K-theory of algebraically closed fields

Know: For all fields F,

$$K_n(F) = \begin{cases} \mathbb{Z}, & n=0 \\ F^\times, & n=1 \\ F^\times \otimes_{\mathbb{Z}} F^\times / \langle a \otimes (1-a) \rangle, & n=2 \end{cases}$$

What about higher K-groups?

Ex (Quillen). $$K_i(\mathbb{F}_q) = \begin{cases} \mathbb{Z}, & i=0 \\ 0, & i=2n>0 \\ \mathbb{Z}/(q^n-1), & i=2n-1 \end{cases}$$

Moreover, a field ext. $\mathbb{F}_q \hookrightarrow \mathbb{F}_{q'}$ induces an incl. $K_i(\mathbb{F}_q) \hookrightarrow K_i(\mathbb{F}_{q'})$

+-construction & htpy groups commute with colimits

$$\Rightarrow \quad K_i(\overline{\mathbb{F}}_p) = \begin{cases} \mathbb{Z}, & i=0 \\ 0, & i=2n>0 \\ \mathbb{Q}/\mathbb{Z}[\tfrac{1}{p}], & i=2n-1 \end{cases}$$

Slogan: "The alg. K-theory of an alg. closed field only depends on the characteristic."

K-theory w/ finite coefficients

X pointed top. space $\quad \pi_i(X) = [S^i, X]_* = [\underset{\text{Moore space}}{M(\mathbb{Z}, i)}, X]_*$

(Moore space written above M.)

Def. If $i \geq 2$, the mod ℓ homotopy "group" of X is
$$\pi_i(X; \mathbb{Z}/\ell) = [M(\mathbb{Z}/\ell, i), X]_*$$

Rmk 1) Only a group for $m \geq 3$, abelian for $m \geq 4$

2) If $X = \Omega^k X_k$, $\pi_i(X; \mathbb{Z}/\ell) = [\Sigma^k M(\mathbb{Z}/\ell, i), X_k]_* \cong [M(\mathbb{Z}/\ell, i+k), X_k]$
$$= \pi_{i+k}(X_k; \mathbb{Z}/\ell)$$

so we still get groups in our setting.

Def. The mod l K-groups of a ring R (resp. an exact category $\mathcal{C}$) are given by

$$K_i(R;\mathbb{Z}/l) = \pi_{i+1}(K(R);\mathbb{Z}/l) \qquad \Big(\text{resp. } \pi_{i+1}(NQ(\mathcal{C});\mathbb{Z}/l)\Big)$$

Universal Coefficient Theorem For all i, l there is a SES

$$0 \to K_i(R)\otimes\mathbb{Z}/l \to K_i(R;\mathbb{Z}/l) \to {}_l K_{i-1}(R) \to 0$$

Ex 1) $K_i(\overline{\mathbb{F}}_p;\mathbb{Z}/p) = \begin{cases} \mathbb{Z}/p & , i=0 \\ 0 & , \text{else} \end{cases}$

$K_i(\overline{\mathbb{F}}_p;\mathbb{Z}/m) = \begin{cases} \mathbb{Z}/m & , i \text{ even} \\ 0 & , i \text{ odd} \end{cases}$ if $(m,p)=1$

2) $K_i(\overline{\mathbb{Q}},\mathbb{Z}/m) = \begin{cases} \mathbb{Z}/m & , i \text{ even} \\ 0 & , i \text{ odd} \end{cases}$

Main thm: If $i: k \hookrightarrow F$ incl. of alg. closed fields, then $i_*: K_n(k;\mathbb{Z}/m) \to K_n(F;\mathbb{Z}/m)$ is an isomorphism.

Main ingredient: Rigidity theorem

Specialization maps

R DVR   Have fiber sequence $G(k) \to G(R) \to G(F)$
$R/m = k$, $\mathrm{Frac}(R) = F$ (coming from localization & dévissage)

R, k, F regular noetherian $\Rightarrow$ K_i- and G_i-groups coincide (resolution)

$$\rightsquigarrow \; \dots \to K_*(k) \to K_*(R) \to K_*(F) \xrightarrow{\partial} K_{*-1}(k) \to \dots$$

Rmk 1) This is a long exact sequence of $K_*(R)$-modules

2) A uniformizer $s \in R$ determines $[s] \in F^\times \simeq K_1(F)$ and

$$\partial([s]) = \underbrace{[R/sR]}_{=k} = 1 \qquad (\text{Weibel V, 6.1})$$

__Def__. For a uniformizer s of R, the specialization map is

$$\lambda_s : K_n(F) \xrightarrow{s\cdot} K_{n+1}(F) \xrightarrow{\partial} K_n(k)$$

__Lemma__ (i) If $k \subset R$, the LES $(*)$ breaks up into split SES

$$0 \to K_n(R) \to K_n(F) \xrightarrow{\partial} K_{n-1}(k) \to 0$$

(ii) Moreover, the natural map $\pi_* : K_n(R) \to K_n(k)$ factors through λ_s.

