)/p). \tag{5.3}$$

It turns out that $T_nf\in M_k$ and the Hecke operators acting on M_k satisfy the formulas (see [Ser73, p. 101])

$$\begin{aligned} T_nT_m &= T_{nm} \quad \text{ if } (n,m)=1,\\ T_{p^n}T_p &= T_{p^{n+1}} + p^{k-1}T_{p^{n-1}}. \end{aligned}$$

Moreover, there are simple formulas which relate the Fourier coefficients of T_nf to the Fourier coefficients of f. In particular, if $f(z)=\sum_{n\geq 0} c_n(f)u^n$ is the expansion of f, then

$$c_1(T_nf) = c_n(f). \tag{5.4}$$

An important consequence of this simple formula is that if $f(z)$ is an eigenform, i.e., that there exists a complex number λ_n such that $T_n f = \lambda_n f$ for all $n \geq 1$, then $c_n(f) = \lambda_n c_1(f)$. Thus, two eigenforms with the same Hecke eigenvalues are scalar multiples of each other.

Finally, for a Hecke eigenform f, Deligne constructed a Galois representation

$$\pi_f : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell),$$

which is everywhere unramified except at the prime ℓ, and $\mathrm{Tr}(\pi_f(\mathrm{Frob}_p)) = \lambda_p$(= eigenvalue of T_p) for any prime $p \neq \ell$.

5.2. **Drinfeld modular forms.** The theory of Drifeld modular forms was initially developed by Goss [Gos80a], [Gos80b], and Gekeler [Gek88]. In our exposition, we will use the previous subsection as a blueprint.

Let $\Omega^2 = \mathbb{C}_\infty - F_\infty$ be the Drinfeld "half-plane". We denote the absolute value on $\mathbb{C}_\infty$ by $|\cdot|$. The group $\mathrm{GL}_2(F_\infty)$ acts on Ω^2 via linear fractional transformations (5.1). The scalar matrices act trivially, and one can show that $\mathrm{PGL}_2(F_\infty) = \mathrm{GL}_2(F_\infty)/F_\infty^\times$ acts faithfully on Ω^2 and all (non-archimedean) analytic automorphisms of Ω^2 are obtained this way; cf. [Ber95]. The analog of the modular group is $\mathrm{GL}_2(A)$. Define the "imaginary part" of $z \in \mathbb{C}_\infty$ as

$$|z|_i = \min_{x \in F_\infty} |z - x|.$$

The set

$$\mathcal{F} = \{z \in \Omega^2 \mid |z| = |z|_i \geq 1\}.$$

is a fundamental domain for the action of $\mathrm{GL}_2(A)$ on Ω^2; cf. [Gek99, Prop. 6.5]. In particular, every element of Ω^2 is $\mathrm{GL}_2(A)$-equivalent to some element of $\mathcal{F}$.

Exercise 5.1. Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(F_\infty)$ and $z \in \Omega^2$. Prove that

$$\left|\frac{az+b}{cz+d}\right|_i = |\det(\gamma)| \frac{|z|_i}{|cz+d|^2}.$$

This is an analog of a well-known formula relating $\mathrm{Im}(z)$ to $\mathrm{Im}(\gamma z)$ over $\mathbb{C}$; see [Ser73, p. 77].

To define a modular form on Ω^2 with respect to $\mathrm{GL}_2(A)$ we have to come up with reasonable analogs for (i)-(iii) in the definition of classical modular forms.

(i) A *holomorphic* function on $\mathbb{C}_\infty$ is a function which around each $z_0 \in \mathbb{C}_\infty$ can be expanded into power series $\sum_{n \geq 0} a_n (z - z_0)^n$, absolutely and uniformly convergent in some open disc $|z - z_0| < d$. For example, the holomorphic functions on the unit disk $|z| \leq 1$ correspond to power series $\sum_{n \geq 0} a_n z^n$ such that $|a_n| \to 0$. Now Ω^2 is a connected "admissible" open subspace of the rigid-analytic space $\mathbb{P}^1(\mathbb{C}_\infty)$; concretely, this means that Ω^2 has a covering by open sets $\{U\}$ each of which is a disc with q smaller disks removed; one such open glued to another along a boundary of a missing disk. (There are nice pictures of this covering in Teitelbaum's paper [Tei92].) With this covering one never quite reaches the boundary $\mathbb{P}^1(F_\infty)$ of Ω^2, which is similar to the hyperbolic geometry on $\mathcal{H}$ where the points in $\mathbb{P}^1(\mathbb{R})$ are "infinitely far away" from the points in $\mathcal{H}$. A function $f : \Omega^2 \to \mathbb{C}_\infty$ is *holomorphic* if its restriction to each U can be written as an appropriate power series; see [GvdP80, IV] for the actual definitions.

(ii) The substitute for this condition might seem clear but since we are working with $\mathrm{GL}_2(A)$ instead of $\mathrm{SL}_2(A)$, one can in fact define *type* m and *weight* k modular forms by requiring

$$f\left(\frac{az+b}{cz+d}\right) = (ad-bc)^{-m}(cz+d)^k f(z).$$

For simplicity, we will only deal with forms type 0, and omit it from notation, although forms of non-zero type exist and are important in the theory.

