CLASSICAL K-THEORY

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ABSTRACT. Unedited live-texed notes from the first meeting of the algebraic K-theory seminar at UMich, Winter 2015. Note taker was Cameron Franc.

1. K_0 OF A RING

Let R be an associative ring with a unit. Recall that a (left) R-module is *projecitve* if there exists a module Q such that $P \oplus Q$ is free. Equivalently, for every diagram

$$\begin{array}{ccc} & & P \\ & & \downarrow \\ M & \longrightarrow & N \end{array}$$

with $M \to N$ surjective, there exists a map $P \to M$ making a commutative triangle. Equivalently, $\mathrm{Ext}^1_R(P, \bullet) = 0$.

The set of isomorphism classes of finitely generated projective R-modules has the structure of an abelian monoid under direct sum. Then $K_0(R)$ is the group completion of this monoid. That is, it is the free-abelian group on the isomorphism classes of finitely generated projective R-modules mod the obvious relations: $[P \oplus Q] - [P] - [Q]$. It is not hard to check that $[P] = [P']$ in $K_0(R)$ if and only if there exists a finitely generated projective R-module Q such that $P \oplus Q \cong P' \oplus Q$. Even more conretely, this is equivalent to having $P \oplus R^n \cong P' \oplus R^n$ for some $n \geq 0$.

Example 1. If R is a local ring then $K_0(R) = \mathbf{Z}$ since finitely generated projective R-modules are free.

Example 2. If R is a PID then $K_0(R) = \mathbf{Z}$ by the classification of finitely generated modules over a PID.

Example 3. If R is a Dedekind domain then $K_0(R) = \mathbf{Z} \oplus \mathrm{Cl}(R)$. To see this, note that a finitely generated projective module over R breaks up into a direct sum of fractional ideals (prove this by induction). Next note that $I_1 \oplus I_2 \cong R \oplus I_1 I_2$, so that if P is finitely generated and projective, $P \cong R^{n-1} \oplus I$ for some ideal I. This decomposition yields the decomposition of $K_0(R)$.

Example 4. Eilenberg-Mazur Swindle: Let R^∞ be an infinitely generated free module. If $P \oplus Q \cong R^n$ then

$$P \oplus R^\infty \cong (P \oplus Q) \oplus (P \oplus Q) \oplus \cdots = R^\infty$$

Thus, if we allow non-finitely generated modules then we obtain a trivial K_0.

We record the following properties:

- K_0 is a covariant functor from rings to abelian groups;
- it respects finite direct products and filtered direct limits.

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2. K_0 OF AN EXACT CATEGORY

Definition 5. An *exact category* is a pair $(\mathcal{C}, \mathcal{E})$ where $\mathcal{C}$ is an additive category that is a full subcategory of an abelian category $\mathcal{A}$, and $\mathcal{E}$ is the family of sequences in $\mathcal{C}$ of the form $0 \to B \to C \to D \to 0$ that are exact in $\mathcal{A}$. Further, we assume that if B and D lie in $\mathcal{C}$, then C is also in $\mathcal{C}$ (that is, $\mathcal{C}$ is closed under extensions). We'll usually say $\mathcal{C}$ is an exact category unless we wish to specify the exact sequences in $\mathcal{E}$.

Definition 6. Let $\mathcal{C}$ be a small exact category. Then $K_0(\mathcal{C})$ is the abelian group generated by the objects of $\mathcal{C}$, and with relations given by the exact sequences.

Example 7. Let $\mathcal{C}$ be the category of finitely generated projective R-modules contained in the category of all R-modules. Then $K_0(\mathcal{C}) = K_0(R)$ because exact sequences of projective modules split.

Example 8. Let X be a quasi-projective scheme over a commutative ring R. Then one is interested in $K_0(\mathrm{VB}(X)) = K_0(X)$, where $\mathrm{VB}(X)$ is the exact category of vector bundles on X, which is a full subcategory of the category of quasicoherent modules on X.

