**K-theory seminar**

Lecture 4 • Andrew Snowden • February 6, 2015

## 1. K-theory of schemes

1.1. **Definitions.** Let $X$ be a scheme. We define $\mathcal{P}(X)$ to be the category of locally free sheaves on $X$ of finite rank, regarded as an exact category. We define $K_i(X)$ to be $K_i(\mathcal{P}(X))$.

Suppose that $X$ is noetherian. Then the category $\mathcal{M}(X)$ of coherent sheaves on $X$ is abelian, and we define $G_i(X) = K_i(\mathcal{M}(X))$. The inclusion $\mathcal{P}(X) \subset \mathcal{M}(X)$ induces a map $K_i(X) \to G_i(X)$. If $X$ is regular, then this map is an isomorphism (since every coherent sheaf then admits a finite length resolution by locally free sheaves).

Give a locally free coherent sheaf $\mathcal{E}$ on $X$, we have an exact functor $[\mathcal{E}] = - \otimes_{\mathcal{O}_X} \mathcal{E}$ on $\mathcal{M}(X)$ and $\mathcal{P}(X)$. If

$$0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3 \to 0$$

is an exact sequence of locally free sheaves, then we have an exact sequence

$$0 \to [\mathcal{E}_1] \to [\mathcal{E}_2] \to [\mathcal{E}_3] \to 0$$

of functors. It follows that $[\mathcal{E}_2] = [\mathcal{E}_1] + [\mathcal{E}_3]$ on $K_i(X)$ and $G_i(X)$. This gives $K_0(X)$ a ring structure and $K_i(X)$ and $G_i(X)$ the structure of a module over $K_0(X)$.

1.2. **Pull-back maps.** Let $f \colon X \to Y$ be a map of schemes. Then there is an exact functor $f^* \colon \mathcal{P}(Y) \to \mathcal{P}(X)$, which induces a homomorphism $f^* \colon K_i(Y) \to K_i(X)$. In this way, $K_i$ is a contravariant functor from schemes to abelian groups.

Now suppose that $f$ is flat and $X$ and $Y$ are noetherian. Then $f^* \colon \mathcal{M}(Y) \to \mathcal{M}(X)$ is an exact functor, and so there is an induced map $f^* \colon G_i(Y) \to G_i(X)$. In this way, $G_i$ is a contravariant functor on the category of noetherian schemes with flat morphisms.

More generally, let $\mathcal{M}(Y, f^*) \subset \mathcal{M}(Y)$ be the category of sheaves $M$ such that $L_i f^*(M) = 0$ for $i > 0$. Then $f^*$ induces an exact functor $\mathcal{M}(Y, f^*) \to \mathcal{M}(X)$, and thus homomorphisms $K_i(\mathcal{M}(Y, f^*)) \to G_i(X)$. If $f$ has finite Tor dimension (i.e., $L_i f^* = 0$ for $i \gg 0$, which holds if $Y$ is regular) and every coherent sheaf on $Y$ is a quotient of a vector bundle (e.g., $Y$ is projective over an affine scheme) then every coherent sheaf on $Y$ admits a finite resolution by sheaves in $\mathcal{M}(Y, f^*)$, and so $K_i(\mathcal{M}(Y, f^*)) = G_i(Y)$. In this case, $f^*$ induces a map $G_i(Y) \to G_i(X)$. For $i = 0$, this is an Euler characteristic construction: for a coherent sheaf $M$ on $Y$ we have

$$f^*([M]) = \sum_{i \geq 0} (-1)^i [L_i f^*(M)],$$

where $[-]$ denotes the class in $G_0$. The sum is finite due to the finiteness of Tor dimension.

1.3. **Push-forward maps.** Let $f \colon X \to Y$ be a finite morphism of noetherian schemes. Then $f_*$ induces an exact functor $\mathcal{M}(X) \to \mathcal{M}(Y)$, and we get homomorphisms $f_* \colon G_i(X) \to G_i(Y)$.