(iii) We have seen that Ω^2 has one cusp ∞ with respect to the action of $\mathrm{GL}_2(A)$ (see Exercise 2.11). We need to define what it means for f to be holomorphic at the cusp ∞. Consider the exponential function

$$e_A(z) = z\prod_{0\neq a\in A}\left(1-\frac{z}{a}\right).$$

We know that $e_A(z)$ is $\mathbb{F}_q$-linear, which implies $e_A(z) = z + \sum_{i\geq 1} a_i x^{q^i}$. Then $e_A(z)' = 1$, so the logarithmic derivative of $e_A(z)$ is

$$u(z) := \frac{1}{e_A(z)} = \frac{e_A(z)'}{e_A(z)} = \sum_{a\in A}\frac{1}{z+a}.$$

This shows that $u(z)$ is A-invariant and has a simple zero at ∞. Now a holomorphic function f on Ω^2 satisfying (ii) is A-invariant, so for $z\in\Omega^2$ with $|z|_i$ large enough f can be expanded into series $f(z) = \sum_{n\in\mathbb{Z}} a_n u(z)^n$; see [Gos80a]. A modular form with respect to $\mathrm{GL}_2(A)$ must have $a_n = 0$ for $n<0$; a cusp form, in addition, must have $a_0 = 0$.

Let f be a Drinfeld modular form of weight k with respect to $\mathrm{GL}_2(A)$. Acting by the scalar matrices, we see that $f(z) = \alpha^k f(z)$ for all $\alpha\in\mathbb{F}_q^\times$. Thus, $f=0$ if $k\not\equiv 0 \pmod{q-1}$. Let $\mathcal{M}_k$ be the $\mathbb{C}_\infty$-vector space of Drinfeld modular forms of weight k, and $\mathcal{M} = \bigoplus_{i\geq 0}\mathcal{M}_{(q-1)i}$. By an argument similar to the classical case, one shows that (see [Gos80a])

$$\dim\mathcal{M}_k = 1 + \left[\frac{k}{q^2-1}\right].$$

(One relates modular forms to sections of line bundles on $X_0(1) = \mathbb{P}^1_{\mathbb{C}}$ and applies the Riemann-Roch theorem.)

Goss [Gos80a] proved that for any positive integer k divisible by $q-1$, the Eisenstein series

$$G_k(z) = \sum_{(0,0)\neq(a,b)\in A^2}\frac{1}{(az+b)^k}$$

is a modular form of weight k, and

$$\mathcal{M} = \mathbb{C}_\infty[G_{q-1}, G_{q^2-1}].$$

Let ϕ^z be the Drinfeld module of rank 2 over $\mathbb{C}_\infty$ corresponding to the lattice $A+Az$. This module is determined by

$$\phi^z_T = T + g(z)\tau + \Delta(z)\tau^2.$$

We have the relations (see [Gos80a])

$$g(z) = (T^q - T)G_{q-1}(z) \quad \text{and}$$
$$\Delta(z) = (T^{q^2} - T^q)G_{q-1}^{q+1} + (T^{q^2} - T)G_{q^2-1}.$$

Hence $g(z) \in \mathcal{M}_{q-1}$ and $\Delta(z) \in \mathcal{M}_{q^2-1}$. Moreover, Δ is a cusp form, q^2-1 is the lowest weight where non-zero cusp forms exist, Δ spans the space of cusp forms of weight q^2-1, and Δ does not vanish on Ω^2 (since it is the leading coefficient of ϕ_T^z).

Let $\rho_a(x)$, $a \in A$, be the cyclotomic polynomials defined by the Carlotz module $\rho_T(x) = Tx + x^q$. Gekeler proved (see [Gek88]) the following product decomposition

$$\Delta(z) = -\boldsymbol{\pi}^{q^2-1}u^{q-1}\prod_{a\in A_+}(\rho_a(u^{-1})u^{q^{\deg(a)}})^{(q^2-1)(q-1)},$$

which is the analog of the product decomposition (5.2) of Ramanujan's Δ if we take into account that $\rho_a(x)$ is the analog of x^n-1 over $\mathbb{Q}$, and $q-1$, q^2-1 play the roles of 2 and 12, respectively, in this context.

For a non-zero ideal $\mathfrak{n} \lhd A$, we define a Hecke operator $T_\mathfrak{n}$ as correspondence on the set of A-lattices in $\mathbb{C}_\infty$ of rank 2, which transforms a lattice Λ to the formal sum $\sum \Lambda'$ over all sublattices $\Lambda' \subset \Lambda$ such that $\Lambda/\Lambda' \cong A/\mathfrak{m} \times A/\mathfrak{m}'$, $\mathfrak{m}\mathfrak{m}' = \mathfrak{n}$. One can use the map $z \mapsto A + Az$ from points on Ω^2 to lattices to define an action of $T_\mathfrak{n}$ on Drinfeld modular forms. For a prime $(\wp) = \mathfrak{p} \lhd A$ generated by a monic irreducible polynomial $\wp \in A$ and $f \in \mathcal{M}_k$, the action of $T_\mathfrak{p}$ is given by the formula (cf. (5.3))

$$(T_\mathfrak{p} f)(z) = \wp^{k-1}f(\wp z) + \frac{1}{\wp}\sum_{\substack{b\in A\\ b<\deg(\wp)}} f((z+b)/\wp)$$