Remark 9. If X is a noetherian scheme, then let $G_0(X) = K_0(\mathrm{Coh}(X))$. There exists a morphism

$$K_0(X) \to G_0(X)$$

called the *Cartan homomorphism*, which Serre proved is an isomorphism when X is regular and quasiprojective over a noetherian ring.

Example 10. One can show that $K_0(\mathbf{P}^1) = \mathbf{Z}^2$. More generally, there is a surjective map

$$\mathrm{rk} \oplus \det \colon K_0(X) \to H^0(X, \mathbf{Z}) \oplus \mathrm{Pic}(X)$$

which is an isomorphism for nonsingular curves.

3. K_1 OF A RING

As above, R is an associative ring with unit.

Definition 11. First define $\mathrm{GL}(R) = \varinjlim \mathrm{GL}_n(R)$ where $\mathrm{GL}_n(R) \to \mathrm{GL}_{n+1}(R)$ is defined by $g \mapsto \left(\begin{smallmatrix} g & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Then $K_1(R) := \mathrm{GL}(R)/[\mathrm{GL}(R), \mathrm{GL}(R)]$.

Note that $K_1(R)$ is an abelian group. It satisfies the following universal property: every homomorphism $\mathrm{GL}(R) \to A$ factors through $K_1(R)$. As with K_0, the association $R \mapsto K_1(R)$ is functorial.

In order to understand $K_1(R)$ it's useful to get a grip first on the commutator subgroup of $\mathrm{GL}(R)$. Define $E_n(R) \subseteq \mathrm{GL}_n(R)$ to be the group generated by the elementary matrices $e_{ij}(r)$ where $r \in R$, $i \neq j$, and $e_{ij}(r)$ is the usual matrix with all entries 0 save for the (i, j)th, which contains r, and the diagonal entries, which are 1. Set $E(R) = \varinjlim_n E_n(R)$. This is the same as the group generated by the images of the $e_{ij}(r)$ in $\mathrm{GL}(R)$.

Lemma 12 (Whitehead)**.** *One has* $E(R) = [\mathrm{GL}(R), \mathrm{GL}(R)]$.

Proof. First show that $E(R) = [E(R), E(R)]$ by proving the same thing for $E_n(R)$ for $n \geq 3$ using a bunch of identities which we'll write down:

$$e_{ij}(r)e_{ij}(s) = e_{ij}(r+s)$$
$$[e_{ij}(r), e_{kl}(s)] = \begin{cases} 1 & j \neq k, i \neq l \\ e_{ik}(rs) & i \neq k. \end{cases}$$

If A, $B \in \mathrm{GL}_n(R)$ then

$$\begin{pmatrix} ABA^{-1}B^{-1} & 0 \\ 0 & 1_n \end{pmatrix} = \begin{pmatrix} AB & 0 \\ 0 & (AB)^{-1} \end{pmatrix} \begin{pmatrix} A^{-1} & 0 \\ 0 & A \end{pmatrix} \begin{pmatrix} B^{-1} & 0 \\ 0 & B \end{pmatrix}.$$

For any $M \in \mathrm{GL}_n(R)$,

$$\begin{pmatrix} M & 0 \\ 0 & M^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ M^{-1}-1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ M-1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -M^{-1} \\ 0 & 1 \end{pmatrix},$$

and each of these is in $E_n(R)$. So $K_1(R) = \mathrm{GL}(R)/E(R)$. □

Definition 13. Define $\mathrm{SL}(R) = \varinjlim \mathrm{SL}_n(R)$.

One has $\mathrm{GL}(R) = \mathrm{SL}(R) \rtimes R^\times$ and there is a determinant map $\det\colon K_1(R) \to R^\times$. Define $\mathrm{SK}_1(R)$ to be the kernel of this map.

Example 14. If F is a field then $K_1(F) = F^\times$. To see this, note that Dickson showed $\mathrm{SL}_n(F) = [\mathrm{GL}_n(F), \mathrm{GL}_n(F)]$ in 1899 except for two specific cases. Or, use elementary row operators to show $E_n(F) = \mathrm{SL}_n(F)$ for all $n \geq 1$.