More generally, suppose that $f$ is proper. Then $R^i f_* = 0$ for $i \gg 0$, and each $R^i f_*$ takes coherent sheaves to coherent sheaves. Let $\mathcal{M}(X, f_*) \subset \mathcal{M}(X)$ be the category of sheaves $M$ on which $R^i f_*(M) = 0$ for $i > 0$. Then $f_*$ induces homomorphisms $K_i(\mathcal{M}(X, f_*)) \to K_i(Y)$. If every coherent sheaf on $X$ embeds into a sheaf in $\mathcal{M}(X, f_*)$ (which is automatic if $X$

admits an ample line bundle) then $K_i(\mathcal{M}(X, f_*)) = K_i(\mathcal{M}(X)) = G_i(X)$, and we get a homomorphism $f_*\colon G_i(X) \to G_i(Y)$. For $i = 0$, this is an Euler characteristic:

$$f_*(M) = \sum_{i\geq 0} (-1)^i [R^i f_*(M)].$$

**Proposition 1.** *Let $X$ and $Y$ be noetherian schemes admitting ample line bundles, and let $f\colon X \to Y$ be a proper map of finite Tor dimension. Then $f_*$ induces a map $K_i(X) \to K_i(Y)$, and the diagram*

$$\begin{array}{ccc} K_i(X) & \longrightarrow & G_i(X) \\ \downarrow f_* & & \downarrow f_* \\ K_i(Y) & \longrightarrow & G_i(Y) \end{array}$$

*commutes.*

*Proof.* Let $\mathcal{P}(X, f_*)$ be the category of vector bundles $\mathcal{E}$ on $X$ such that $R^i f_*(\mathcal{E}) = 0$ for all $i > 0$. Using the ample line bundle on $X$, every vector bundle on $X$ embeds into one in $\mathcal{P}(X, f_*)$ such that the quotient is also a vector bundle. Thus $K_i(\mathcal{P}(X, f_*)) = K_i(\mathcal{P}(X)) = K_i(X)$.

Let $\mathcal{H}(Y) \subset \mathcal{M}(Y)$ be the category of sheaves of finite Tor dimension. We claim that for $\mathcal{E} \in \mathcal{P}(X, f_*)$, we have $f_*(\mathcal{E}) \in \mathcal{H}(Y)$. This is local on $Y$, so we can assume $Y$ is affine. Let $\{U_i\}$ be a finite open cover of $X$. Then the Cech complex

$$0 \to \mathrm{H}^0(X, \mathcal{E}) \to \bigoplus_i \mathrm{H}^0(U_i, \mathcal{E}) \to \bigoplus_{ij} \mathrm{H}^0(U_{i,j}, \mathcal{E}) \to \cdots$$

is exact (since $\mathcal{E} \in \mathcal{P}(X, f_*)$) and of finite length, and its terms have finite Tor dimension (since $f$ does). This shows that $\mathrm{H}^0(X, \mathcal{E}) = f_*(\mathcal{E})$ has finite Tor dimension.

Since $Y$ admits an ample line bundle, every coherent sheaf on $Y$ is a quotient of a vector bundle, and every sheaf in $\mathcal{H}(Y)$ admits a finite length resolution by vector bundles. Thus $K_i(\mathcal{H}(Y)) = K_i(\mathcal{P}(Y)) = K_i(Y)$. We have thus obtained our map $f_*\colon K_i(X) \to K_i(Y)$. □

**Proposition 2** (Projection formula)**.** *Let $f$ be above. Then the equation*

$$f_*(x \cdot f^*(y)) = f_*(x) \cdot y$$

*holds in the following cases: (i) $x \in K_0(X)$, $y \in G_i(Y)$; (ii) $x \in K_0(X)$, $y \in K_i(Y)$; (iii) $x \in G_i(X)$, $y \in K_0(Y)$.*

### 1.4. **Open and closed subschemes.**

**Proposition 3.** *Let $X$ be a noetherian scheme. Let $X_{\mathrm{red}}$ be the reduced subscheme of $X$. Then the map $f_*\colon G_i(X_{\mathrm{red}}) \to G_i(X)$ induced by the finite map $f\colon X_{\mathrm{red}} \to X$ is an isomorphism.*

*Proof.* Every coherent sheaf on $X$ has a finite length filtration (given by powers of the nilradical) where the graded pieces belong to $\mathcal{M}(X_{\mathrm{red}})$. □

**Proposition 4.** *Let $X$ be a noetherian scheme and let $Z$ and $Z'$ be closed subschemes with the same underlying topological space. Then there is a natural isomorphism $G_i(Z) = G_i(Z')$.*

*Proof.* We have $Z_{\mathrm{red}} = Z'_{\mathrm{red}}$. Now use the previous proposition. □

We can therefore define $G_i(Z)$ for a Zariski closed subspace $Z$ of $X$.