As expected $T_\mathfrak{p} f \in \mathcal{M}_k$, and is cuspidal if f is. But unlike the classical case, we have

$$T_{\mathfrak{n}\mathfrak{m}} = T_\mathfrak{n} T_\mathfrak{m}$$

for all ideals $\mathfrak{n}, \mathfrak{m} \lhd A$, i.e., not necessarily coprime $\mathfrak{n}$ and $\mathfrak{m}$. The reason for this is that $T_{\mathfrak{p}^n}T_\mathfrak{p} = T_{\mathfrak{p}^{n+1}} + q^{\deg(\wp)k-1}T_{\mathfrak{p}^{n-1}} = T_{\mathfrak{p}^{n+1}}$ as $q = 0$ in $\mathbb{C}_\infty$.

So far everything seems to work just as over $\mathbb{C}$. The first major difference with the classical case emerges when one tries to relate the u-expansion of $T_\mathfrak{n} f$ to the u-expansion of f. The relationship between the coefficients of the expansion of $T_\mathfrak{n} f$ and f are given by very complicated and poorly understood formulas, cf. [Gos80b], [Gek88]. For example, let $\mathbb{T}_k$ denote the $\mathbb{C}_\infty$ algebra generated by Hecke operators acting on $\mathcal{M}_k$, and denote by $\sum_{i\geq 0} c_i(f)u^i$ the expansion of $f \in \mathcal{M}_k$. We can consider the pairing

$$\mathbb{T}_k \times \mathcal{M}_k \to \mathbb{C}_\infty,$$
$$T_\mathfrak{n}, f \mapsto c_1(T_\mathfrak{n} f).$$

The analog of this pairing over $\mathbb{C}$ is perfect, thanks to (5.4). On the other hand, over $\mathbb{C}_\infty$ it is expected not to be perfect, although this still seems to be an open problem; cf. [Arm11].

Next, we have (see [Gos80a], [Gos80b])

$$T_\mathfrak{p} G_k = \wp^{k-1} G_k$$
$$T_\mathfrak{p} \Delta = \wp^{q-2} \Delta$$
$$T_\mathfrak{p} g = \wp^{q-2} g.$$

Hence $\Delta \in \mathcal{M}_{q^2-1}$ has the same Hecke eigenvalues as $g \in \mathcal{M}_{q-1}$, which is completely different from the classical situation, where Hecke eigenvalues uniquely determine the eigenform, up to scaling. It is an open problem whether "Hecke eigenvalues plus weight" suffices to determine the eigenform.

For a cuspidal eigenform f with $T_\mathfrak{p} f = \lambda_\mathfrak{p} f$, Böckle constructed a Galois representation

$$\pi_f : \mathrm{Gal}(F^{\mathrm{sep}}/F) \to \mathbb{C}_\infty^\times$$

such that $\pi_f(\mathrm{Frob}_\mathfrak{p}) = \lambda_\mathfrak{p}$ for all but finitely many primes $\mathfrak{p}$; cf. [Böc15]. Note that this representation has abelian image; it is in fact a Hecke character! Moreover, this representation can be ramified at places other than those dividing the level of f. This is very different from the classical situation. It is not clear yet how to describe exactly the Hecke characters which arise from cuspidal Hecke eigenforms.

Remark 5.2. Böckle's construction is quite technical, but in a special case it can be described in simpler terms. Let $J_0(\mathfrak{n})$ denote the Jacobian of $X_0(\mathfrak{n})$. One can deduce from (5.5) that there is a Hecke equivariant isomorphism (which is a special case of Böckle's Eichler-Shimura isomorphism for Drinfeld modular forms)

$$J_0(\mathfrak{n})[p](\overline{F}) \otimes_{\mathbb{F}_p} \mathbb{C}_\infty \cong \underline{H}_!(\Gamma_0(\mathfrak{n}), \mathbb{Z}) \otimes \mathbb{C}_\infty.$$

The space $\underline{H}_!(\Gamma_0(\mathfrak{n}), \mathbb{Z}) \otimes \mathbb{C}_\infty$ can be identified with a subspace of weight 2 and type 1 Drinfeld cusp forms on $\Gamma_0(\mathfrak{n})$. Thus, for a Hecke eigenform $f \in \underline{H}_!(\Gamma_0(\mathfrak{n}), \mathbb{Z}) \otimes \mathbb{C}_\infty$ we get a 1-dimensional Hecke invariant subspace V_f of $J_0(\mathfrak{n})[p](\overline{F}) \otimes_{\mathbb{F}_p} \mathbb{C}_\infty$, which also carries a Galois action. The character π_f is the representation $\mathrm{Gal}(F^{\mathrm{sep}}/F) \to \mathrm{Aut}(V_f)$; see [Böc15, Prop. 8.5]. In some sense, the reason that π_f is one-dimensional, instead of two-dimensional, is related to the fact that $\dim_{\mathbb{F}_p} J_0(\mathfrak{n})[p] = \frac{1}{2} \dim_{\mathbb{F}_\ell} J_0(\mathfrak{n})[\ell]$ for $\ell \neq p$, so half of the p-torsion of $J_0(\mathfrak{n})$ is "missing".

