K-theory (3)

Q-construction (continuation)

Setup $(\mathcal{C}, \mathcal{E})$ exact category

Example: $\mathcal{C} = \mathrm{Mod}_R^{fg}$ $R$ = Noeth ring

or

$\mathrm{Vect}_R := \{\text{f.g proj } R\text{-modules}\}$

Notation: $X \hookrightarrow Y$ = admissible mono
$X \twoheadrightarrow Z$ = — " — epi

Def. $Q\mathcal{C}$ = categ. def by

- obj = obj$(\mathcal{C})$

- $X, Y \in \mathrm{obj}(\mathcal{C}) \Rightarrow$

$$\mathrm{Hom}_{Q\mathcal{C}}(X,Y) = \{ X \twoheadleftarrow Z \hookrightarrow Y \} / \text{obvious isom}$$

- composition:

$$(X \twoheadleftarrow Z \hookrightarrow Y) \circ (Y \twoheadleftarrow W \hookrightarrow U)$$

given by

$$\begin{tikzcd}
X & Z \arrow[l, two heads] \arrow[r, hook] & Y \\
& Z \times_Y W \arrow[ul] \arrow[u, two heads] \arrow[r, hook] \arrow[dr, hook] & W \arrow[u, two heads] \arrow[d, hook] \\
& & U
\end{tikzcd}$$

Notation: $\underset{\text{monic in } \mathcal{C}}{i : X \hookrightarrow Y} \Rightarrow i_! : X \to Y$ in $QC$

$$X \twoheadleftarrow X \hookrightarrow Y$$

2) $\underset{\text{epi in } \mathcal{C}}{j : Y \twoheadrightarrow Z} \Rightarrow j^! : Z \to Y$ in $QC$

$$Z \twoheadleftarrow Y \hookrightarrow Y$$

3) Any $f : X \to Z$ in $QC$

corresp to $X \overset{j}{\twoheadleftarrow} Y \overset{i}{\hookrightarrow} Z$

$$\Rightarrow f = i_! \circ j^!$$

<u>Rmk</u> $N(QC)$ is connected since $X \in \mathcal{C}$

$\Rightarrow i_X : 0 \to X$, $j_X : X \to 0$

$(i_X)_! : 0 \to X$

$(j_X)^! : 0 \to X$

## Universal property of QC

$F : Q\mathcal{C} \to \mathcal{D} \iff$ a) $\bullet$ $F(X) \in \mathcal{D}$ $\forall X \in Ob(\mathcal{C})$

b) $\bullet$ $\forall\, i : X \hookrightarrow Y$ in $\mathcal{C}$, need $F(i_!) : F(X) \to F(Y)$

c) $\bullet$ $\forall j : Y \twoheadrightarrow Z$ in $\mathcal{C}$, need $F(j^!) : F(Z) \to F(Y)$

s.t $\bullet$ b), c) compatible with compositio

$\bullet$ $\forall$ Cartesian square in $\mathcal{C}$

$$\begin{tikzcd} Y \arrow[r, "j"] & Z \\ Y \times_Z W \arrow[u, hook, "i_1"] \arrow[r, "j_1"'] & W \arrow[u, hook, "i"'] \end{tikzcd} \qquad \begin{array}{c} F(j^!) \circ F(i_!) \\ \| \\ F((i_1)_!) \circ F(j_1^!) \end{array}$$

**Theorem** $\pi_1(N(Q\mathcal{C}), 0) \simeq K_0(\mathcal{C})$

Idea: $X \in \mathcal{C} \rightsquigarrow$ get $0 \overset{(i_X)_!}{\underset{j_X^!}{\rightrightarrows}} X$ in $Q\mathcal{C}$

$\Rightarrow$ loop based at $0$ in $N(Q\mathcal{C})$

Details of proof:

Enough to show:

$$\pi_1(N(Q\mathcal{C}))\text{-sets} \xrightarrow{\sim} \{K_0(\mathcal{C})\text{-sets}\}$$

proved last time $\leftarrow \wr\wr$

$$\mathrm{Fun}(Q\mathcal{C}, \mathrm{Sets}^{\simeq})$$

$$\wr\wr$$

$$\left\{ F \in \mathrm{Fun}(Q\mathcal{C}, \mathrm{Sets}^{\simeq}) \,\middle/\, \begin{array}{l} F(X) = F(0) \quad \forall X \in \mathcal{C} \\ F((i_X)_!) = id \end{array} \right\}$$

Constr. of "$\Leftarrow$": $S \in \{K_0(\mathcal{C})\text{-sets}\}$

Define $F(X) = S \qquad \forall X$

$$F(i_!) = id \qquad \forall i : X \hookrightarrow Y$$

$$F(j^!) = \ker(j) \circ (-)$$

for $j : Y \twoheadrightarrow Z$ in $\mathcal{C}$

Universal property $\Rightarrow$ get $F : Q\mathcal{C} \to \mathrm{Sets}^{\simeq}$

"$\Longrightarrow$" Fix $F : Q\mathcal{C} \to Sets^{\cong}$ s.t

$F(X) = F(0)$

$F((i_X)_!) = id \quad \forall i : 0 \hookrightarrow X$

Set $S = F(0)$

Goal: make $K_0(\mathcal{C})$ act on $S$

Given $X \in \mathcal{C}$, set $[X] : S \xrightarrow{\sim} S$ to be

$$[X] = F(j_X^!) : \underbrace{F(0)}_{= S} \longrightarrow \underbrace{F(X)}_{= S}$$