__Pf__ (i) Since $k \subset R$, $(*)$ is a ~~SES~~ sequence of $K_*(k)$-modules, so a splitting is given by
$$K_{n-1}(k) \to K_n(F)$$
$$a \longmapsto s\cdot a$$

$$(\ \partial(s\cdot a) = \partial(s)\cdot a = 1\cdot a = a)$$

(ii)
$$\begin{tikzcd} K_n(R) \arrow[rr, "\pi_*"] \arrow[dr, "i_*"'] & & K_n(k) \\ & K_n(F) \arrow[ur, "\lambda_s"'] & \end{tikzcd}$$

$$\lambda_s(i_* a) = \partial(s\cdot i_* a) = \underbrace{\partial(s)}_{=1}\cdot \pi_* a = \pi_* a$$

__Lemma__. If k alg. closed, $m \in \mathbb{Z}_{\geq 2}$, the mod m specialization maps are independent of the choice of the uniformizer s.

__Pf__. s, s' two uniformizers, $s' = us$, $u \in R^\times$

Then for all $a \in K_n(F)$:

$$\lambda_{s'}(a) - \lambda_s(a) = \partial(s'\cdot a) - \partial(s\cdot a)$$
$$= \partial(\cancel{s\cdot a}) + \partial(u\cdot a) - \partial(\cancel{s\cdot a})$$
$$= \bar{u}\cdot\partial(a)$$

, $\bar{u}$ = image of u under $$\begin{tikzcd} & \overset{F^\times}{\overset{\shortparallel}{K_1(F;\mathbb{Z}/m)}} \arrow[dr] & \\ \overset{R^\times}{\overset{\shortparallel}{K_1(R;\mathbb{Z}/m)}} \arrow[ur] \arrow[rr] & & K_1(k;\mathbb{Z}/m) \end{tikzcd}$$

$\Rightarrow$ ~~[illegible]~~ $\operatorname{im}(\lambda_{s'} - \lambda_s) \subseteq \operatorname{im}(K_1(k;\mathbb{Z}/m) \otimes K_{*-1}(k;\mathbb{Z}/m) \to K_*(k;\mathbb{Z}/m))$

But $K_1(k) = k^\times$ is div. (k alg. closed!) $\Rightarrow K_1(k;\mathbb{Z}/m) = 0$

$\Rightarrow \lambda_s = \lambda_{s'}$

Families of specialization maps

k alg. closed, $m \in \mathbb{Z}_{\geq 2}$, C smooth curve$/k$, $F = k(C)$

$\forall c \in C$ closed get specialization map

$$\lambda_c : K_n(F; \mathbb{Z}/m) \xrightarrow{s_c \cdot} K_{n+1}(F; \mathbb{Z}/m) \xrightarrow{\partial} K_n(\underset{\mathcal{O}_{C,c}/(s_c)}{\underset{\parallel}{k}})$$

(indep. of uniformizer)

$\rightsquigarrow$ "global specialization map" $\lambda : \mathcal{D}iv(C) \to \mathcal{H}om(K_*(F; \mathbb{Z}/m), K_*(k; \mathbb{Z}/m))$

Prop. The specialization map factors as

$$\begin{tikzcd}
\mathcal{D}iv(C) \arrow[rr] \arrow[dr] & & \mathcal{H}om(K_*(F; \mathbb{Z}/m), K_*(k; \mathbb{Z}/m)) \\
& \mathcal{P}ic(C) \arrow[ur] &
\end{tikzcd}$$

Pf. Case 1: $C = \mathbb{P}^1$ WTS: λ_c does not depend on c

$F = k(t)$

Use analog of localization sequence for Dedekind domain R with fraction field F:

$$\ldots \to \bigoplus_{0 \neq \mathfrak{p} \subset R} K_n(R/\mathfrak{p}R; \mathbb{Z}/m) \xrightarrow{\oplus \pi_{\mathfrak{p}}^{i}} K_n(R; \mathbb{Z}/m) \to K_n(F; \mathbb{Z}/m) \to \bigoplus_{0 \neq \mathfrak{p} \subset R} K_{n-1}(R/\mathfrak{p}; \mathbb{Z}/m)$$