Example 15. In 1941, Dieudonne proved that if D is a division ring then $K_1(D) = D^\times/[D^\times, D^\times]$. This isomorphism is given by the so-called Dieudonne determinant $\mathrm{GL}(D) \to (D^\times)/[D^\times, D^\times]$.

Example 16. Since GL commutes with products, one can show $K_1(R_1 \times R_2) = K_1(R_1) \oplus K_1(R_2)$.

Example 17. One has $\mathrm{GL}(R) \cong \mathrm{GL}(M_n(R))$ so that $K_1(R) \cong K_1(M_n(R))$.

Example 18. Bass-Milnor-Serre proved that if R is euclidean or a maximal order in a number field K, then $\mathrm{SK}_1(R) = 0$ and $K_1(R) = K^\times$.

Given $a, b \in R$ with $(a, b) = 1$, choose b and c so that $ad - bc = 1$. Then let $[a, b]$ denote the class of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $\mathrm{SK}_1(R)$. This is well-defined and one has the relations

- $[a, b] = [b, a]$,
- $[a, b] = 1$ for all $b \in R$ if $a \in R^\times$,
- $[a_1a_2, b] = [a_1, b][a_2, b]$
- $[a, b] = [a + rb, b]$ for all $r \in R$.

These symbols generate $\mathrm{SK}_1(R)$ under certain conditions (e.g. R is noetherian of dimension ≤ 1, plus more).

4. K_2 OF A RING

Definition 19. Let R be an associataive unital ring and let $n \geq 3$ be an integer. The Steinberg group $\mathrm{St}_n(R)$ is genereated by symbols $x_{ij}(r)$ with $1 \leq i \lneq j \leq n$, $r \in R$, modulo the relations $x_{ij}(r)x_{ij}(s) = x_{ij}(r+s)$ and

$$[x_{ij}(r), x_{kl}(s)] = \begin{cases} 1 & j \neq k, i \neq l, \\ x_{ij}(rs) & j = k, i \neq l, \\ x_{kl}(-sr) & j \neq k, i = l. \end{cases}$$

There exists a map $\phi_n\colon \mathrm{St}_n(R) \to E_n(R)$ sending $x_{ij}(r)$ to $e_{ij}(r)$. Define $\mathrm{St}(R) = \varinjlim \mathrm{St}_n(R)$. There is a natural map $\phi = \varinjlim \phi_n\colon \mathrm{St}(R) \to E(R)$. Set $K_2(R) = \ker \phi$.

There exists an exact sequence

$$1 \to K_2(R) \to \mathrm{St}(R) \to \mathrm{GL}(R) \to K_1(R).$$

One can show:

Theorem 20. *The group $K_2(R)$ is the center of $\mathrm{St}(R)$. In particular, $K_2(R)$ is abelian.*

Proof. If $x \in Z(\mathrm{St}(R))$ then $\phi(x) \in Z(E(R))$, so $x \in \ker \phi$. Let $x \in K_2(R)$, so that $x \in \mathrm{St}(R)$ and $\phi(x) = 1$. Note that for all elements $y \in \mathrm{St}(R)$ we have $\phi([x, y]) = 1$. Choose a large n such that x can be written as a word in $x_{ij}(r)$s with $i, j < n$. Then for all $y = x_{kn}(s)$ with $k < n$, the Steinberg relations give allow one to write $[x, y]$ as a word in $x_{in}(r)$s for $i < n$. But the subgroup generated by the $x_{in}(r)$s wth $i < n$ maps *injectively* by ϕ into $E(R)$. Since $\phi([x, y]) = 1$, it follows that $[x, y] = 1$. Hence x commutes with all $x_{kn}(s)$s with $k < n$. An analogous argument allows one to show it commutes with all $x_{nk}(s)$. Then relations also allow one to show that x commutes with $x_{ij}(s)$ with both $i, j < n$. This is enough to show that x commutes with everything (make n even larger if necessary). □

Example 21. One can show $K_2(\mathbf{Z}) = \mathbf{Z}/2\mathbf{Z}$, $K_2(\mathbf{Z}[\sqrt{-7}]) = \mathbf{Z}/2\mathbf{Z}$, $K_2(\mathbf{Z}[i]) = 1$, and there are a bunch of other examples in the literature. But in general it's pretty hard to compute $K_2(R)$.

