INTERSECTION THEORY ON ARITHMETIC SURFACES

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1. ARITHMETIC SURFACES

Fix a Dedekind domain[1] R with fraction field K, and set $S = \operatorname{Spec}(R)$.

Definition 1. An ARITHMETIC SURFACE over S is an integral, normal and excellent scheme $\mathcal{C} \to S$ that is flat and of finite type over S, whose generic fiber is a smooth connected curve over K, and whose special fibers are (possibly reducible, nonreduced and/or nonsmooth) curves over the corresponding residue fields.

Normality ensures that $\mathcal{C}$ is regular in codimension 1, that is, the singularities of $\mathcal{C}$ should be "honest points". Our assumptions imply that at most a finite number of the special fibers could be singular. Properness ensures that all fibers are projective curves over the residue fields.

Example 2. Let R denote the localisation of $\mathbf{Z}$ at a prime ideal (p). Thus,

$$R = \left\{ \frac{a}{b} \in \mathbf{Q} \mid p \nmid b \right\},$$

and $K = \mathbf{Q}$. Let $\mathcal{C} = \operatorname{Spec}(R[x,y]/(xy-p))$. This is regular in codimension one. The generic fiber is $\operatorname{Spec}(\mathbf{Q}[x,y]/(xy-p))$, which is a nonsingular curve over $\mathbf{Q}$. The special fiber is $\operatorname{Spec}(\mathbf{F}_p[x,y]/(xy))$, which is singular at the origin. Note that the special fiber also has two irreducible components. This surface is not proper over R. If we had instead taken $\mathcal{C} = \operatorname{Proj}(R[x,y,z]/(xy-pz^2))$, then we would have obtained a regular and proper arithmetic surface over R.

2. WEIL DIVISORS

Let us now fix $\mathcal{C} \to S$ a proper arithmetic surface that is regular in codimension one. In this section we recall the theory of Weil divisors on $\mathcal{C}$, cf. [1].

Definition 3. A PRIME DIVISOR on $\mathcal{C}$ is a closed integral subscheme of codimension one. Let $\operatorname{Div}(\mathcal{C})$ denote the free abelian group generated by the prime divisors. Then $\operatorname{Div}(\mathcal{C})$ is the (WEIL) DIVISOR GROUP of $\mathcal{C}$, and elements of $\operatorname{Div}(\mathcal{C})$ are called (WEIL) DIVISORS.

In this note we only deal with Weil divisors, so we'll simply refer to them as divisors. A divisor D is said to be EFFECTIVE if it can be expressed in the form $D = \sum_i n_i Y_i$ for integers $n_i \geq 0$ and prime divisors Y_i.

Definition 4. Let $\mathcal{C} = \operatorname{Spec}(\mathbf{Z}[x])$ and let $R = \mathbf{Z}$, so that $\mathcal{C}$ is the affine line over $\mathbf{Z}$. It looks like a plane made up of all the affine lines over the possible base fields, fibered over $\operatorname{Spec}(\mathbf{Z})$ in the obvious way. Let $Y_1 = \operatorname{Spec}(\mathbf{Z}[x]/(p)) \cong \operatorname{Spec}(\mathbf{F}_p[x])$ for

[1]That is, R is a field, or a noetherian ring such that the localisation at each maximal ideal is a DVR.

some prime number p, and let $Y_2 = \mathrm{Spec}(\mathbf{Z}[x]/(x)) \cong \mathrm{Spec}(\mathbf{Z})$. These are both prime divisors on $\mathcal{C}$. The divisors Y_1 is an example of a *fibral divisor*, while Y_2 is *horizontal*.

Recall that an integral scheme has a unique generic point, which is a dense point corresponding to the zero ideal in any affine open subset of the scheme. Let $Y \subseteq \mathcal{C}$ be a prime divisor and let $\eta \in X$ be the generic point of Y. Since $\mathcal{C}$ is regular and Y is of codimension 1, the localisation $\mathcal{O}_{\mathcal{C},\eta}$ is a regular local ring of dimension 1. In particular, it is an integrally closed noetherian local ring of Krull dimension 1, which is one of the equivalent conditions of being a DVR. Let K denote the function field of $\mathcal{C}$. This can be realized as the fraction field of $\mathcal{O}_{\mathcal{C},\eta}$. Thus, Y gives rise to a valuation v_Y on K.[2] It is known (cf. Lemma 6.1 of II.6 in [1]) that if $f \in K^\times$ is nonzero, then $v_Y(f) \neq 0$ for only finitely many prime divisors Y on $\mathcal{C}$.