(Use: all f.g. torsion modules$/R$ are of the form $\bigoplus_{i=1}^{k} R/\mathfrak{p}_i^{r_i} R$)

For $R = k[t]$, sequence is split: (Weibel V.6)

$$\begin{tikzcd}
K_n(k[t]; \mathbb{Z}/m) \arrow[r, "i_*"] & K_n(k(t); \mathbb{Z}/m) \arrow[r] & \bigoplus_{a \in k} K_{n-1}(k[t]/(t-a); \mathbb{Z}/m) \arrow[l, bend right, "\bigoplus_{a \in k} ((t-a) \cdot \_)"'] \arrow[d, phantom, "\rotatebox{90}{$\cong$}"] \\
K_n(k; \mathbb{Z}/m) \arrow[u, "j_*", "\wr"'] & & \bigoplus_{a \in k} K_{n-1}(k; \mathbb{Z}/m)
\end{tikzcd}$$

$\rightsquigarrow$ get $i_* j_* \oplus \bigoplus_{a \in k} ((t-a) \cdot \_) : K_n(k; \mathbb{Z}/m) \oplus \bigoplus_{a \in k} K_{n-1}(k[t]/(t-a); \mathbb{Z}/m) \xrightarrow{\cong} K_n(k(t); \mathbb{Z}/m)$

$\Rightarrow$ STS: $\lambda_c \circ (i_* j^*)$, $\lambda_c \circ ((t-a)\cdot -)$ are the same

$\underline{\lambda_c \circ ((t-a)\cdot -)}$: For $b \in K_{n-1}(k[t]/(t-a);\mathbb{Z}/m) \cong K_{n-1}(k;\mathbb{Z}/m)$,

$$\lambda_c((t-a)\cdot b) = \partial(s_c\cdot(t-a)\cdot b) \underset{K_0(k;\mathbb{Z}/m)\text{-linearity}}{=} \underbrace{\partial(s_c\cdot(t-a))}_{\in K_1(k;\mathbb{Z}/m)=0}\cdot b = 0$$

$\underline{\lambda_c \circ (i_* j^*)}$: Suffices to check $\lambda_c \circ (i_* j^*)(\overset{\in K_0(k;\mathbb{Z}/m)}{1})$

$$\lambda_c \circ (i_* j^*)(1) = \lambda_c(\underset{K_0(k(t);\mathbb{Z}/m)}{\overset{\uparrow}{1}}) = \partial(s_c\cdot 1) = \partial(s_c) = 1$$

Case 2: C general curve

Let $0 \neq f \in F =: k(C)$ Want: $\lambda(\mathrm{div}(f)) = 0$

f corresponds to finite map $f: C \longrightarrow \mathbb{P}^1$

STS: For all $a \in K_n(F;\mathbb{Z}/m)$: $\lambda(\mathrm{div}(f))(a) = \sum_{f(c)=0} e_c\lambda_c(a) - \sum_{f(c)=\infty} e_c\lambda_c(a) = 0$

Let $\mu \in \mathbb{P}^1$, t_μ uniformizer of $\mathcal{O}_{\mathbb{P}^1,\mu}$ (later: $\mu = 0,\infty$)

s_c —— " —— $\mathcal{O}_{C,c}$ for $f(c)=\mu$ $t_\mu = u_c\cdot s_c^{e_c}$

The natural maps $k(t) \longrightarrow F$, $\mathcal{O}_{\mathbb{P}^1,\mu} \longrightarrow \mathcal{O}_{C,c}$ induce transfer maps $N_{F/k(t)}: K_n(F;\mathbb{Z}/m) \to K_n(k(t);\mathbb{Z}/m)$, $N_c: K_n(k;\mathbb{Z}/m) \to K_n(k;\mathbb{Z}/m)$