Remark 5.3. What about higher dimensional Ω^r? It is possible to define modular forms on Ω^r for $r \geq 2$. Moreover, the constructions discussed in this subsection generalize to Ω^r. For example, higher rank lattices in $\mathbb{C}_\infty$ give Eisenstein series on Ω^r and the leading coefficient $\Delta(z)$ of the Drinfeld module $\phi_T^z = T + \cdots + \Delta(z)\tau^r$ uniformized by the lattice corresponding to $z \in \Omega^r$ is a cusp form of weight $q^r - 1$. There is even a product decomposition for this Δ, similar to $r = 2$ case. We refer to [BB17], [Gek17] for the details.

5.3. **Drinfeld automorphic forms.** There is a different way of looking at classical modular forms, which starts with the observation that $\mathrm{SL}_2(\mathbb{R})$ acts transitively on the upper half-plane $\mathcal{H}$ and the stabilizer of the point $i \in \mathcal{H}$ is the rotation group $\mathrm{SO}_2(\mathbb{R})$. Hence

$$\mathcal{H} \cong \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R}).$$

Modular forms then correspond to complex-valued functions on $\mathrm{SL}_2(\mathbb{R})$ which are left invariant under $\mathrm{SL}_2(\mathbb{Z})$, behave in a prescribed manner under the right translations by elements of the maximal compact subgroup $\mathrm{SO}_2(\mathbb{R})$, and are harmonic in a suitable sense; see, for example, [Bum97, §3.2].

Now replace $\mathrm{SL}_2(\mathbb{R})$ by $\mathrm{GL}_2(F_\infty)$, and $\mathrm{SO}_2(\mathbb{R})$ by a compact subgroup $\mathcal{K}$ of $\mathrm{GL}_2(F_\infty)$; for example, we can take $\mathcal{K} = \mathrm{GL}_2(\mathcal{O}_\infty)$, where $\mathcal{O}_\infty$ is the ring of integers in F_∞. Then our second version of the upper half-plane $\mathcal{H}$ over F could be the Bruhat-Tits tree $\mathcal{B}^2$, since $\mathrm{GL}_2(F_\infty)$ acts transitively on the similarity classes of $\mathcal{O}_\infty$-lattices in $F_\infty \oplus F_\infty$, with $F_\infty^\times \mathrm{GL}_2(\mathcal{O}_\infty)$ being the stabilizer of $\mathcal{O}_\infty \oplus \mathcal{O}_\infty$. Hence

$$\mathrm{Vertices}(\mathcal{B}^2) \longleftrightarrow \mathrm{GL}_2(F_\infty)/F_\infty^\times \mathrm{GL}_2(\mathcal{O}_\infty)$$

Assume the compact subgroup $\mathcal{K}$ is fixed. Let Γ be a congruence subgroup of $\mathrm{GL}_2(A)$, and C be an abelian group equipped with a (possibly trivial) action of Γ. A C-valued *automorphic form* with respect to Γ is a function

$$f : \mathrm{GL}_2(F_\infty) \to C$$

which is right $F_\infty^\times\mathcal{K}$-invariant ($f(zg) = f(z)$ for $g \in F_\infty^\times\mathcal{K}$), left Γ-equivariant ($f(\gamma z) = \gamma f(z)$ for $\gamma \in \Gamma$), and which satisfies an appropriate harmonicity condition.

The relevant compact subgroup to the context of Drinfeld's paper [Dri74] is the Iwahori group $\mathcal{I} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_\infty) \mid c \in \mathfrak{p}_\infty \right\}$, where $\mathfrak{p}_\infty$ is the maximal ideal of $\mathcal{O}_\infty$. In this case, $\mathrm{GL}_2(F_\infty)/F_\infty^\times\mathcal{I}$ can be identified with the set of (oriented) edges of $\mathcal{B}^2$, since $F_\infty^\times\mathcal{I}$ is the stabilizer of the directed edge corresponding to the inclusion of lattices $\mathcal{O}_\infty \oplus \mathcal{O}_\infty \supset \mathcal{O}_\infty \oplus \mathfrak{p}_\infty$. (The reversely oriented edge corresponds to $\mathcal{O}_\infty \oplus \mathfrak{p}_\infty \supset \mathfrak{p}_\infty \oplus \mathfrak{p}_\infty$. Note that $\mathfrak{p}_\infty \oplus \mathfrak{p}_\infty = \wp_\infty(\mathcal{O}_\infty \oplus \mathcal{O}_\infty) \sim \mathcal{O}_\infty \oplus \mathcal{O}_\infty$, where $\wp_\infty$ is a uniformizer of $\mathcal{O}_\infty$.)