Example 5. Again take $\mathcal{C}$ to be the affine line over $\mathbf{Z}$, where $Y_1 = \mathrm{Spec}(\mathbf{F}_p[x])$, and where $Y_2 = \mathrm{Spec}(\mathbf{Z}) \subseteq \mathcal{C}$. In this case $K = \mathrm{Frac}(\mathbf{Z}[x]) = \mathbf{Q}(x)$, and any $f \in K^\times$ can be written in the form $f = p^n a(x)/b(x)$ where $n \in \mathbf{Z}$ and $a(x)$ and $b(x)$ are polynomials over $\mathbf{Z}$, each of which has at least one coefficient that is not divisible by p. The integer n is uniquely determined by f and $n = v_{Y_1}(f)$. There is a similar description for v_{Y_2}.

Definition 6. If K is the fraction field of $\mathcal{C}$ and $f \in K^\times$, then $\mathrm{div}(f)$ denotes the PRINCIPAL WEIL DIVISOR defined by

$$\mathrm{div}(f) = \sum_{\text{prime } Y \in \mathrm{Div}(\mathcal{C})} v_Y(f) \cdot Y \in \mathrm{Div}(\mathcal{C}).$$

Mapping nonzero rational functions to their divisors defines a homomorphism $K^\times \to \mathrm{Div}(\mathcal{C})$, and the DIVISOR CLASS GROUP of $\mathcal{C}$, denoted $\mathrm{Cl}(\mathcal{C})$, is the cokernel of this map. Two divisors with the same image in the class group are said to be LINEARLY EQUIVALENT. In other words, divisors D_1 and D_2 are linearly equivalent if and only if $D_1 - D_2 = (f)$ for some nonzero $f \in K^\times$.

3. LOCAL INTERSECTION MULTIPLICITY

Let K denote the fraction field of $\mathcal{C}$, let $x \in \mathcal{C}$ be a closed point, and let Y be a prime divisor on $\mathcal{C}$.

Definition 7. A LOCAL UNIFORMIZER for Y at x is a function $f \in \mathcal{O}_{\mathcal{C},x}$ such that $v_Y(f) = 1$ and $v_{Y'}(f) = 0$ for all prime divisors $Y' \neq Y$ with $x \in Y'$.

Remark 8. Note that if $x \notin Y$ and if f is a local uniformizer for Y at x, then $v_{Y'}(f) = 0$ for all prime divisors Y' containing x. It follows that f is a unit in $\mathcal{O}_{\mathcal{C},x}$. (*Proof*: if f is not a unit, Krull's principal ideal theorem says that (f) is height one, so there's a minimal associated prime of (f) of height one. It defines a divisor Y' containing x with $v_{Y'}(f) \neq 0$.) One can show similarly the a local uniformizer is well-defined up to a unit.

Why do local uniformizers exist? If $x \notin Y$ then we're free to take $f = 1$. If we assume that $\mathcal{C}$ is regular, then for $x \in Y$ we appeal to the fact that $\mathcal{O}_{\mathcal{C},x}$ is a regular local ring and thus a UFD (Auslander-Buchsbaum theorem), so that the ideal defining Y in $\mathcal{O}_{\mathcal{C},x}$ is principal. A generator for this ideal will be a uniformizer. Silverman gives an argument in [4] assuming only normality, but I wasn't able to follow it. I think we'll be working with regular surfaces though, so this isn't a huge issue.

[2] Since $\mathcal{C}$ is proper and hence separated, the closed integral subscheme Y is uniquely determined by the valuation v_Y.

Definition 9. Let $\pi\colon \mathcal{C} \to \operatorname{Spec}(R)$ be an arithmetic surface. Let x be a closed point of $\mathcal{C}$, and let Y_1 and Y_2 be two prime divisors of $\mathcal{C}$. Then the LOCAL INTERSECTION MULTIPLICITY of Y_1 and Y_2 at x is the quantity

$$(Y_1 \cdot Y_2)_x := \operatorname{length}_{\mathcal{O}_{S,\pi(x)}} \mathcal{O}_{\mathcal{C},x}/(f_1, f_2)$$

where f_1 and f_2 denote uniformizers for Y_1 and Y_2, respectively, at x.

