

K-theory of algebraically closed fields

Know: For all fields F ,

$$K_n(F) = \begin{cases} \mathbb{Z} & , n=0 \\ F^\times & , n=1 \\ F^\times \otimes_2 F^\times / \langle a \otimes (1-a) \rangle & , n=2 \end{cases}$$

What about higher K-groups?

Ex (Quillen). $K_i(\mathbb{F}_q) = \begin{cases} \mathbb{Z} & , i=0 \\ 0 & , i=2n > 0 \\ \mathbb{Z}/q^{n-1} & , i=2n-1 \end{cases}$

Moreover, a field ext. $\mathbb{F}_q \hookrightarrow \mathbb{F}_{q^r}$ induces an incl. $K(\mathbb{F}_q) \hookrightarrow K(\mathbb{F}_{q^r})$

+ construction & homotopy groups commute with colimits

$$\Rightarrow K_i(\overline{\mathbb{F}}_p) = \begin{cases} \mathbb{Z} & , i=0 \\ 0 & , i=2n > 0 \\ \mathbb{Q}/\mathbb{Z}[\frac{1}{p}] & , i=2n-1 \end{cases}$$

Slogan: "The alg. K-theory of an alg. closed field only depends on the characteristic."

K-theory w/ finite coefficients

X pointed top. space

$$\pi_i(X) = [S^i, X] = \overset{\text{above page}}{=} [M(\mathbb{Z}, i), X]$$

Def. If $i \geq 2$, the mod l homotopy "group" of X is

$$\pi_i(X; \mathbb{Z}/l) = [M(\mathbb{Z}/l, i), X]$$

Prop 1) Only a group for $m \geq 3$, abelian for $m \geq 4$

2) If $X = \Omega^k X_0$, $\pi_i(X; \mathbb{Z}/l) = [\Sigma^k M(\mathbb{Z}/l, i); X_0] \cong [M(\mathbb{Z}/l, i+k), X_0]$

$$= \pi_{i+k}(X_0; \mathbb{Z}/l)$$

so we still get groups in our setting.

Def. The mod l K -groups of a ring R (resp. an exact category \mathcal{C}) are given by

$$K_i(R; \mathbb{Z}/l) = \pi_{i+1}(K(R); \mathbb{Z}/l) \quad (\text{resp. } \pi_{i+1}(NQ(\mathcal{C}); \mathbb{Z}/l))$$

Universal Coefficient Theorem For all i, l there is a SES

$$0 \rightarrow K_i(R) \otimes \mathbb{Z}/l \rightarrow K_i(R; \mathbb{Z}/l) \rightarrow {}_l K_{i-1}(R) \rightarrow 0$$

Ex. 1) $K_i(\overline{\mathbb{F}}_p; \mathbb{Z}/p) = \begin{cases} \mathbb{Z}/p, & i=0 \\ 0, & \text{else} \end{cases}$

$$K_i(\overline{\mathbb{F}}_p; \mathbb{Z}/m) = \begin{cases} \mathbb{Z}/m, & i \text{ even} \\ 0, & i \text{ odd} \end{cases} \quad \text{if } (m, p) = 1$$

$$2) K_i(\overline{\mathbb{Q}}, \mathbb{Z}/m) = \begin{cases} \mathbb{Z}/m, & i \text{ even} \\ 0, & i \text{ odd} \end{cases}$$

Main thm: If $i: k \hookrightarrow F$ incl. of alg. closed fields, then

$$i_*: K_n(k; \mathbb{Z}/m) \rightarrow K_n(F; \mathbb{Z}/m) \text{ is an isomorphism.}$$

Main ingredient: Rigidity theorem

Specialization maps

R DVR Have fiber sequence $G_*(k) \rightarrow G_*(R) \rightarrow G_*(F)$
 $R/m = k, \text{Frac}(R) = F$ (coming from localization & divisors)

R, k, F regular noetherian $\Rightarrow K_*$ - and G_* -groups coincide (resolution)

$$\dots \rightarrow K_n(k) \rightarrow K_n(R) \rightarrow K_n(F) \xrightarrow{\partial} K_{n-1}(k) \rightarrow \dots$$

Remark 1) This is a long exact sequence of K_* -modules

2) A uniformizer $s \in R$ determines $[s] \in F^\times \simeq K_1(F)$ and

$$\partial([s]) = [R/sR] = 1 \quad (\text{Nürbel } \text{IV}, 6.1)$$

$= k$

Def. For a uniformiser s of R , the specialisation map is

$$\lambda_s: K_n(F) \xrightarrow{s} K_{n+1}(F) \xrightarrow{\partial} K_n(k)$$

Lemma (i) If $k \subset R$, the LES (*) breaks up into split SES

$$0 \rightarrow K_n(R) \rightarrow K_n(F) \xrightarrow{\partial} K_{n+1}(k) \rightarrow 0$$

(ii) Moreover, the natural map $\pi_s: K_n(R) \rightarrow K_n(k)$ factors through λ_s .