**Proposition 5.** *Let $X$ be a noetherian scheme, let $Z$ be a closed subscheme, and let $U$ be its complement. Then there is a long exact sequence*

$$\cdots \to G_i(Z) \to G_i(X) \to G_i(U) \to G_{i-1}(Z) \to \cdots$$

*Proof.* Give $Z$ the reduced scheme structure. Let $\mathcal{M}_Z(X) \subset \mathcal{M}(X)$ be the category of sheaves supported on $Z$. Identify $\mathcal{M}(Z)$ with the subcategory of $\mathcal{M}(X)$ on objects killed by the ideal sheaf $I_Z$. Then every object of $\mathcal{M}_Z(X)$ admits a finite length filtration (by powers of $I_Z$) where the quotients belong to $\mathcal{M}(Z)$, and so $K_i(\mathcal{M}_Z(X)) = K_i(\mathcal{M}(Z)) = G_i(Z)$. It is a standard fact that $\mathcal{M}(U)$ is the Serre quotient $\mathcal{M}(X)/\mathcal{M}_Z(X)$. The result now follows from the localization sequence. □

1.5. **Limits.** Let $\{X_i\}_{i\in I}$ be a filtered inverse system of schemes where the transition maps $X_i \to X_j$ are affine. Then the inverse limit $X$ exists as a scheme. We have the following result in this situation:

**Proposition 6.** *The natural map $\varinjlim K_q(X_i) \to K_q(X)$ is an isomorphism. If $X$ and the $X_i$ are noetherian and the transition maps are flat, the same is true for $G_q$.*

*Proof.* We have an equivalence $\mathcal{P}(X) = \varinjlim \mathcal{P}(X_i)$, and the Q-construction and $\pi_i$ commute with filtered colimits. Similarly for $\mathcal{M}$. □

## 2. Homotopy invariance

2.1. **Graded modules over polynomial rings.** Let $A$ be a noetherian ring, and let $B = A[t_1, \dots, t_n]$ be the graded ring where $\deg(t_i) = 1$. Let $\mathcal{M}_{\mathrm{gr}}(B)$ be the category of finitely generated non-negatively graded $B$-modules. This has a functor $(-1)$ (shift the grading), which gives $K_i(\mathcal{M}_{\mathrm{gr}}(B))$ the structure of a $\mathbf{Z}[t]$-module. There is an exact functor $\mathcal{M}(A) \to \mathcal{M}_{\mathrm{gr}}(B)$ given by $M \mapsto M \otimes_A B$, which induces group homomorphism $G_i(A) \to K_i(\mathcal{M}_{\mathrm{gr}}(B))$, and therefore a $\mathbf{Z}[t]$-module homomorphism $G_i(A) \otimes \mathbf{Z}[t] \to K_i(\mathcal{M}_{\mathrm{gr}}(B))$.

**Proposition 7.** *The canonical homomorphism $\psi\colon G_i(A) \otimes \mathbf{Z}[t] \to K_i(\mathcal{M}_{\mathrm{gr}}(B))$ is an isomorphism.*

*Proof.* Let $\mathcal{N} \subset \mathcal{M}_{\mathrm{gr}}(B)$ be the category of modules $M$ where $\mathrm{Tor}_i^B(M, A) = 0$ for $i > 0$. The functors $\mathrm{Tor}^i_B(-, A) = 0$ for $i > n$ (since $A$ is the quotient of $B$ by a regular sequence of length $n$), and so $K_i(\mathcal{N}) = K_i(\mathcal{M}_{\mathrm{gr}}(B))$.

Let $\mathcal{N}_p \subset \mathcal{N}$ be the subcategory consisting of modules generated in degrees $\leq p$. There are exact functors

$$\mathcal{M}(A) \xrightarrow{a} \mathcal{N}_p \xrightarrow{b} \mathcal{M}(A)^{p+1}$$

give by

$$a(M_0, \dots, M_p) = \bigoplus_{i=0}^{p} B(-i) \otimes_A M_i$$

and

$$b(N) = ((N/B_+N)_0, \dots, (N/B_+N)_p).$$

Note that $N/B_+N = N \otimes_B A$ is exact on $\mathcal{N}$. Clearly, $b \circ a$ is the identity functor, and thus induces the identity on $K$-theory.

For a graded $B$-module $N$, let $F_qN$ be the submodule generated by elements of degree $\leq q$, and let $F_{-1}N = 0$. Regarding $F_q$ as a functor $\mathcal{N}_p \to \mathcal{N}_p$, the chain $F_0 \subset \cdots \subset F_p$

is a filtration of the identity functor $I$. One shows that for $N \in \mathcal{N}$, there is a canonical isomorphism

$$F_q N / F_{q-1} N = B(-p) \otimes (N/B_+N)_p,$$

and so $N \mapsto F_q N/F_{q-1}N$ is exact as well. Thus $\sum_{i=0}^{p} (F_q/F_{q-1})_* = I_*$ induces the identity on $K_i(\mathcal{N}_p)$. However, from the above identification, we see that $\sum_{i=0}^{p}(F_q/F_{q-1})_*$ induces $ab$ on $K$-thoery. Thus $a$ and $b$ are isomorphisms.

