HIGHER K-THEORY

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Abstract. Notes from the second meeting of the algebraic K-theory seminar at UMich, Winter 2015. Note taker was Takumi Murayama.

Motivation

One reason we would like to define higher K-groups is that with K_0, we get certain "half-exact" sequences that seem to be analogues of the Mayer-Vietoris or the localization long exact sequences, that we would like to actually turn into long exact sequences. Historically, Atiyah and Hirzebruch applied K_0, which was first defined in algebraic geometry, to topology. It was realized that in this context, K_0 was π_0 of a space $K(X)$, and so Atiyah and Hirzebruch defined $K_i(X)$ as $\pi_i(K(X))$. This construction has since been reincorporated into algebraic geometry via the work of Quillen.

There are two constructions of higher K-groups that are useful. We first define the "Plus-construction," which works well with affines and is useful computationally, but has some drawbacks. The second construction is the "Q-construction," which is better suited for theory. It is not known if "Q" stands for Quillen.

1. The Plus-construction

The definition is purely algebro-topological.

Definition 1.1. Let X be a connected topological space. Let a perfect normal subgroup $N \leqslant \pi_1(X)$ be given. Then, $X \xrightarrow{\varphi} X^+$ is a *plus-construction (relative to N)* if it satisfies the following:

1. φ induces an isomorphism $\pi_1(X_+) = \pi_1(X)/N$.
2. $H_\bullet(\mathrm{fib}(\varphi), \mathbf{Z}) = H_\bullet(\mathrm{pt}, \mathbf{Z})$, where $\mathrm{fib}(\varphi)$ is the homotopy fibre of the map φ.

Remark. We can assume φ is a Serre fibration, we can compute the homotopy fibre as an actual fibre, and the Serre spectral sequence gives that $H_\bullet(X, \mathbf{Z}) \simeq H_\bullet(X^+, \mathbf{Z})$.

We will keep in mind the following two examples:

Example 1.2.

1. Let R be a (unital, associative) ring, and let $X = B\mathrm{GL}(R)$, where $B-$ denotes the classifying space, and $\mathrm{GL}(R)$ is the infinite general linear group. Note that $\pi_1(B\mathrm{GL}(R)) = \mathrm{GL}(R)$, while its higher homotopy groups vanish. We can then let $N = \mathrm{E}(R)$, the subgroup of elementary matrices, which we saw were a perfect normal subgroup of $\mathrm{GL}(R)$ last time.

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(2) Let $X = B\Sigma_\infty$, the classifying space for the infinite symmetric group, defined as

$$\Sigma_\infty \coloneqq \operatorname{colim}(\Sigma_1 \hookrightarrow \Sigma_2 \hookrightarrow \Sigma_3 \hookrightarrow \cdots)$$

and $N = A_\infty$ is the infinite alternating group, which is a perfect normal subgroup of S_∞, with quotient $\mathbf{Z}/2$.

Our first goal is to actually show X^+ exists.

Theorem 1.3 (Quillen)**.** *Let X, N as before. Then,*

(1) *There exists a plus construction $\varphi\colon X \to X^+$.*
(2) *If $\psi\colon X \to Y$ induces a map on π_1 that kills N, then ψ factors uniquely (up to pointed homotopy) through φ.*

Note that (2) quantifies the uniqueness statement in Definition 1.1(1).

We proceed in two steps. First, we assume that $N = \pi_1(X)$, and that in particular, $\pi_1(X)$ is perfect. Note this holds for $\mathrm{E}(R)$ but not for $\mathrm{GL}(R)$. We then deduce the full result by passing to covering spaces $\tilde{X}$ corresponding to N.

Step 1. Assume $N = \pi_1(X)$; in particular, this means that $\pi_1(X)$ is perfect. Then, set

$$X_1 \coloneqq \operatorname{colim}\left(\begin{tikzcd} \bigvee_{\alpha\in N} S^1 \arrow[r, "\alpha"] \arrow[d] & X \\ \bigvee_{\alpha\in N} D^2 & \end{tikzcd}\right)$$

i.e., glue two-cells along the one-cells in X in order to kill the subgroup Nin π_1. Note H_1 is unaffected since all elements in N are commutators, hence are 0 in H_1 already. Then, by the Seifert-van Kampen theorem,

$$\pi_1(X_1) = \pi_1(X) *_{\pi_1(\bigvee S^1)} \pi_1\Big(\bigvee D^2\Big) = \pi_1(X)/N = 0$$

by assumption on N.

