Gross–Zagier reading seminar

Lecture 1 • Andrew Snowden • September 9, 2014

1. Introduction

Let $E/\mathbf{Q}$ be an elliptic curve. The Birch–Swinnerton-Dyer conjecture predicts that the rank of E is the order of vanishing of the L-function of E at $s = 1$. One of the hard parts of this conjecture is constructing the required points on E. For example, if the L-function vanishes, one must somehow show that E has a point of infinite order.

It is now known that for every elliptic curve E (defined over $\mathbf{Q}$) there is a surjective map $\pi\colon X \to E$ from a modular curve $X = X_0(N)$. One strategy for finding points on E is to take the image of points on X. This is a useful idea because the points of X have a meaning: they correspond to degree N isogenies of elliptic curves with cyclic kernel. One can use this to write down at least some explicit points on X. The easiest points to write down are the so-called Heegner points, corresponding to elliptic curves with complex multiplication.

Let K be an imaginary quadratic field of discriminant D, relatively prime to N. A **Heegner point** of X is an isogeny $E \to E'$ such that E and E' both have complex multiplication by $\mathcal{O}_K$. (One can define more general Heegner points, but these are the only ones we will consider.) It turns out that Heegner points exist if and only if D is congruent to a square modulo $4N$, in which case there are $2^s h$ of them, where s is the number of distinct primes dividing N and h is the class number of K. The theory of complex multiplication for elliptic curves shows that every Heegner point is defined over the Hilbert class field H of K.

Let $x \in X(H)$ be a Heegner point, let $\pi(x) \in E(H)$ be its image under π, and let $y \in E(K)$ be the trace of $\pi(x)$ down to K. It turns out y is independent of the choice of x, up to sign. Since we've taken D to be a square modulo $4N$, the sign in the functional equation for $L(E/K, s)$ is -1, and so $L(E/K, s)$ vanishes at $s = 1$. It therefore "makes sense" to look at the value of the derivative at $s = 1$. (Of course, one does not need vanishing to look at the value of the derivative, but without vanishing one does not expect a nice answer.) The Gross–Zagier formula is:

$$L'(E/K, 1) = (\text{easy stuff}) \times (\text{period}) \times \widehat{h}(y). \tag{1}$$

Here the "easy stuff" is made up of things like the degree of π, the number of units of $\mathcal{O}_K$, etc.; it's always non-zero. The period is an integral of a rational holomorphic 1-form on E over $E(\mathbf{R})$, and is a non-zero transcendental number. Finally, $\widehat{h}(y)$ denotes the Néron–Tate height of the point y on $E(K)$. This is zero if and only if y is a torsion point.

One can use (1) to obtain information about E over $\mathbf{Q}$ (instead of K) in some instances:

Theorem 2. *Suppose that $L(E/\mathbf{Q}, 1) = 0$ but $L'(E/\mathbf{Q}, 1) \neq 0$. Then there is a point in $E(\mathbf{Q})$ of infinite order.*

Proof. Let E' be the quadratic twist of E corresponding to the field K. We then have a factorization

$$L(E/K, s) = L(E/\mathbf{Q}, s)L(E'/\mathbf{Q}, s).$$

By a theorem of Waldspurger, one can choose K so that $L'(E'/\mathbf{Q}, 1) \neq 0$; fix such a K. It follows then that $L'(E/K, 1) \neq 0$, and so $\widehat{h}(y) \neq 0$ by (1), and so $y \in E(K)$ has infinite order. One can furthermore show that y belongs to $E(\mathbf{Q})$ in this case (one knows $y = \pm y^c$ in general, where c is complex conjugation, and our choice of K forces a $+$ here). □