Now
$$\lambda_\mu(N_{F/k(t)}a) = \partial_\mu(t_\mu\cdot N_{F/k(t)}a)$$
$$\overset{\text{Proj. formula}}{=} \partial_\mu(N_{F/k(t)}(f^* t_\mu\cdot a))$$
$$\underset{\text{of }\partial}{\overset{\text{naturality}}{=}} \sum_{f(c)=\mu} N_c\,\partial_c(u_c\cdot s_c^{e_c}\cdot a)$$
$$= \sum_{f(c)=\mu} N_c\cdot e_c\,\partial_c(s_c\cdot a) = \sum_{f(c)=\mu} N_c\,\lambda_c(a)$$

$$\rightsquigarrow \lambda(\mathrm{div}(f))(a) = \sum_{f(c)=0} e_c\lambda_c(a) - \sum_{f(c)=\infty} e_c\lambda_c(a) = \lambda_0(N_{F/k(t)}a) - \lambda_\infty(N_{F/k(t)}a) \overset{\text{case 1}}{=} 0$$

□

Thm (Rigidity). If c_0, c_1 are two closed points of C, then

$$\lambda_{c_0} = \lambda_{c_1} : K_*(F; \mathbb{Z}/m) \longrightarrow K_*(k; \mathbb{Z}/m)$$

Pf.

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Pic}^\circ(C) \arrow[r] \arrow[dr, "\Rightarrow = 0"'] & \mathrm{Pic}(C) \arrow[r, "\deg"] \arrow[d, "\lambda"] & \mathbb{Z} \arrow[r] & 0 \\
& & \underbrace{\mathrm{Hom}(K_*(F; \mathbb{Z}/m), \underbrace{K_*(k; \mathbb{Z}/m)}_{m^2\text{-torsion}})}_{m^2\text{-torsion}} & &
\end{tikzcd}$$

group of closed points of an abelian variety over an alg. closed field $\Rightarrow$ divisible

Thm. If $i : k \hookrightarrow F$ is an inclusion of alg. closed fields, then

$$i_* : K_n(k; \mathbb{Z}/m) \xrightarrow{\sim} K_n(F; \mathbb{Z}/m)$$

Pf. $F = \underset{\substack{A \subset F \\ \text{f.g. subalg.}}}{\mathrm{colim}}\, A \Rightarrow K_n(F; \mathbb{Z}/m) = \underset{A \subset F}{\mathrm{colim}}\, K_n(A; \mathbb{Z}/m)$

Injectivity: Enough to show injectivity of $K_n(k, \mathbb{Z}/m) \longrightarrow K_n(A; \mathbb{Z}/m)$
But this map has a split induced by $A \longrightarrow A/m = k$
(for ~~some~~ any maximal ideal $m \subset A$)

Surjectivity: STS: For a f.g. subalg. $A \subset F$ $\begin{matrix} h_0 : A \hookrightarrow F \\ h_1 : A \to A/m = k \hookrightarrow F \end{matrix}$ induce the same map on K-groups

WLOG: A smooth (after passing to a localization of A if necessary)

h_0 and h_1 factor as
$$h_0 : A \longrightarrow A \otimes_k F \xrightarrow{\tilde{h}_0} F$$
$$h_1 : A \longrightarrow A \otimes_k F \xrightarrow{\tilde{h}_1} F$$

~~By~~ Since smoothness is stable under base change, $A \otimes_k F$ is smooth $/F$, and $\tilde{h}_0$ and $\tilde{h}_1$ define points $p_0, p_1 \in \mathrm{Spec}(A \otimes_k F)(F)$

$\Rightarrow \exists$ affine smooth curve $C = \mathrm{Spec}(R)/F$ with $C \longrightarrow \mathrm{Spec}\, A \otimes_k F$ st $\begin{matrix} c_0 \mapsto p_0 \\ c_1 \mapsto p_1 \end{matrix}$

$$\begin{tikzcd}
K_n(R; \mathbb{Z}/m) \arrow[rr] \arrow[drr, "c_0"'] \arrow[drr, shift left, "c_1"] & \arrow[dr, phantom, "\circlearrowleft \text{ by prev. lemma}"] & K_n(E; \mathbb{Z}/m) \arrow[d, shift right, "\lambda_{c_0}"'] \arrow[d, shift left, "\lambda_{c_1}"] \\
K_n(A \otimes_k F; \mathbb{Z}/m) \arrow[u, "\circlearrowleft \text{ by choice of } R"'] \arrow[rr, shift left, "\tilde{h}_0"] \arrow[rr, shift right, "\tilde{h}_1"'] & & K_n(F; \mathbb{Z}/m)
\end{tikzcd}$$

But $\lambda_{c_0} = \lambda_{c_1}$ by the rigidity lemma
$\Rightarrow \tilde{h}_0 = \tilde{h}_1$