Next, attaching two-cells affects H_2, so we have to fix our construction. By the Eilenberg-Steenrod axioms, we get the following short exact sequence on (reduced) singular chain complexes:

$$0 \longrightarrow C_\bullet\Bigg(\bigvee_{\alpha\in N} S^1\Bigg) \longrightarrow C_\bullet(X) \longrightarrow C_\bullet(X_1) \longrightarrow 0.$$

The associated long exact sequence on homology gives that $H_i(X) \simeq H_i(X_1)$ for all $i > 2$, and for $i = 1$ we have the exact sequence

$$\mathbf{Z}^{\oplus N} \xrightarrow{0} H_1(X) \longrightarrow H_1(X_1) \longrightarrow 0$$

where $\mathbf{Z}^{\oplus N}$ denotes the direct sum of $\mathbf{Z}$ with itself, indexed by the elements of N. Then, the map $\mathbf{Z}^{\oplus N} \to H_1(X)$ is the zero map by the fact that N is perfect, hence every element $\alpha \in N$ is 0 in homology. Thus, the long exact sequence at $i = 2$ gives the short exact sequence

$$0 \longrightarrow H_2(X) \longrightarrow H_2(X_1) \longrightarrow \mathbf{Z}^{\oplus N} \longrightarrow 0. \tag{1}$$

$\pi_1(X_1) = 0$ by construction, hence X_1 is simply connected, and the Hurewicz theorem implies $\pi_2(X_1) = H_2(X_1)$. We then set

$$X^+ := \operatorname{colim}\left(\begin{array}{c} \displaystyle\bigvee_{\alpha\in N} S^2 \xrightarrow{\alpha} X_1 \\ \downarrow \\ \displaystyle\bigvee_{\alpha\in N} D^3 \end{array}\right)$$

to kill the elements of H_2 generated by the two-cells corresponding to $\mathbf{Z}^{\oplus N}$ in the exact sequence (1). By the Seifert-van Kampen theorem, $0 = \pi_1(X_1) \xrightarrow{\sim} \pi_1(X^+)$, since attaching higher dimensional cells does not affect π_1, hence X^+ has property (1). By the Eilenberg-Steenrod axioms, we also have the short exact sequence

$$0 \longrightarrow C_\bullet\left(\bigvee_{\alpha\in N} S^2\right) \longrightarrow C_\bullet(X_1) \longrightarrow C_\bullet(X^+) \longrightarrow 0$$

which maps the elements of $C_\bullet(\bigvee_{\alpha\in N} S^2)$ generating H_2 to the generators of $H_2(X_1)$ we want to kill in (1). The associated long exact sequence then gives that $H_i(X_1) \simeq H_i(X_+)$ for all $i \neq 2$, and for $i = 2$, we have the short exact sequence

$$0 \longrightarrow \mathbf{Z}^{\oplus N} \longrightarrow H_2(X_1) \longrightarrow H_2(X^+) \longrightarrow 0.$$

Thus, the composite map $X \to X_1 \to X^+$ is a plus construction, where we note we can use the Serre spectral sequence to deduce $C_\bullet(\mathrm{fib}(\varphi)) = C_\bullet(\mathrm{pt})$. □

Step 2. Now consider a general perfect normal subgroup $N \leqslant \pi_1(X)$, and let $\tilde{X}$ be the covering space of X corresponding to the subgroup N. Then, $\tilde{X} \to X$ is a $(\pi_1(X)/N)$-cover, so $\pi_1(\tilde{X}) = N$ is perfect. By the previous step, there exists a plus-construction $\tilde{X}^+$ for $\tilde{X}$. We then set

$$X^+ = \operatorname{colim}\left(\begin{array}{ll} \tilde{X} \hookrightarrow & \tilde{X}^+ \\ \downarrow & \\ X & \end{array}\right)$$

and we claim this defines a plus-construction for X. By the Seifert-van Kampen theorem,

$$\pi_1(X^+) = \pi_1(X) *_N 0 = \pi_1(X)/N.$$

By the Eilenberg-Steenrod axioms, the isomorphism $C_\bullet(\tilde{X}) \xrightarrow{\sim} C_\bullet(\tilde{X}^+)$ induces an isomorphism $C_\bullet(X) \xrightarrow{\sim} C_\bullet(X^+)$. Finally, to get that $\mathrm{fib}(\varphi)$ is $H_\bullet$-acyclic, we can use the same argument as before using $\mathscr{L} \in \mathrm{Loc}(X^+)$, a local system of coefficients on X^+. □