Remark 10. If $x \notin Y_1 \cap Y_2$ then at least one of f_1 or f_2 is a unit in $\mathcal{O}_{\mathcal{C},x}$ and $(Y_1 \cdot Y_2)_x = 0$.

Example 11. Again take $\mathcal{C} = \operatorname{Spec}(\mathbf{Z}[x])$, let $Y_1 = \operatorname{Spec}(\mathbf{F}_p[x])$ and let $Y_2 = \operatorname{Spec}(\mathbf{Z}[x]/(x))$. These should intersect to order 1 at the closed point $(p, x) \in \mathcal{C}$. But note that p is a local uniformizer for Y_1 and x is a local uniformizer for Y_2. We see that

$$(Y_1 \cdot Y_2)_{(p,x)} = \operatorname{length}_{\mathbf{Z}_{(p)}} \mathcal{O}_{\mathbf{Z}[x],(p,x)}/(p, x) = \dim_{\mathbf{F}_p} \mathbf{F}_p = 1$$

as expected.

Example 12. Let's try a more arithmetic example. Once again set $\mathcal{C} = \operatorname{Spec}(\mathbf{Z}[x])$, and let $Y_1 = \operatorname{Spec}(\mathbf{F}_p[x])$ for some *odd* prime p. Let Y_2 be the divisor defined by the prime ideal $(x^2 + 1)$, so that $Y_2 = \operatorname{Spec}(\mathbf{Z}[x]/(x^2 + 1)) = \operatorname{Spec}(\mathbf{Z}[i])$. If $p \equiv 1 \pmod 4$ then $x^2 + 1 \equiv (x - a)(x - b) \pmod p$ for two distinct integers a and b. Consider the closed point $(p, x - a) \in \mathcal{C}$. Note that as an ideal in $\mathcal{O}_{\mathcal{C},(p,x-a)}$, we have $(p, x^2 + 1) = (p, x - a)$, since $x - b$ is a unit in the localisation at $(p, x - a)$ (here we use that $a \neq b$!). To compute the local intersection multiplicity we use p and $x^2 + 1$ as uniformizers and deduce that

$$(Y_1 \cdot Y_2)_{(p,x-a)} = \dim_{\mathbf{F}_p} \mathbf{Z}[x]_{(p,x-a)}/(p, x - a) = 1.$$

Similarly $(Y_1 \cdot Y_2)_{(p,x-b)} = 1$. On the other hand, if $p \equiv 3 \pmod 4$, then $(p, x^2 + 1)$ is a closed point of $\mathcal{C}$. We deduce that

$$(Y_1 \cdot Y_2)_{(p,x^2+1)} = \dim_{\mathbf{F}_p} \mathbf{Z}[x]_{(p,x^2+1)}/(p, x^2 + 1) = 2,$$

since $\mathbf{Z}[x]_{(p,x^2+1)}/(p, x^2 + 1) \cong \mathbf{Z}[i]/(p) \cong \mathbf{F}_{p^2}$.

4. GLOBAL INTERSECTION MULTIPLICITY

We'd like to define a global intersection multiplicity on an arithmetic surface by adding up the local multiplicities. But we'd also like the result to have nice properties (e.g. multiplicites should only depend on linear equivalence classes of divisors). It turns out that since our base schemes are arithmetic and not projective, such a definition will not produce a theory with nice properties.

Example 13. This is example 7.2 of Chapter IV of [4]. Let R denote the localisation of $\mathbf{Z}$ at a prime number p, and let $\mathcal{C} = \operatorname{Proj}(R[x, y])$ denote the projective line over R. Let Y_1 denote the zero locus of x, and let Y_2 denote the zero locus of $x + p^n y$ for some integer $n \geq 1$. These divisors only intersect at the closed point P defined by $x = 0$ on the special fiber $\mathbf{P}^1_{\mathbf{F}_p}$ of $\mathbf{P}^1_R$. To compute the multiplicity, we may use the affine chart $y = 1$ near P, and we deduce that

$$(Y_1, Y_2)_P = \operatorname{length}_{\mathbf{Z}_{(p)}} \mathbf{Z}[x]_{(x,p)}/(x, x + p^n) = \operatorname{length}_{\mathbf{Z}_{(p)}} \mathbf{Z}/(p^n) = n.$$

But note that $\frac{x+p^n y}{y}$ is an element of the fraction field of K of R. Set $Y_3 = Y_2 + \operatorname{div}\left(\frac{x+p^n y}{y}\right)$. Then Y_3 is the zero locus of y, and hence it has no intersection with Y_1. We've changed the intersection multiplicity by changing a divisor within a linear equivalence class!