A C-valued *Drinfeld automorphic form* with respect to Γ is a function

$$f : \{\text{oriented edges of } \mathcal{B}^2\} \to C$$

such that

(i) $f(\gamma e) = \gamma f(e)$ for all edges e and all $\gamma \in \Gamma$;
(ii) $f(\bar{e}) = -f(e)$, where $\bar{e}$ is e with reverse orientation;
(iii) $\sum_{o(e)=v} f(e) = 0$, where the sum is over all edges originating at same vertex v.

Here (ii) and (iii) are the "harmonicity conditions" and are related to a certain combinatorial Laplacian on $\mathcal{B}^2$.

Denote the group of Drinfeld automorphic form by $\underline{H}(\Gamma, C)$. A detailed explanation of why $\underline{H}(\Gamma, C)$ corresponds to automorphic forms in Drinfeld's paper [Dri74] (besides [Dri74] itself) can be found in [vdPR97].

Remark 5.4. André Weil might have been the first to observe that automorphic forms over $\mathbb{F}_q(T)$ have an elementary combinatorial interpretation as functions on $\mathcal{B}^2$; cf. [Wei70], [Wei71]. He also developed a theory of Fourier expansions of these automorphic forms, Hecke operators, and L-series. This all was subsumed into the fundamental work of Jacquet and Langlands [JL70] from the same period.

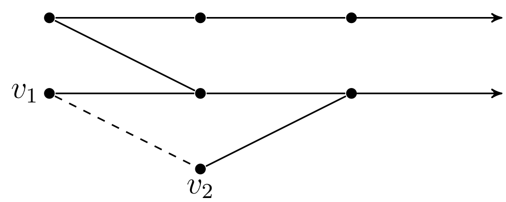

FIGURE 1

First, let $C = V(k)$ be the $\mathbb{C}_\infty$-vector space of polynomials in X and Y of degree $\leq k-2$, with an action of $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ given by

$$X^i Y^{k-2-i} \mapsto (aX+bY)^i (cX+dY)^{k-2-i} \det(\gamma).$$

In [Tei91], using residues of differential forms arising from Drinfeld modular forms, Teitelbaum constructs a homomorphism

$$\mathcal{M}_k(\Gamma) \to \underline{H}(\Gamma, V(k))$$

and shows that this is an isomorphism when restricted to Drinfeld cusp forms of weight k; the inverse map is given by a non-archimedean integration with respect to a measure associated to a given automorphic form. (See also [Tei92] for an exposition of this work, and [ST97] for its generalization to modular forms on Ω^r, $r \geq 2$.)

Now assume $C = \mathbb{C}$ is equipped with trivial action of Γ. The quotient graph $\Gamma \setminus \mathcal{B}^2$ is a finite graph with finitely many infinite half-lines attached to it. For example, for a prime $\mathfrak{p} \lhd A$ of degree 3, the quotient $\Gamma_0(\mathfrak{p}) \setminus \mathcal{B}^2$ looks like the graph in Figure 1, where the dashed edge indicates that the vertices v_1 and v_2 are connected by q edges, and the arrows indicate the infinite half-lines. (The graph is drawn in a somewhat strange way because there is an algorithm for computing these graphs layer-by-layer, and the vertices on the same vertical line represent one layer; see [GN95].) Since $f \in \underline{H}(\Gamma, \mathbb{C})$ is Γ-invariant, it defines a function on the quotient graph $\Gamma \setminus \mathcal{B}^2$. We say that f is a *Drinfeld automorphic cusp form* if it vanishes on the half-lines. Denote the vector space of cusp forms by $\underline{H}_!(\Gamma, \mathbb{C})$.

Exercise 5.5. Prove that

$$\underline{H}_!(\Gamma, \mathbb{C}) \cong H_1(\Gamma \setminus \mathcal{B}^2, \mathbb{C})$$

where the second group is the usual simplicial homology group. (See [GR96, p. 49] for the solution.) In particular, $\underline{H}_!(\Gamma, \mathbb{C})$ is finite dimensional. Also prove that

$$\dim \underline{H}(\Gamma, \mathbb{C}) = \dim \underline{H}_!(\Gamma, \mathbb{C}) + (\#\text{ half-lines}) - 1.$$

Remark 5.6. The infinite half-lines of $\Gamma \setminus \mathcal{B}^2$ naturally correspond to the cusps (="missing points") of the affine variety $\Gamma \setminus \Omega^2$. In fact, the rigid-analytic structure of $\Gamma \setminus \Omega^2$ is such that there is an infinite sequence of nested annuli around each cusp, and each annulus under the building map λ mentioned earlier in the paper maps to an edge on the infinite half-line. (A typical annulus is $\{z \mid 1 \leq |z| \leq q\}$.)