While homologically X^+ does not differ from X, its higher homotopy groups somewhat magically contain a wealth of information.

Example 1.4. Let R be a ring, $X = B\mathrm{GL}(R)$, and $N = \mathrm{E}(R)$, the subgroup of elementary matrices from before in Example 1.2(1). By Theorem 1.3, we then get a plus-construction $B\mathrm{GL}(R)^+$. Then, $\pi_1(B\mathrm{GL}(R)) = \mathrm{GL}(R)/\mathrm{E}(R) = K_1(R)$ by definition from last time. To calculate $\pi_2(B\mathrm{GL}(R)^+)$, we first note

$$\begin{tikzcd} B\mathrm{E}(R) \arrow[r] \arrow[d, "f"'] & B\mathrm{E}(R)^+ \arrow[d] \\ B\mathrm{GL}(R) \arrow[r] & B\mathrm{GL}(R)^+ \arrow[ul, phantom, "\lrcorner", very near start] \end{tikzcd}$$

is a pushout diagram in the category of based topological spaces. The issue is that while pushouts are useful for computing homology, we need to instead use pullbacks to compute homotopy. So instead, we replace $B\mathrm{E}(R)^+$ with another space $Y = B\mathrm{E}(R)^{+\prime}$, which is defined to be the iterated plus-construction relative to $\pi_1(B\mathrm{E}(R), x)$ for all $x \in f^{-1}(\text{base point})$. This gives an explicit construction of an $\mathrm{E}(R)$-covering space for $B\mathrm{GL}(R)^+$, hence we get the pullback diagram

$$\begin{tikzcd} B\mathrm{E}(R) \arrow[r] \arrow[d, "f"'] \arrow[dr, phantom, "\ulcorner", very near start] & Y \arrow[d, "f^+"] \\ B\mathrm{GL}(R) \arrow[r] & B\mathrm{GL}(R)^+ \end{tikzcd}$$

where both f and f^+ are covering spaces with group $\mathrm{E}(R)$. Note that $\pi_1(Y) = 0$ since it was obtained by attaching higher dimensional cells to $B\mathrm{E}(R)^+$. By the same argument as before, we also have $C_\bullet(B\mathrm{E}(R)) \xrightarrow{\sim} C_\bullet(Y)$. Since this diagram is a pullback, we get a map of fibre sequences

$$\begin{tikzcd} F(R) \arrow[r] \arrow[d, equal] & B\mathrm{E}(R) \arrow[r] \arrow[d] & Y \arrow[d] \\ F(R) \arrow[r] & B\mathrm{GL}(R) \arrow[r] & B\mathrm{GL}(R)^+ \end{tikzcd}$$

where the equality on the left denotes an isomorphism in the homotopy category. The corresponding maps on the long exact sequences for homotopy give the commutative diagram

$$\begin{tikzcd} \pi_2(F(R)) \arrow[r] \arrow[d, equal] & 0 \arrow[r] \arrow[d, equal] & \pi_2(Y) \arrow[r] \arrow[d] & \pi_1(F(R)) \arrow[r] \arrow[d, equal] & \mathrm{E}(R) \arrow[r] \arrow[d] & 0 \arrow[d] \\ \pi_2(F(R)) \arrow[r] & 0 \arrow[r] & \pi_2(B\mathrm{GL}(R)^+) \arrow[r] & \pi_1(F(R)) \arrow[r] & \mathrm{GL}(R) \arrow[r] & \mathrm{GL}(R)/\mathrm{E}(R) \end{tikzcd}$$