Pf (i) Since $k \subset R$, (*) is a ~~SES~~^{sequence} of $K_n(k)$ -modules, so a splitting is given by $K_{n+1}(k) \rightarrow K_n(F)$

$$a \longmapsto s \cdot a$$

$$(\partial(s \cdot a) = \partial(s) \cdot a = 1 \cdot a = a)$$

(ii)

$$K_n(R) \xrightarrow{\pi_s} K_n(k) \quad \lambda_s(i_s a) = \partial(s \cdot i_s a)$$

$$\searrow i_s \quad \swarrow \lambda_s \quad = \partial(s) \cdot \pi_s a = \pi_s a$$

$= 1$

Lemma. If k alg. closed, $m \in \mathbb{Z}_{\geq 2}$, the mod m specialisation maps are independent of the choice of the uniformiser s .

Pf. s, s' two uniformisers, $s' = us$, $u \in R^\times$

Then for all $a \in K_n(F)$:

$$\begin{aligned} \lambda_{s'}(a) - \lambda_s(a) &= \partial(s' \cdot a) - \partial(s \cdot a) \\ &= \partial(s \cdot a) + \partial(ua) - \partial(s \cdot a) \\ &= \bar{u} \cdot \partial(a) \end{aligned}$$

$\bar{u} = \text{image of } u \text{ under } K_n(R, \mathbb{Z}/m) \rightarrow K_n(k, \mathbb{Z}/m)$

$$\Rightarrow \text{im}(\lambda_{s'} - \lambda_s) \subseteq \text{im}(K_n(k, \mathbb{Z}/m) \otimes K_{n+1}(k, \mathbb{Z}/m) \rightarrow K_n(k, \mathbb{Z}/m))$$

But $K_n(k) = k^\times$ is div. (k alg. closed!) $\Rightarrow K_n(k, \mathbb{Z}/m) = 0$

$$\Rightarrow \lambda_s = \lambda_{s'}$$

Families of specialization maps

k alg. closed, $m \in \mathbb{Z}_{\neq 2}$, C smooth curve/ k , $F = k(C)$

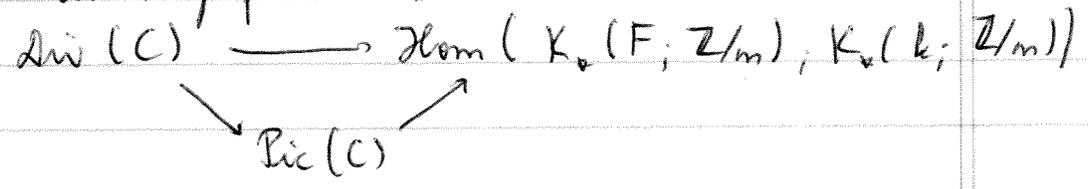
$\forall c \in C$ closed get specialization map

$$\lambda_c: K_n(F; \mathbb{Z}/m) \xrightarrow{\sim} K_{n+1}(F; \mathbb{Z}/m) \xrightarrow{\cong} K_n(k) \cong \mathcal{O}_{C,c}/(s_c)$$

(indep. of uniformizer)

\leadsto "global specialization map" $\lambda: \text{Div}(C) \rightarrow \text{Hom}(K_n(F; \mathbb{Z}/m), K_n(k; \mathbb{Z}/m))$

Prop. The specialization map factors as



Pf. Case 1: $C = \mathbb{P}^1$

WTS: λ_c does not depend on c

$$F = k(t)$$

Use analogy of localization sequence for Dedekind domain R with fraction field F :

$$\rightarrow \bigoplus_{0 \neq \mathfrak{p} \subset R} K_n(R/\mathfrak{p}R; \mathbb{Z}/m) \xrightarrow{\oplus \pi_j^i} K_n(R; \mathbb{Z}/m) \rightarrow K_n(F; \mathbb{Z}/m) \rightarrow \bigoplus_{0 \neq \mathfrak{p} \subset R} K_{n-1}(R/\mathfrak{p}R; \mathbb{Z}/m)$$