2. Overview of the proof

2.1. **Reformulation using modular forms.** Let J be the Jacobian of $X_0(N)$. Given a normalized eigenform $f \in S_2^{\text{new}}(N)$, there is a corresponding quotient E_f of J, and every E of interest is isogeneous to an E_f. Define

$$\mu(f) = L'(E_f/K, 1)$$

and

$$\nu(f) = \widehat{h}(y)(f, f).$$

Here $\widehat{h}(y)$ is the height of $y \in E_f(K)$ and (f, f) is the Petersson inner product of f with itself (which is roughly the period in the Gross–Zagier formula). We want to show $\mu = \nu$, up to some easy factors. (In fact, these easy factors will not depend on f, only K.) We can extend μ and ν uniquely to linear functions on the space of newforms. The non-degeneracy of the Petersson inner product implies that they are represented by cusp forms. That is, we have cusp forms F and G such that

$$\mu(f) = (f, F), \qquad \nu(f) = (f, G)$$

for all normalized eigenforms $f \in S_2^{\text{new}}(N)$. Furthermore, F and G are well-defined up to oldforms. (We could specify F and G uniquely by taking them to be newforms, but prefer not to.) It thus suffices to show $F = G$ up to oldforms, and for this, it is enough to show that their prime-to-N Fourier coefficients agree. This is accomplished by computing the Fourier coefficients of F and G in closed form and directly comparing. These two computations are completely independent of one another.

Remark 3. The quotient E_f is not always an elliptic curve: sometimes it is a higher dimensional abelian variety. Nonetheless, one can make sense of $\widehat{h}(y)$. □

2.2. **The form F.** Let $f \in S_2^{\text{new}}(N)$ be a normalized eigenform, and write $f = \sum_{n\geq 1} a_n q^n$. Then

$$L(E_f/\mathbf{Q}, s) = \sum_{n\geq 1} a_n n^{-s}.$$

A simple computation shows that

$$L(E_f/K, s) = \sum_{n\geq 1} a_n r(n) n^{-s},$$

where $r(n)$ is the number of integral ideals in K of norm n. In other words, nth coefficient in the above Dirichlet series is the product of the nth coefficient in $L(E_f/\mathbf{Q}, s)$ and the nth coefficient of the Dedekind zeta function of K. The way to understand this type of product of L-series is through Rankin's method.

Define

$$\theta = \sum_{n\geq 0} r(n) q^n$$

where $r(0)$ is roughly the class number of K. We have

$$\sum_{n\geq 0} a_n r(n) e^{-2\pi n y} = \int_0^1 f(x+iy)\overline{\theta(x+iy)}dx$$

for any $y > 0$. (Here f and θ are functions of $q = e^{2\pi i z}$.) Multiplying by y^{s-1} and integrating from 0 to ∞, we find

$$L(E_f/K, s) = \int_0^\infty \int_0^1 f(x+iy)\overline{\theta(x+iy)}y^{s-1}dxdy$$

(up to some easy factors). We can rewrite this as

$$L(E_f/K, s) = \int_{\Gamma_\infty\backslash\mathfrak{h}} f(z)\overline{\theta(z)}y^{s-1}dxdy$$

Here $\Gamma_\infty \subset \Gamma_0(N)$ is the group of linear fractional transformations generated by $z \mapsto z+1$. We now use the fact that $f(z)$ and $\theta(z)$ are invariant under all of $\Gamma_0(N)$ to write this integral as

$$L(E_f/K, s) = \int_{\Gamma_0(N)\backslash\mathfrak{h}} f(z)\overline{\theta(z)E_{\overline{s}}(z)}dxdy = (f, \theta E_{\overline{s}})$$

where $E_s(z)$ is the non-holomorphic Eisenstein series

$$E_s(z) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(N)} (\gamma\cdot y)^{s-1}.$$

(In fact, there was a lie here: $\theta(z)$ is only invariant under a subgroup of $\Gamma_0(N)$ of the form $\Gamma_0(NM)$, so θE_s is a form on this smaller group. However, one can trace down to $\Gamma_0(N)$ to get a form on the larger group, and it has the same inner product with f.) We can now take the derivative at $s = 1$ to get a formula for $L'(E_f/K, 1)$. However, the result still has a deficiency: E_s is non-holomorphic. To fix this, one applies a holomorphic projection operator.