Taking the limit as $p \to \infty$, we see that the functor

$$\mathcal{M}(A)^{\bigoplus \infty} \to \mathcal{N}, \qquad (M_0, M_1, \dots) \mapsto \sum_{i=0}^{\infty} B(-i) \otimes_A M_i$$

induces an isomorphism on $K$-theory. But this map is exactly $\psi$, after identifying $K_i(\mathcal{N})$ with $K_i(\mathcal{M}_{\mathrm{gr}}(B))$. □

2.2. **Modules over polynomial rings.** Let $A$ be a noetherian ring. We aim to prove the following result:

**Proposition 8.** *The functor $M \mapsto M \otimes_A A[x]$ induces an isomorphism $G_i(A) \to G_i(A[x])$.*

*Proof.* The idea is to reduce to the case of graded modules over polynomial rings. To do this, we use the following observation: an $A[x]$-module is the same as a $\mathbf{G}_m$-equivariant module on $\mathbf{A}^2_A \setminus \mathbf{A}^1_A$, where $\mathbf{A}^1_A$ is the $y$-axis in $\mathbf{A}^2_A$. Via the localization sequence, we can understand the $\mathbf{G}_m$-equivariant K-theory of $\mathbf{A}^2_A \setminus \mathbf{A}^1_A$ from that of $\mathbf{A}^2_A$ and $\mathbf{A}^1_A$, which is simply the K-theory of graded modules over polynomial rings over $A$.

Let us now translate this to ring theory. We have $\mathbf{A}^2_A = \operatorname{Spec}(A[t,u])$, $\mathbf{A}^1_A = \operatorname{Spec}(A[t])$ and $\mathbf{A}^2_A \setminus \mathbf{A}^1_A = \operatorname{Spec}(A[t,u,u^{-1})$. We have an equivalence of categories $\mathcal{M}_{\mathrm{gr}}(A[t,u,u^{-1}]) = \mathcal{M}(A[x])$, and so the localization sequence gives a long exact sequence [fix: need to use $\mathbf{Z}$-graded modules, not just $\mathbf{Z}_{\geq 0}$-graded ones. doesn't change much]

$$\cdots \to K_i(\mathcal{M}_{\mathrm{gr}}(A[t])) \to K_i(\mathcal{M}_{\mathrm{gr}}(A[t,u])) \to K_i(\mathcal{M}(A[x])) \to \cdots,$$

which translates to

$$\cdots \to G_i(A) \otimes \mathbf{Z}[t] \to G_i(A) \otimes \mathbf{Z}[t] \to G_i(A[x]) \to \cdots,$$

The result now follows from the following lemma. □

**Lemma 9.** *The following diagram commutes:*

$$\begin{array}{ccc} K_i(\mathcal{M}_{\mathrm{gr}}(A[t])) & \longrightarrow & K_i(\mathcal{M}_{\mathrm{gr}}(A[t,u])) \\ \| & & \| \\ K_i(A) \otimes \mathbf{Z}[t] & \xrightarrow{t-1} & K_i(A) \otimes \mathbf{Z}[t] \end{array}$$

*Here the top horizontal map comes from treating $A[t]$-modules as $A[t,u]$-modules where $u$ acts by 0.*

*Proof.* Let $i\colon \mathcal{M}(A) \to \mathcal{M}_{\mathrm{gr}}(A[t])$ be $i(M) = M \otimes_A A[t]$, and let $j\colon \mathcal{M}(A) \to \mathcal{M}_{\mathrm{gr}}(A[t,u])$ be $j(M) = M \otimes_A A[t,u]$. Tensoring the exact sequence

$$0 \to A[t,u](-1) \xrightarrow{u} A[t,u] \to A[t] \to 0$$

over $A$ with $M$, we obtain an exact sequence of functors

$$0 \to j(-1) \to j \to i \to 0.$$

Thus $i = (t-1)j$ on $K$-theory. □

### 2.3. **Affine bundles.**

**Proposition 10.** *Let $X$ be a noetherian scheme and let $f\colon E \to X$ be a flat map whose fibers are affine spaces. Then $f^*\colon G_i(X) \to G_i(E)$ is an isomorphism.*

*Proof.* Given $T \to X$, we say that "the proposition holds for $T$" if the maps $G_i(T) \to G_i(E_T)$ are isomorphisms for all $i$, where $E_T = E \times_X T$. Let $Z$ be a closed subscheme of $X$ with complement $U$. We then obtain a diagram