Remark 22. The extension

$$0 \to K_2(R) \to \mathrm{St}(R) \to E(R) \to 0$$

is the universal central extension of $E(R)$. Thus $K_2(R) = H_2(E(R), \mathbf{Z})$.

5. Products

If R is a *commutative* ring then one has a product map $K_0(R) \otimes_{\mathbf{Z}} K_0(R) \to K_0(R)$, and similarly for $K_1(R)$ and $K_2(R)$[1]. One even has a map

$$K_1(R) \otimes_{K_0(R)} K_1(R) \to K_2(R)$$

Map $g \otimes h \mapsto \{g, h\}$ as follows. First, suppose that $\alpha, \beta \in E(R)$ commute. Then define a product $\alpha \star \beta \in K_2(R)$ by setting $\alpha \star \beta = [\tilde{\alpha}, \tilde{\beta}]$ where $\tilde{\alpha}$, $\tilde{\beta}$ are lifts of α and β in $\mathrm{St}(R)$. Now regard $g \in \mathrm{GL}_n(R)$ and $h \in \mathrm{GL}_m(R)$. Then define

$$\{g, h\} = \begin{pmatrix} g \otimes 1_m & 0 & 0 \\ 0 & g^{-1} \otimes 1_m & 0 \\ 0 & 0 & 1_{mn} \end{pmatrix} \star \begin{pmatrix} 1_n \otimes h & 0 & 0 \\ 0 & 1_{mn} & 0 \\ 0 & 0 & 1_n \otimes h^{-1} \end{pmatrix}$$

[1]The existence of these maps is not obvious, although if one realizes $K_1(R)$ as K_0 of projective modules along with an automorphism, then the product in K_1 is given by tensor product.

Theorem 23 (Matsumoto)**.** *Let F be a field. Then $K_2(F)$ is the free abelian group on the symbols $\{a,b\}$ with $a,b \in F^\times$ subject to the relations*

$$\{a_1a_2,b\} = \{a_1,b\}\{a_2,b\},$$
$$\{a,b\} = \{b,a\}^{-1},$$
$$\{a,1-a\} = 1.$$

That is, $K_2(F) \cong F^\times \otimes F^\times / \langle a \otimes (1-a) \rangle$.

Corollary 24. *One has $K_2(\mathbf{F}_q) = 1$.*

Proof. Let x be a generator of $\mathbf{F}_q^\times$. If q is even then $\{x,x\} = \{x,-x\} = 1$. If q is odd then $\{x, xx^{\frac{q-1}{2}}\} = \{x,-x\} = 1$. So $\{x,x\}$ has order 1 or 2 by skew symmetry. The set $\mathbf{F}_q^\times - \{1\}$ is invariant under $z \mapsto 1-z$. It contains $\frac{q-1}{2}$ nonsquares and $\frac{q-3}{2}$ squares. Thus, there exists a nonsquare z such that $1-z$ is also nonsquare. Write $z = x^i$ and $1-z = x^j$ for odd i and j. Then one checks that $1 = \{z, 1-z\} = \{x,x\}^{ij} = \{x,x\}$ since ij is odd. □

Remark 25. Milnor K-theory is defined using the tensor algebra and the Steinberg symbols. For a field F one checks that $K_0^M(F) = \mathbf{Z}$, $K_1^M(F) = F^\times$ and $K_2^M(F) = K_2(F)$.

Remark 26. The theorem of Merkurjev-Suslin says that $K_2(F)/nK_2(F)$ is the n-torsion in the Brauer group of F.