From now on assume that $\Gamma = \Gamma_0(\mathfrak{n})$ for some ideal $\mathfrak{n} \lhd A$. For an ideal $0 \neq \mathfrak{m} \lhd A$, there is a Hecke operator $T_\mathfrak{m}$ acting on $\underline{H}(\Gamma, \mathbb{C})$. Assume $\mathfrak{m}$ is coprime to $\mathfrak{n}$. Considering $f \in \underline{H}(\Gamma, \mathbb{C})$ as a function on $\mathrm{GL}_2(F_\infty)$, we define

$$(T_\mathfrak{m} f)(g) = \sum f\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} g\right),$$

where the sum is over $a, b, d \in A$ such that a, d are monic, $(ad) = \mathfrak{m}$, $\deg(b) < \deg(d)$. (If $\mathfrak{m}$ is not coprime to $\mathfrak{n}$, the definition has to be slightly modified.) These Hecke operators preserve $\underline{H}(\Gamma, \mathbb{C})$ and $\underline{H}_!(\Gamma, \mathbb{C})$, commute with each other, and satisfy recursion formulas similar to those that appear in the classical theory of (weight-2) modular forms. For example, $T_{\mathfrak{p}^2} = T_\mathfrak{p}^2 + |\mathfrak{p}| T_\mathfrak{p}$. The automorphic forms in $\underline{H}(\Gamma, \mathbb{C})$ have "Fourier expansions", which are no longer related to Ω^2, but rather have a group-theoretic nature. (These expansions were intoroduced by Weil [Wei71]. See [Gek95] for a nice exposition and simplified formulae.) One can give formulas for the effect of the Hecke operators on the Fourier expansions, which are similar to the classical case. In fact, one can even show that the pairing between the Hecke algebra and $\underline{H}(\Gamma, \mathbb{C})$ constructed from the analog of (5.4) is perfect in this case.

Remark 5.7. For the existence of Fourier expansions, Hecke operators and etc. the fact that our coefficient ring is $\mathbb{C}$ is important insofar the characteristic of $\mathbb{C}$ is not p, since p is the only prime that shows up in the denominators of Weil's formulae. Hence, as long as our coefficient ring R is commutative, unitary and has characteristic $\neq p$ (e.g. $R = \mathbb{Z}/n\mathbb{Z}$, $p \nmid n$), one should expect a reasonably well behaved theory of automorphic forms $\underline{H}(\Gamma, R)$, similar to the classical theory of modular forms of weight 2 on $\Gamma_0(N)$. This is important in the theory of the Eisenstein ideal over F; see [Pál05], [PW17].

Let $X_0(\mathfrak{n})$ be the curve from Definition 2.8. It follows from Drinfeld's results in [Dri74] that $X_0(\mathfrak{n})$ can be defined over F. Now the correspondence $T_\mathfrak{m}$ may be defined on $X_0(\mathfrak{n})$ using the interpretation of $Y_0(\mathfrak{n})$ a moduli scheme of Drinfeld modules of rank 2. Thus, $T_\mathfrak{m}$ induces an endomorphism of $J_0(\mathfrak{n})$, and of the ℓ-adic cohomology $H^1(X_0(\mathfrak{n}) \otimes F^{\mathrm{sep}}, \mathbb{Q}_\ell)$ $(\ell \neq p)$. The following deep result follows from [Dri74, Prop. 10.3], and can be thought of as the analog of the Eichler-Shimura isomorphism:

Theorem 5.8. *Let* sp *be the two-dimensional special ℓ-adic representation of* $\mathrm{Gal}(F_\infty^{\mathrm{sep}}/F_\infty)$. *There is a canonical isomorphism*

$$H^1(X_0(\mathfrak{n}) \otimes F_\infty^{\mathrm{sep}}, \mathbb{Q}_\ell) \cong \underline{H}_!(\Gamma_0(\mathfrak{n}), \mathbb{Q}_\ell) \otimes \mathrm{sp}$$

compatible with the action of the Hecke operators and $\mathrm{Gal}(F_\infty^{\mathrm{sep}}/F_\infty)$.

Remark 5.9. The special representation sp is the representation one obtains from the action of $\mathrm{Gal}(F_\infty^{\mathrm{sep}}/F_\infty)$ on the ℓ-adic Tate module of an elliptic curve over F_∞ with split multiplicative reduction.

Let $f \in \underline{H}_!(\Gamma_0(\mathfrak{n}), \mathbb{C})$ be an eigenform for all Hecke operators $T_\mathfrak{p} f = \lambda_\mathfrak{p} f$, $\mathfrak{p} \lhd A$ prime. The main result in [Dri74] implies that there is an irreducible representation

$$\pi_f : \mathrm{Gal}(F^{\mathrm{sep}}/F) \to \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell),$$

which is unramified at all $\mathfrak{p} \nmid \mathfrak{n}$ and $\mathrm{Tr}(\pi_f(\mathrm{Frob}_\mathfrak{p})) = \lambda_\mathfrak{p}$. In fact, the vector space of this representation is a direct summand of the semi-simplification of $H^1(X_0(\mathfrak{n}) \otimes F^{\mathrm{sep}}, \overline{\mathbb{Q}}_\ell)$ as $\mathbb{T}_\mathfrak{n} \times \mathrm{Gal}(F^{\mathrm{sep}}/F)$-module, where $\mathbb{T}_\mathfrak{n}$ is the Hecke algebra generated by $T_\mathfrak{m}$'s acting on $\underline{H}_!(\Gamma_0(\mathfrak{n}), \mathbb{Q}_\ell)$. This is an analog of Deligne's theorem for classical modular forms.