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & G_i(Z) & \longrightarrow & G_i(X) & \longrightarrow & G_i(U) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & G_i(E_Z) & \longrightarrow & G_i(E) & \longrightarrow & G_i(E_U) & \longrightarrow & \cdots \end{array}$$

Thus if the proposition holds for two of $Z$, $U$, or $X$, then it holds for the third as well. By noetherian induction, we can assume that the proposition holds for $E_Z \to Z$ for all proper closed subschemes $Z$ of $X$. If $X$ is reducible, say $X = Z_1 \cup Z_2$, then the proposition holds for $Z_1$ and $Z_2$ and $Z_1 \cap Z_2$, and thus for $X \setminus Z_1 = Z_2 \setminus (Z_1 \cap Z_2)$, and therefore for $X$. We can therefore assume $X$ is irreducible. Since $G_i$ is insensitive to nilpotents, we can assume $X$ is integral. Now take the direct limit of the above diagram over all proper closed subschemes of $X$, to obtain a diagram

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & \varinjlim G_i(Z) & \longrightarrow & G_i(X) & \longrightarrow & \varinjlim G_i(U) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & \varinjlim G_i(E_Z) & \longrightarrow & G_i(E) & \longrightarrow & \varinjlim G_i(E_U) & \longrightarrow & \cdots \end{array}$$

It thus suffices to show that the map

$$\varinjlim G_i(U) \to \varinjlim G_i(E_U) \tag{11}$$

is an isomorphism. We have

$$\varinjlim G_i(U) = G_i(\varprojlim U) = G_i(K),$$

where $K$ is the function field of $X$. Similarly,

$$\varinjlim G_i(E_U) = G_i(\varprojlim E_U) = G_i(K[x_1, \ldots, x_n]).$$

Thus (11) is an isomorphism by Proposition 8. □

## 3. Filtration by codimension and the BGQ spectral sequence

### 3.1. **Preliminaries.**

If $X \to Y$ is a map of topological spaces with homotopy fiber $F$ then there is a long exact sequence of homotopy groups

$$\cdots \to \pi_i(F) \to \pi_i(X) \to \pi_i(Y) \to \pi_{i-1}(F) \to \cdots$$

If $\mathcal{A}$ is an abelian category and $\mathcal{B}$ is a Serre subcategory then the map $N(Q(\mathcal{A})) \to N(Q(\mathcal{A}/\mathcal{B}))$ has homotopy fiber $N(Q(\mathcal{B}))$, and the resulting long exact sequence is the localization sequence in K-theory.

One can think of $\mathcal{B} \subset \mathcal{A}$ as a 1-step filtration of $\mathcal{A}$. There is a version of localization for longer filtrations, where the long exact sequence is replaced by a spectral sequence. We now explain how this works.

First, suppose that we have maps of topological spaces

$$Y = Y_n \to Y_{n-1} \to \cdots \to Y_0$$

Let $X_0 = Y_0$ and for $1 \le i \le n$ let $X_i$ be the homotopy fiber of $Y_i \to Y_{i-1}$. One would like to say that there is a spectral sequence with $E_1^{p,q} = \pi_{p-q}(X_q)$ that converges to $\pi_{p-q}(Y)$. This is essentially the case, except for the fact that $\pi_0$ and $\pi_1$ cause problems (because they're not abelian groups). However, if the $Y$'s are all H-spaces, and the maps are maps of H-spaces, then this problem goes away, and there is indeed such a spectral sequence.

Now suppose that $\mathcal{A}$ is an abelian category and

$$0 = F^n\mathcal{A} \subset \cdots \subset F^0\mathcal{A} \subset \mathcal{A}$$

is a decreasing filtration by Serre subcategories. For $1 \le i \le n$ put $\mathcal{B}_i = F^{i-1}\mathcal{A}/F^i\mathcal{A}$, and let $\mathcal{B}_0 = \mathcal{A}/F^0\mathcal{A}$. For $0 \le i \le n$, let $Y_i = N(Q(\mathcal{A}/F^i\mathcal{A}))$. Then for $1 \le i \le n$ the map $Y_i \to Y_{i-1}$ has homotopy fiber $X_i = N(Q(\mathcal{B}_i))$, and $X_0 = Y_0 = N(Q(\mathcal{B}_0))$. We thus have a spectral sequence with $E_1^{p,q} = \pi_{p-q}(X_q) = K_{p-q-1}(\mathcal{B}_q)$ that converges to $\pi_{p-q}(Y) = K_{p-q-1}(\mathcal{A})$.

## 4. Severi–Brauer varieties and projective bundles

[to add]