Thus F is obtained by taking the product of an explicit theta function and non-holomorphic Eisenstein series on $\Gamma_0(NM)$, tracing down to $\Gamma_0(N)$, taking the derivative at $s = 1$, and then applying a holomorphic projection operator. Working through these operations gives an explicit (and long) expression for the Fourier coefficients of F. This is a long computation, but fairly elementary.

A few remarks:

- The identity $L(E_f/K, s) = (f, \theta E_{\overline{s}})$ implies that $L(E_f/K, s)$ has a functional equation, since E_s has a functional equation in s. Using the sign of the functional equation, one can see that $L(E_f/K, 1)$ vanishes in cases of interest.
- We will actually need to work with a more general L-series. Let $\mathscr{A}$ be an ideal class of K and let $r_{\mathscr{A}}(n)$ be the number of integral ideals of K in the class $\mathscr{A}$ with norm n. Put

  $$L_{\mathscr{A}}(f, s) = \sum_{n\geq 1} r_{\mathscr{A}}(n)a_n n^{-s}.$$

  It is this series we will need to work with. The above results go through for it, and we get a modular form $F_{\mathscr{A}}$.
- All the above goes through for higher weight forms. In fact, it is easier for higher weight forms because Eisenstein series in weight 2 are subtle.

2.3. **The form G.** Assume for the moment that N is prime, so there are no oldforms. Let $f_1, \dots, f_r$ be a basis for $S_2(N)$ consisting of normalized eigenforms. Then, somewhat obviously, we have

$$G = \sum_{i=1}^{r} \nu(f_i) \frac{f_i}{(f_i, f_i)} = \sum_{i=1}^{r} \widehat{h}(y_i) f_i,$$

where y_i is the projection of x to $E_i = E_{f_i}$. (Let's forget about the trace from H to K for the moment.) We can therefore write

$$a_n(G) = \sum_{i=1}^{r} \widehat{h}(y_i) a_n(f_i).$$

Note that $\widehat{h}(y_i) = \langle y_i, y_i \rangle_{E_i}$, where $\langle , \rangle_{E_i}$ denotes the bilinear heigh pairing on E_i. Furthermore, the Hecke algebra $\mathbf{T}$ acts on J, and on each factor E_i. In fact, it acts on E_i in the same way it acts on f_i, that is, $T_n P = a_n(f_i) P$ for $P \in E_i$. Thus we can rewrite the above as

$$a_n(G) = \sum_{i=1}^{r} \langle y_i, T_n y_i \rangle_{E_i}.$$

But this is just $\langle x, T_n x \rangle_J$ (the Néron–Tate height pairing on J) since J is the product of the E_i and the E_i are orthogonal under the height pairings. This, in turn, is equal to $\langle c, T_n c \rangle_X$ (the Néron height pairing on the curve X), where c is the degree 0 divisor $(x) - (\infty)$. We have thus have the formula

$$G = \sum_{n \geq 1} \langle c, T_n c \rangle_X \cdot q^n$$

When N is composite, nothing much of substance changes. It's no longer true the J is the product of the E_i, so the above "derivation" of G doesn't quite work. But if we just start by defining G by the above formula, then it is true that $(f, G) = \nu(f)$ for normalized eigenforms $f \in S_2^{\text{new}}(N)$, and that's all we need.

The real problem, then, is to compute the height pairing $\langle c, T_n c \rangle_X$. Néron's theory factors this pairing into a product of local height pairings, so it suffices to compute each of these separately. At the archimedean places, this involves explicit special functions. At the non-archimedean places, the local height is defined in terms of intersection theory of divisors on $X_0(N)$, and the computations boil down to deformation theory of elliptic curves. In the end, one obtains a complicated, though explicit, formula for the height.

Remark 4. I ignored the tracing from H to K above. To accomodate that, we consider the more general series

$$G_\sigma = \sum_{n \geq 1} \langle c, T_n c^\sigma \rangle_X q^n$$

for $\sigma \in \mathrm{Gal}(H/K)$. One then shows that $F_{\mathscr{A}}$ and G_σ coincide up to oldforms when $\mathscr{A}$ and σ correspond under the class field theory isomorphism $\mathrm{Cl}(K) = \mathrm{Gal}(H/K)$. $\square$