Overall, we see that the part of the theory of $\mathbb{C}$-valued Drinfeld automorphic forms related to Hecke operators and Galois representations is quite similar to the classical theory.

Remark 5.10. Drinfeld's fundamental theorem from [Dri74] has been generalized to higher dimensional modular varieties $M^r(\mathfrak{n}) = \Gamma(\mathfrak{n}) \setminus \Omega^r$, $r \geq 3$, by Laumon [Lau96], [Lau97]. Laumon describes the decomposition of the ℓ-adic cohomology groups with compact supports $H^i_c(M^r(\mathfrak{n}) \otimes F^{\mathrm{sep}}, \overline{\mathbb{Q}}_\ell)$ in terms of automorphic representations of GL_r over F.

Using ℓ-adic cohomology group of moduli spaces of shtukas, L. Lafforgue [Laf02] was able to attach an r-dimensional ℓ-adic representation of $\mathrm{Gal}(F^{\mathrm{sep}}/F)$ to every cuspidal automorphic representation of GL_r in such a way that their local L-factors agree, which gives the direction "Automorphic $\to$ Galois" of the Langlands correspondence for GL_r over F. He then proves the direction "Galois $\to$ Automorphic" using Grothendieck's work on L-functions of ℓ-adic Galois representations over function fields and the converse theorems for GL_r of Cogdell and Piatetski-Shapiro, thus settling the global Langlands correspondence for GL_r over function fields. More recently, V. Lafforgue [Laf12] established the direction "Automorphic $\to$ Galois" of the global Langlands correspondence for any reductive group G over a function field using moduli spaces of G-shtukas.

5.4. **Modularity of elliptic curves.** Let E be a non-isotrivial elliptic curve over F, so $j(E) \notin \mathbb{F}_q$. Let $\mathfrak{N}_E$ be the conductor of E. For a place $\mathfrak{p} \nmid \mathfrak{N}_E$ of F define

$$a_\mathfrak{p} = |\mathfrak{p}| + 1 - \#E(\mathbb{F}_\mathfrak{p}),$$

where $\#E(\mathbb{F}_\mathfrak{p})$ is the number of rational points on the reduction of E at $\mathfrak{p}$. For a place $\mathfrak{p} \mid \mathfrak{N}_E$ define

$$a_\mathfrak{p} = 0, 1, -1$$

according to whether E has additive, split multiplicative, or non-split multiplicative reduction, respectively. Let

$$L(E,s) = \prod_{\mathfrak{p} \mid \mathfrak{N}_E} \left(1 - \frac{a_\mathfrak{p}}{|\mathfrak{p}|^s}\right)^{-1} \prod_{\mathfrak{p} \nmid \mathfrak{N}_E} \left(1 - \frac{a_\mathfrak{p}}{|\mathfrak{p}|^s} + \frac{1}{|\mathfrak{p}|^{2s-1}}\right)^{-1}.$$

Grothendieck's theory of L-functions implies that $L(E,s)$ is a polynomial in q^{-s} with $\mathbb{Z}$-coefficients, constant term 1, and degree $\deg(\mathfrak{N}_E) - 4$.

Remark 5.11. This implies that the degree of the conductor of E is least 4, so, for example, there are no elliptic curves over F with multiplicative reduction at T and $(T-1)$ and good reduction everywhere else. On the other hand, there are elliptic curves with conductor of degree 4, for example, the Drinfeld modular curve $X_0(T^3)$ over $\mathbb{F}_2(T)$ is an elliptic curve of conductor $T^3 \cdot \infty$, given by the Weierstrass equation $y^2 + Txy = x^3 + T^2x$. Its L-function is $L(X_0(T^3), s) = 1$.

$L(E,s)$ has a functional equation:

$$L(E,2-s) = \pm q^{(s-1)(\deg(\mathfrak{N}_E)-4)} L(E,s).$$

The L-functions of the twists of E have similar properties. This allows one to apply Weil-Jacquet-Langlands converse theorem [JL70, Thm. 11.5] to conclude that $L(E,s)$ is "automorphic"; see [Del73, p. 577]. Instead of explaining what this means exactly in general, assume for simplicity that E has split multiplicative reduction at $\infty = 1/T$, so its conductor is $\mathfrak{N}_E = \mathfrak{n} \cdot \infty$ for some $\mathfrak{n} \lhd A$. The automorphy of E in this case means that there is a Hecke eigenform $f \in \underline{H}_!(\Gamma_0(\mathfrak{n}), \mathbb{C})$ such that $T_\mathfrak{p} f = a_\mathfrak{p} f$ for all $\mathfrak{p} \nmid \mathfrak{n}$. In particular, the information about the number of rational points on the reductions of E at primes of A can be recovered from a function on a finite graph!

Exercise 5.12. Prove that a non-isotrivial elliptic curve over a function field has at least one place of potentially multiplicative reduction, so the assumption on E having split multiplicative reduction at ∞ is not very restrictive, since this can be achieved by passing to a finite extension of F and choosing ∞ appropriately.