where $\pi_2(B\mathrm{E}(R)) = \pi_2(B\mathrm{GL}(R)) = 0$ since they are classifying spaces for discrete groups. By the five lemma, we then have $\pi_2(B\mathrm{GL}(R)^+) = \pi_2(Y)$, which is in turn isomorphic to $H_2(Y)$ by the Hurewicz theorem. But $H_2(Y) \simeq H_2(B\mathrm{E}(R)) = H_2(\mathrm{E}(R), \mathbf{Z})$ by construction, which is isomorphic to $K_2(R)$ as we saw last time, since the extension

$$0 \longrightarrow K_2(R) \longrightarrow \mathrm{St}(R) \longrightarrow \mathrm{E}(R) \longrightarrow 0$$

is the universal central extension of $\mathrm{E}(R)$. Note in particular that, looking at the top row of the diagram above, we have $\pi_1(F(R)) \simeq \mathrm{St}(R)$.

We therefore define the following:

Definition 1.5. We let $K(R) \coloneqq B\mathrm{GL}(R)^+ \times K_0(R)$, where $K_0(R)$ has the discrete topology, and $K_i(R) \coloneqq \pi_i(K(R))$.

Remark. $K(R)$ is an H-space, and is even a group object in the homotopy category, as well as an infinite loop space. The fact that $K(R)$ is a group object shouldn't be too surprising, seeing as a good generalization of K-theory should have some analogue of the direct sum operation on projective modules we saw from last time.

We now mention the following example due to Quillen, which was part of the original motivation to define higher K-groups.

Example 1.6 (Quillen)**.** Let $R = \mathbf{F}_q$. Quillen found the fibre sequence

$$BGL(\mathbf{F}_q)^+ \longrightarrow BU \xrightarrow{1-\psi^q} BU$$

where U is the infinite unitary group, and ψ^q is the qth Adams operation. The higher homotopy groups of BU are controlled by Bott periodicity, and so Quillen calculated the higher K-groups for $\mathbf{F}_q$ as follows:

$$K_i(\mathbf{F}_q) = \begin{cases} \mathbf{Z}/(q^n - 1) & \text{for } i = 2(n-1) \\ 0 & \text{for } i = 2n \\ \mathbf{Z} & \text{for } i = 0 \end{cases}$$

We return to our other main example.

Example 1.7. Let $X = B\Sigma_\infty$, and $N = A_\infty$. We have the following

Theorem (Baratt-Priddy-Quillen)**.** $X^+ \simeq \Omega^\infty(\Sigma^\infty(S^0)) = \operatorname{colim} \Omega^n(\Sigma^n(S^0))$.

This implies $\pi_i(X^+) = \pi_i^{\text{st}}$. In particular, $\pi_1(X^+) = \pi_1^{\text{st}} = \mathbf{Z}/2$ from the Hopf fibration, as expected. This can be interpreted as follows. First, there is a natural map

$$\begin{array}{ccc} \Sigma_\infty & \longrightarrow & GL(\mathbf{Z}) \\ \cup\!| & & \cup\!| \\ A_\infty & \longrightarrow & E(R) \end{array}$$

so there is an induced map on plus-constructions $X^+ \to BGL(\mathbf{Z})^+$. Then, we get the commutative diagram

$$\begin{array}{ccc} \pi_1(X^+) & \longrightarrow & \pi_1(BGL(\mathbf{Z})^+) \\ \| & & \| \\ \pi_1^{\text{st}} & & K_1(\mathbf{Z}) \\ \| & & \| \\ \mathbf{Z}/2 & = & \mathbf{Z}/2 \end{array}$$

In this way, we can get elements of K-groups from stable homotopy groups of spheres.

Before we move on to the Q-construction, we note a few defects of the plus-construction:
(1) $\pi_0(K(R)) = K_0$, but $\pi_0(K(R))$ does not have a natural group structure.
(2) $K(R)$ hence $K_i(R)$ is only defined for rings R—how would we define K-groups for $\mathbf{P}^1$, for example?
(3) There are no exact sequences relating the different K_i's.

This motivates the Q-construction, which gives another construction for higher K-groups that ends up matching the plus-construction from before.

2. The Q-construction

2.1. **Classifying spaces.** In the plus-construction, we used repeatedly the fact that to a group G, we can associate a classifying space BG such that $\pi_1(BG) = G$. This is in fact a special case of a more general construction: a group is a special case of a groupoid, and a groupoid is in turn a special case of a category. Our first goal is to define a space $N\mathscr{C}$ to a category $\mathscr{C}$ that generalizes the construction of BG.