Rather than sacrifice having our intersection theory be independent of linear equivalence class representatives,[3] one can instead restrict the divisors on which the intersection multiplicity is defined. In order to explain this, let us set $S = \mathrm{Spec}(R)$ and fix a closed point s of S. Let k denote the residue field of s and as usual let K denote the fraction field of R. If $\mathcal{C} \to S$ is a proper and regular arithmetic surface, then a Weil divisor will be called FIBRAL over s if it is contained in the fiber $\mathcal{C}_s$ of $\mathcal{C}$ over $s \in S$. Let $\mathrm{Div}_s(\mathcal{C})$ denote the subgroup of $\mathrm{Div}(\mathcal{C})$ generated by the fibral prime divisors over s.

The fibral divisors are rather simple, in a certain sense. They are curves on $\mathcal{C}$ restricted to a single fiber, which is itself a curve. Thus, $\mathrm{Div}_s(\mathcal{C})$ is the free group on the irreducible components of the curve $\mathcal{C}_s \to \mathrm{Spec}(k)$.

The main theorem that we wish to state is as follows.

Theorem 14. *Let $\mathcal{C} \to S$ be a proper arithmetic surface that is regular in codimension 1, where S is Dedekind and $s \in S$ is a closed point. Then there exists a unique bilinear pairing*

$$\mathrm{Div}(\mathcal{C}) \times \mathrm{Div}_s(\mathcal{C}) \to \mathbf{Z}$$

denoted $(Y, Z) \mapsto (Y \cdot Z)$ that is characterized by the following two properties:

(1) *if Y and Z are distinct prime divisors, then*

$$(Y \cdot Z) = \sum_{x \in Y \cap Z} (Y \cdot Z)_x;$$

(2) *If Y_1, $Y_2 \in \mathrm{Div}(\mathcal{C})$ are linearly equivalent and $Z \in \mathrm{Div}_s(\mathcal{C})$, then $(Y_1 \cdot Z) = (Y_2 \cdot Z)$.*

Furthermore, this pairing satisfies the following symmetry property: if Y and Z are both *fibral over s, then $(Y \cdot Z) = (Z \cdot Y)$.*

Proof. See [2] or Theorem 1.12 of Chapter 9 in [3]. □

Example 15. Let p be an odd prime and let R denote the localisation of $\mathbf{Z}$ at p. Let $\mathcal{C} = \mathrm{Proj}(R[x, y, z]/(xy - pz^2)$. The special fiber is the reducible curve $\mathcal{C}_p = \mathrm{Proj}(\mathbf{F}_p[x, y, z]/(xy))$ which is the union of the x and y axes over $\mathbf{F}_p$. Let X be the prime fibral divisor which is the x-axis, and let Y be the prime fibral divisor which is the y-axis. One computes that $(Y \cdot Z) = 1$, as expected, using (1) of the theorem. On the other hand, the sum $X + Y$ is the principal divisor defined by the function p, as $X + Y$ is the whole fiber over p. It follows that $(X \cdot X) = -(Y \cdot X) = -1$.

Remark 16. In general, one can show that if Y is a fibral divisor, then $(Y \cdot Y) \leq 0$.

References

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[2] S. Lichtenbaum. Curves over discrete valuation rings. *Amer. J. Math.*, 90:380–405, 1968.

[3] Q. Liu. *Algebraic geometry and arithmetic curves*, volume 6 of *Oxford Graduate Texts in Mathematics*. Oxford University Press, Oxford, 2002. Translated from the French by Reinie Erné, Oxford Science Publications.

[4] J. H. Silverman. *Advanced topics in the arithmetic of elliptic curves*, volume 151 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1994.

[3]Being able to change representatives is useful for computing self-intersections.