As good as it is, such an automorphy of elliptic curves is not very satisfactory. To study the geometry of E one would like a geometric parametrization of E by a curve with "modular interpretation". This is where Drinfeld's fundamental theorem comes into play. Let

$$\mathrm{Ta}_\ell(E) = \varprojlim E[\ell^n] \cong \mathbb{Z}_\ell \times \mathbb{Z}_\ell,$$

and $V_\ell(E) = \mathrm{Ta}_\ell(E) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$. Let $V_\ell(J_0(\mathfrak{n}))$ be defined similarly. Since $H^1(X_0(\mathfrak{n}) \otimes F^{\mathrm{sep}}, \mathbb{Q}_\ell)$ is the dual of $V_\ell(J_0(\mathfrak{n}))$, the automorphy of E and Drinfeld's theorem imply that $V_\ell(E)$ is a quotient of $V_\ell(J_0(\mathfrak{n}))$ as a $\mathrm{Gal}(F^{\mathrm{sep}}/F)$-module. Zarhin's isogeny theorem then implies that E is quotient $J_0(\mathfrak{n})$; composing the canonical map $X_0(\mathfrak{n}) \to J_0(\mathfrak{n})$ with $J_0(\mathfrak{n}) \to E$, we get:

Theorem 5.13. *Let E be an elliptic curve over F of conductor $\mathfrak{n}\cdot\infty$ having split multiplicative reduction at ∞. There is a non-constant morphism $X_0(\mathfrak{n}) \to E$ defined over F.*

The corresponding theorem over $\mathbb{Q}$ is the modularity theorem of Wiles and others, whose proof is quite different, and much more complicated than the proof of the above theorem.

Theorem 5.13 can be stated in a more precise form as a bijection between the sets:

(i) normalized Hecke eigenforms in $\underline{H}_!(\Gamma_0(\mathfrak{n}), \mathbb{C})$ with integer eigenvalues;
(ii) one-dimensional isogeny factors of the "new" part of $J_0(\mathfrak{n})$, i.e., one-dimensional factors which do not occur in some $J_0(\mathfrak{m})$ with $\mathfrak{m}$ strictly dividing $\mathfrak{n}$;
(iii) isogeny classes of elliptic curves over F with conductor $\mathfrak{n} \cdot \infty$, and split multiplicative reduction at ∞.

In [GR96], Gekeler and Reversat gave a beautiful analytic construction of $J_0(\mathfrak{n})$ and of the optimal elliptic curve E associated to an eigenform $f \in \underline{H}_!(\Gamma_0(\mathfrak{n}), \mathbb{C})$ with integer eigenvalues. (In each isogeny class (iii) there is a unique E for which the quotient map $J_0(\mathfrak{n}) \to E$ has connected reduced kernel; this is the *optimal* curve.) To simplify the notation, assume $\mathfrak{n}$ is fixed and denote $H = \underline{H}_!(\Gamma_0(\mathfrak{n}), \mathbb{Z})$. Note that H is a free $\mathbb{Z}$-module of rank equal to the genus of $X_0(\mathfrak{n})$, and it is a lattice in $\underline{H}_!(\Gamma_0(\mathfrak{n}), \mathbb{C})$ in the sense that $H \otimes_\mathbb{Z} \mathbb{C} = \underline{H}_!(\Gamma_0(\mathfrak{n}), \mathbb{C})$.

The action of Hecke operators on $\underline{H}_!(\Gamma_0(\mathfrak{n}),\mathbb{C})$ preserve the lattice H. Assume $f \in H$ is primitive (i.e. not a multiple of another element), and is an eigenform of all Hecke operators (necessarily with integer eigenvalues). Let E_f be the associated optimal elliptic curve over F. Let $\mathrm{ev}_f : \mathrm{Hom}(H, \mathbb{C}_\infty^\times) \to \mathbb{C}_\infty^\times$ be the evaluation on f, i.e., $\varphi \in \mathrm{Hom}(H, \mathbb{C}_\infty^\times)$ maps to $\varphi(f)$.

The Jacobian $J_0(\mathfrak{n})$ has split purely multiplicative reduction at ∞, hence by a theorem of Mumford and Tate it can be uniformized as a quotient of a multiplicative torus by a lattice. Using non-archimedean theta functions, Gekeler and Reversat construct an embedding $H \to \mathrm{Hom}(H, \mathbb{C}_\infty^\times)$ and show that the quotient is isomorphic to $J_0(\mathfrak{n})$:

$$0 \to H \to \mathrm{Hom}(H, \mathbb{C}_\infty^\times) \to J_0(\mathfrak{n}) \to 0 \tag{5.5}$$

Moreover, if Ξ is the image of H under $H \to \mathrm{Hom}(H, \mathbb{C}_\infty^\times) \xrightarrow{\mathrm{ev}_f} \mathbb{C}_\infty^\times$, then $\Xi = \omega_E^{\mathbb{Z}}$ is cyclic, generated by some ω_E with $|\omega_E| < 1$, and, as a Tate curve, $\mathbb{C}_\infty^\times/\Xi \cong E_f$.

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