Definition 2.1. We define a functor

$$\begin{aligned} N\colon \{\text{categories}\} &\longrightarrow \{\text{simplicial sets}\} \\ \mathscr{C} &\longmapsto N\mathscr{C} \end{aligned}$$

associating a category $\mathscr{C}$ with its *nerve* $N\mathscr{C}$, as follows. The n-simplices are defined to be collections of "n-composable morphisms," that is

$$\begin{aligned} N\mathscr{C}_0 &= \mathsf{Fun}([0], \mathscr{C}) = \mathsf{obj}(\mathscr{C}) \\ N\mathscr{C}_1 &= \mathsf{Fun}([0 \to 1], \mathscr{C}) = \mathsf{ar}(\mathscr{C}) \\ &\ \vdots \\ N\mathscr{C}_n &= \mathsf{Fun}([0 \to 1 \to \cdots \to n], \mathscr{C}) \end{aligned}$$

where $[n] \coloneqq [0 \to 1 \to \cdots \to n]$ denotes the category with $n+1$ objects with the specified arrows between the objects, together with face and degeneracy maps $N\mathscr{C}_n \to N\mathscr{C}_m$ coming from the functors $[m] \to [n]$. By composing with the *geometric realization* functor $|-|\colon \{\text{simplicial sets}\} \to \{\text{spaces}\}$, we also denote the geometric realization of this simplicial set as $N\mathscr{C}$ when no confusion can arise.

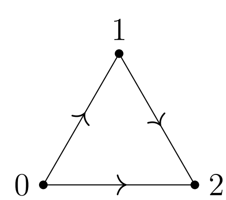
Remark. When visualizing a nerve, it is often useful to instead picture the geometric realization of that nerve; however, such a realization forgets some of the information contained in the face and degeneracy maps of the nerve. In low-dimensional cases, we can at least label vertices and edges with their corresponding objects and arrows, as we do below.

Example 2.2.

(1) The nerve associated to $[1]$ is

$$N([1]) = N([0 \to 1]) = \overset{0}{\bullet} \longrightarrow \overset{1}{\bullet} = I$$

In fact, $N([n]) = N([0 \to 1 \to \cdots \to n])$ is the n-simplex. For example, when $n = 2$ we get the 2-simplex



(2) If $\mathscr{C} = BG = [\mathrm{pt}/G]$, the *category* BG, then the nerve $N\mathscr{C}$ is the *space* BG (Borel).

We have the following (functorial) properties of $N(-)$.

Property 1. $N(-)$ commutes with products: $N(\mathscr{C}_1 \times \mathscr{C}_2) = N\mathscr{C}_1 \times N\mathscr{C}_2$.

Example 2.3. $N(\mathscr{C} \times [0 \to 1]) = N\mathscr{C} \times I$.

Property 2. If F and G are both functors $\mathscr{C} \to \mathscr{D}$, and $\eta\colon F \Rightarrow G$ is a natural transformation of these functors, then $\eta_*\colon F_* \xrightarrow{\sim} G_*$, i.e., η induces a homotopy equivalence between F_* and G_* as maps $N\mathscr{C} \to N\mathscr{D}$.

Proof. Giving a natural transformation η is equivalent to giving a functor $\tilde{\eta}\colon \mathscr{C} \times [0 \to 1] \to \mathscr{D}$ such that $\tilde{\eta}|_{\mathscr{C}\times[0]} = F$ and $\tilde{\eta}|_{\mathscr{C}\times[1]} = G$. Thus, $\tilde{\eta}$ gives a homotopy $F_* \simeq G_*$. □

Property 3. If $F\colon \mathscr{C} \rightleftharpoons \mathscr{D} :G$ is an adjoint pair of functors, then F_* and G_* are mutually inverse homotopy equivalences.

Proof. Given an adjoint pair of functors, we get two natural transformations: the unit $\mathrm{id}_{\mathscr{C}} \Rightarrow G \circ F$ and the counit $F \circ G \Rightarrow \mathrm{id}_{\mathscr{D}}$. Using Property 2, we are done. □

In particular, we have the following

Corollary 2.4. If $\mathscr{C}$ has an initial or final object, then $N\mathscr{C}$ is contractible.