(Use: all f.g. torsion modules $/R$ are of the form $\bigoplus_{i=1}^h R/\mathfrak{p}_i^i R$)

For $R = k[t]$, sequence is split:

$$\begin{array}{ccc} K_n(k[t]; \mathbb{Z}/m) & \xrightarrow{i_0} & K_n(k(t); \mathbb{Z}/m) \xrightarrow{\oplus_{a \in k} ((t-a)_-)} \bigoplus_{a \in k} K_{n-1}(k[t]/(t-a); \mathbb{Z}/m) \\ \uparrow j_0 & & \uparrow \cong \\ K_n(k; \mathbb{Z}/m) & & \bigoplus_{a \in k} K_{n-1}(k; \mathbb{Z}/m) \end{array}$$

(Wibel V.6)

$$\leadsto \text{get } i_0 j_0 \oplus \bigoplus_{a \in k} ((t-a)_-): K_n(k; \mathbb{Z}/m) \oplus \bigoplus_{a \in k} K_{n-1}(k[t]/(t-a); \mathbb{Z}/m) \xrightarrow{\cong} K_n(k(t); \mathbb{Z}/m)$$

\Rightarrow STS: $\lambda_c \circ (i_0 j_+)$, $\lambda_c \circ ((t-a) \cdot -)$ are the same

$\lambda_c \circ ((t-a) \cdot -)$: For $b \in K_{n-1}(k[t]/(t-a); \mathbb{Z}/m) \cong K_{n-1}(k; \mathbb{Z}/m)$,
 $\lambda_c((t-a) \cdot b) = \partial(s_c \cdot (t-a) \cdot b) = \partial(s_c \cdot (t-a)) \cdot b = 0$
 $K_0(k; \mathbb{Z}/m) \in K_1(k; \mathbb{Z}/m) = 0$

$\lambda_c \circ (i_0 j_+)$: Suffices to check $\lambda_c \circ (i_0 j_+)(1) \in K_0(k; \mathbb{Z}/m)$
 $\lambda_c \circ (i_0 j_+)(1) = \lambda_c(1) = \partial(s_c \cdot 1) = \partial(s_c) = 1$
 $K_0(k(t); \mathbb{Z}/m)$

Case 2: C general curve

Let $0 \neq f \in F = k(C)$ Want: $\lambda(\text{div}(f)) = 0$

f corresponds to finite map $f: C \rightarrow \mathbb{P}^1$

STS: For all $a \in K_n(F; \mathbb{Z}/m)$: $\lambda(\text{div}(f))(a) = \sum_{f(c)=0} e_c \lambda_c(a) - \sum_{f(c)=\infty} e_c \lambda_c(a) = 0$

Let $\mu \in \mathbb{P}^1$, t_μ uniformizer of $\mathcal{O}_{\mathbb{P}^1, \mu}$ (later: $\mu = 0, \infty$)
 $s_c \xrightarrow{\quad} \mathcal{O}_{C, c}$ for $f(c) = \mu$ $t_\mu = u_c \cdot s_c^{e_c}$

The natural maps $k(t) \rightarrow F$ induce transfer maps $N_{F/k(t)}: K_n(F; \mathbb{Z}/m) \rightarrow K_n(k(t); \mathbb{Z}/m)$
 $\mathcal{O}_{\mathbb{P}^1, \mu} \rightarrow \mathcal{O}_{C, c}$ $N_C: K_n(k; \mathbb{Z}/m) \rightarrow K_n(k; \mathbb{Z}/m)$