We need to show: given

$$0 \to X \xrightarrow{i} Y \xrightarrow{j} Z \to 0 \quad \text{exact sequence}$$

get

$$\begin{tikzcd} X \arrow[r, hook, "i"] \arrow[d, "j_X"'] & Y \arrow[d, "j"] \\ 0 \arrow[r, hook, "i_Z"'] & Z \end{tikzcd} \quad \Rightarrow F(j_X^!) = F(j^!)$$

Now use

$$\begin{tikzcd} Y \arrow[rr, "j"] \arrow[dr, "j_Y"'] & & Z \arrow[dl, "j_Z"] \\ & 0 & \end{tikzcd}$$

$$\Rightarrow \quad \underbrace{F(j^!)}_{\parallel} \circ F(j_Z^!) = F(j_Y^!)$$

$$F(j_X^!) \circ F(j_Z^!)$$

We get in this way a $K_0(\mathcal{C})$-action

## Fundamental theorems

**Thm. 1** (Additivity)

$(\mathcal{C}, \mathcal{E})$ exact cat $\Rightarrow$ $\mathcal{E}$ is also an exact categ (termwise)

i.e. the obvious functors $\mathcal{E} \to \mathcal{C}$ are exact

$$\Rightarrow \quad N(Q\mathcal{E}) \xrightarrow{(s,t)} N(Q\mathcal{C}) \times N(Q\mathcal{C})$$

homotopy equiv

where $s$ = "source", $t$ = "target"

**Cor.** $F : \mathcal{C} \to \mathcal{C}'$ exact functor

$F = \bigcup_{i=0}^{n} F_i$ s.t $F_i \hookrightarrow F_{i+1}$ admissible

$$\Rightarrow \quad F_* = \sum_{i=0}^{n-1} F_{i+1}/F_i : K_*(\mathcal{C}) \to K_*(\mathcal{C}')$$

where $K_i(\mathcal{C}) = \pi_{i+1}(N(Q\mathcal{C}), 0)$

<u>Example</u> $A = \bigoplus_{i \geq 0} A_i$ comm. graded ring

$\mathscr{C} = Vect_A^{gr}$

Graded Nakayama Lemma $\Rightarrow$

$$\mathbb{Z}[t, t^{-1}] \otimes_{\mathbb{Z}} k_i(Vect_{A_0}) \xrightarrow{\sim} k_i(\mathscr{C})$$

<u>Thm 2</u> (Devissage)

$A$ = abelian cat, $B \subset A$ full abelian subcat
s.t $B \to A$ exact
and $\forall x \in A$, $\exists$ finite filtration
$X = \bigcup_{i=0}^{n} X_i$ s.t $X_{i+1}/X_i \in B$ $\forall i$
$\Rightarrow N(QB) \longrightarrow N(QA)$ is a homot. equiv.

(e.g $X$ variety, $Z \hookrightarrow X$ closed subvar
$B = Coh(Z)$, $A = Coh_Z(X)$ coh sheaves supp on $Z$)

<u>Thm 3</u> (Localization)
$A$ = abelian categ, $B \subset A$ Serre subcateg
$\Rightarrow$ Get quotient $A/B$
Then the natural maps give a fibre sequence:

$$N(QB) \longrightarrow N(QA) \longrightarrow N(Q\,A/B)$$

<u>Example</u> $X$ = variety, $Z \hookrightarrow X$ closed subvar

$A = Coh(X)$, $B = Coh_Z(X)$,

$A/B = Coh(X \setminus Z)$

<u>Thm. 4</u> (Resolution)

$A$ = exact categ, $B \subset A$ exact full subcat

Assume: 1) $B$ closed under kernels of surjections in $A$

2) $\forall X \in A$, $\exists$ exact sequence

$$\ldots \to Y_1 \to Y_0 \to X \to 0 \quad Y_i \in B$$

$\Rightarrow N(QB) \longrightarrow N(QA)$ homot equiv

$\Rightarrow K_i(B) \simeq K_i(A)$

<u>Example</u> $X$ smooth quasiproj var

$A = Coh(X)$

$B = Vect(X) \Rightarrow K_i(X) = G_i(X)$

where $K_i(X) := K_i(Vect(X))$

$G_i(X) := K_i(Coh(X))$

## Main tools for proving the theorems

Thm. A Say $f : \mathcal{C} \to \mathcal{C}'$ is a functor, $Y \in \mathcal{C}'$

Set $Y \backslash f = \{ (X, v) \mid X \in Ob(\mathcal{C}), v : Y \to f(X) \}$

If $N(Y \backslash f)$ is contractible $\forall Y \in \mathcal{C}'$,

then $N(\mathcal{C}) \longrightarrow N(\mathcal{C}')$ is a homot. equivalence.

Rmk. In Thm A, fix $Y \in Q(A)$

Apply thm for $f : QB \to QA$

$$Y \backslash f = \{ X \in B, \; X \underset{\text{subobject}}{\rightarrow} Y \}$$

Work by induction on the length of a filtration needed to build $Y$ from $B$

If $Y \in B \Rightarrow N(Y \backslash f)$ contractible since $Y \backslash f$ has final object

Then use Thm A to get Thm. 2.

<u>**Thm. B**</u> $f: \mathcal{C} \to \mathcal{C}'$ is a functor

Assume: $\forall\, Y \to Y'$ in $\mathcal{C}'$, the induced map $N(Y' \backslash f) \to N(Y \backslash f)$ is a homotopy equivalence

Then $\forall\, Y \in \mathcal{C}'$ get a fibre sequence

$$N(Y \backslash f) \longrightarrow N(\mathcal{C}) \longrightarrow N(\mathcal{C}')$$