Thus, most categories we consider normally have uninteresting nerves.

Remark. The proof above only needed the existence of natural transformations $\mathrm{id}_{\mathscr{C}} \Rightarrow G \circ F$ and $F \circ G \Rightarrow \mathrm{id}_{\mathscr{D}}$, and not anything about how they compose with F or G which is part of the definition for an adjoint pair. This is one way in which a lot of information is lost when working with nerves of categories instead of the categories themselves.

Property 4. If $\mathscr{C}$ is a *filtered* category, then $N\mathscr{C}$ is contractible.

Proof. We have the following equivalence of categories:

$$\operatorname*{colim}_{x\in\mathscr{C}} \mathscr{C}/x \xrightarrow{\sim} \mathscr{C}$$

where $\mathscr{C}/x$ is the *slice category* formed by taking objects and morphisms to be over a fixed object $x \in \mathscr{C}$. The colimit above is filtered since $\mathscr{C}$ is filtered. Then, we have that

$$N\mathscr{C} \cong N\left(\operatorname*{colim}_{x\in\mathscr{C}} \mathscr{C}/x\right) \cong \operatorname*{colim}_{x\in\mathscr{C}} N(\mathscr{C}/x)$$

since functors $[n] \to \mathscr{C}$ factor through some $\mathscr{C}/x$ by the fact that $\mathscr{C}$ is filtered. Finally, filtered colimits are exact, hence since $N(\mathscr{C}/x)$ is contractible for any $x \in \mathscr{C}$ by Corollary 2.4, we have that $N\mathscr{C}$ is contractible. □

Property 5. For a category $\mathscr{C}$, associate to it another category $G(\mathscr{C})$, the *universal groupoid under* $\mathscr{C}$, defined to be $\mathscr{C}[\mathsf{ar}(\mathscr{C})^{-1}]$, i.e., the category obtained by formally inverting every arrow in $\mathscr{C}$. Then, we have the following

Theorem 2.5. *There is an equivalence of categories*

$$\left\{\begin{matrix}\text{covering spaces}\\ \text{of } N\mathscr{C}\end{matrix}\right\} \simeq \mathsf{Fun}(G(\mathscr{C}), \mathsf{Sets}) = \mathsf{Fun}(\mathscr{C}, \mathsf{Sets}^{\simeq}),$$

where $\mathsf{Sets}^{\simeq}$ *denotes the category of sets with isomorphisms. In particular, if* $\mathscr{C}$ *is connected and* $x \in \mathscr{C}$*, then* $\{\pi_1(N\mathscr{C}, x)\text{-}\mathsf{Sets}\} \simeq \mathsf{Fun}(G(\mathscr{C}), \mathsf{Sets})$.

Proof idea. $\to$. Let $f\colon Y \to N\mathscr{C}$ be a covering space. We then get a functor

$$\begin{aligned} F_Y\colon \mathscr{C} &\longrightarrow \mathsf{Sets}^{\simeq} \\ Z &\longmapsto f^{-1}(Z) \\ (Z_1 \to Z_2) &\longmapsto (f^{-1}(Z_1) \to f^{-1}(Z_2)) \end{aligned}$$

where the arrow on the right is given by the fact that a path in $N\mathscr{C}$ gives rise to a unique path in Y by covering space theory; the uniqueness of this path implies the association above indeed defines a functor.

$\leftarrow$. Let a functor $F\colon \mathscr{C} \to \mathsf{Sets}^{\simeq}$ be given. Then, consider the category

$$\mathscr{C}/F := \left\{ (X,x) \;\middle|\; \begin{array}{c} X \in \mathscr{C} \\ x \in F(X) \end{array} \right\}$$

with arrows being the obvious maps induced by F. We then have the forgetful functor $\mathscr{C}/F \to \mathscr{C}$, which induces a map $N(\mathscr{C}/F) \to N\mathscr{C}$ of nerves; this map is a covering space. $\square$

Remark. The construction $\mathscr{C}/F$ above is due to Grothendieck and was first defined in the context of stacks.

We will continue our discussion of the Q-construction next time.