Now $\lambda_\mu(N_{F/k(t)} a) = \partial_\mu(t_\mu \cdot N_{F/k(t)} a)$

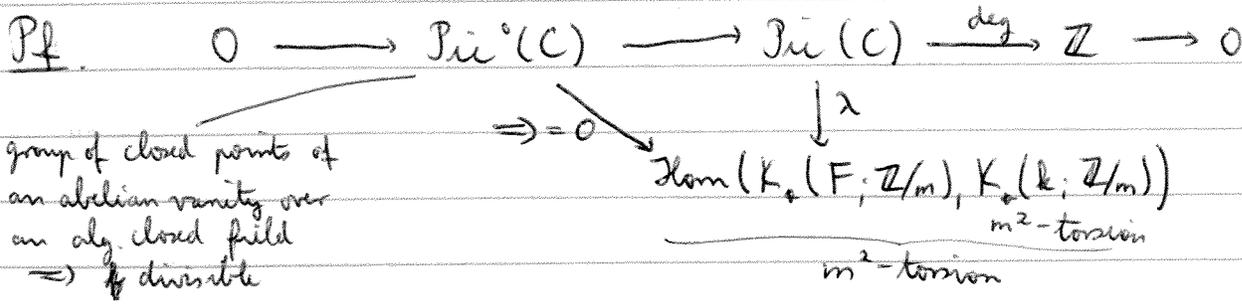
$$\begin{aligned} & \text{Proj. formula} \\ &= \partial_\mu(N_{F/k(t)}(f^* t_\mu \cdot a)) \\ & \text{naturality} \\ & \text{of } \partial \\ &= \sum_{f(c)=\mu} N_c \partial_c(u_c \cdot s_c^{e_c} \cdot a) \end{aligned}$$

$$= \sum_{f(c)=\mu} N_c \cdot e_c \partial_c(s_c \cdot a) = \sum_{f(c)=\mu} N_c \lambda_c(a)$$

$$\leadsto \lambda(\text{div}(f))(a) = \sum_{f(c)=0} e_c \lambda_c(a) - \sum_{f(c)=\infty} e_c \lambda_c(a) = \lambda_0(N_{F/k(t)} a) - \lambda_\infty(N_{F/k(t)} a)$$

case 1 = 0 □

Thm (Rigidity). If c_0, c_1 are two closed points of C , then
 $\lambda_{c_0} = \lambda_{c_1} : K_n(F; \mathbb{Z}/m) \longrightarrow K_n(k; \mathbb{Z}/m)$



Thm. If $i: k \hookrightarrow F$ is an inclusion of alg. closed fields, then
 $i_*: K_n(k; \mathbb{Z}/m) \xrightarrow{\sim} K_n(F; \mathbb{Z}/m)$

Pf. $F = \text{colim}_{A \subset F \text{ f.g. subalg.}} A \Rightarrow K_n(F; \mathbb{Z}/m) = \text{colim}_{A \subset F} K_n(A; \mathbb{Z}/m)$

Injectivity: enough to show injectivity of $K_n(k; \mathbb{Z}/m) \rightarrow K_n(A; \mathbb{Z}/m)$
 But this map has a split induced by $A \rightarrow A/m = k$
 (for ~~maximal~~ maximal ideal $m \subset A$)

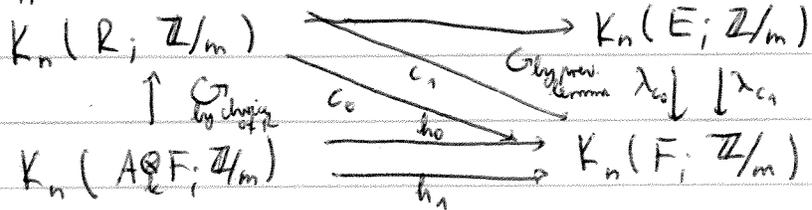
Surjectivity: STS: For a f.g. subalg. $A \subset F$ $\begin{matrix} h_0: A \hookrightarrow F \\ h_1: A \rightarrow A/m = k \hookrightarrow F \end{matrix}$ induce the same map on K -groups

WLOG: A smooth (after passing to a localization of A if necessary)
 h_0 and h_1 factor as

$$\begin{matrix} h_0: A \longrightarrow A \otimes_k F \xrightarrow{\tilde{h}_0} F \\ h_1: A \longrightarrow A \otimes_k F \xrightarrow{\tilde{h}_1} F \end{matrix}$$

Since smoothness is stable under base change, $A \otimes_k F$ is smooth/ F , and \tilde{h}_0 and \tilde{h}_1 define points $p_0, p_1 \in \text{Spec}(A \otimes_k F)(F)$

$\Rightarrow \exists$ affine smooth curve $C = \text{Spec}(R)/F$ with $C \rightarrow \text{Spec} A \otimes_k F$ at $\begin{matrix} c_0 \mapsto p_0 \\ c_1 \mapsto p_1 \end{matrix}$



But $\lambda_{c_0} = \lambda_{c_1}$ by the rigidity lemma $\Rightarrow \tilde{h}_0 = \tilde{h}_1$