Δ-modules

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References

- Syzygyies of Segre embeddings and Δ-modules
  arXiv:1006.5248
- Introduction to twisted commutative algebras (w/S. Sam)
  arXiv:1209.5122
- GL-equivariant modules over polynomial rings in infinitely many variables (w/S. Sam) arXiv:1206.2233
- These slides:
  http://math.mit.edu/~asnowden/

We cite the three papers as [S], [SS1] and [SS2] in the following.

§1. Introduction

$$V_1^* \otimes \cdots \otimes V_n^*$$

$$\|$$

$$\mathbf{V}_n(V_1, \ldots, V_n)$$

The variety **V** has three pieces of structure of interest:

(A1) **Naturality.** Given linear maps $f_i \colon V_i \to V_i'$, there is an induced map

$$f^* \colon \mathbf{V}_n(V_1', \ldots, V_n') \to \mathbf{V}_n(V_1, \ldots, V_n).$$

(A2) **Symmetry.** Given $\sigma \in S_n$, there is an induced isomorphism

$$\sigma^* \colon \mathbf{V}_n(V_{\sigma(1)}, \ldots, V_{\sigma(n)}) \to \mathbf{V}_n(V_1, \ldots, V_n).$$

(A3) **Flattening.** There is a natural isomorphism

$$\mathbf{V}_{n+1}(V_1, \ldots, V_{n+1}) = \mathbf{V}_n(V_1, \ldots, V_{n-1}, V_n \otimes V_{n+1})$$

A Δ-**variety** is a subvariety of $\mathbf{V}$ which respects this structure.

Precisely, a Δ-variety is a rule X which assigns to each $(V_1, \ldots, V_n)$ a closed subvariety

$$X_n(V_1, \ldots, V_n) \subset \mathbf{V}_n(V_1, \ldots, V_n)$$

such that:

(B1) Given linear maps f_i as in (A1), f^* carries $X_n(V_1', \ldots, V_n')$ into $X_n(V_1, \ldots, V_n)$.

(B2) Given $\sigma \in S_n$ as in (A2), σ^* carries $X_n(V_{\sigma(1)}, \ldots, V_{\sigma(n)})$ into $X_n(V_1, \ldots, V_n)$.

(B3) The flattening isomorphism (A3) induces an inclusion

$$X_{n+1}(V_1, \ldots, V_{n+1}) \subset X_n(V_1, \ldots, V_{n-1}, V_n \otimes V_{n+1}).$$

Note: a Δ-variety is not a single variety, but

an interrelated system of varieties.

Example: the Segre variety

Define

$$X_n(V_1, \ldots, V_n) \subset \mathbf{V}_n(V_1, \ldots, V_n)$$

to be the set of pure tensors. This is the **Segre variety**, and is the motivating example of a Δ-variety.

- Conditions (B1) and (B2): linear maps and permutations carry pure tensors to pure tensors.
- Condition (B3): the inclusion

  $$X_{n+1}(V_1, \ldots, V_{n+1}) \subset X_n(V_1, \ldots, V_{n-1}, V_n \otimes V_{n+1})$$

  simply means we can regard an $(n+1)$-fold tensor as an n-fold tensor:

  $$v_1 \otimes \cdots \otimes v_{n+1} = v_1 \otimes \cdots \otimes v_{n-1} \otimes (v_n \otimes v_{n+1}).$$

Other examples

There are many other examples of Δ-varieties:

- Higher subspace varieties. These directly generalize Segre varieties.
- The tangent and secant varieties of a Δ-variety is a Δ-variety.
- The sum, union and intersection of two Δ-varieties is a Δ-variety.

In particular, the secant varieties of the Segre are Δ-varieties.

A Δ-**module** is the result of taking a linear invariant of a Δ-variety.

Precisely, a Δ-module is a rule F which assigns to each $(V_1, \dots, V_n)$ a vector space $F_n(V_1, \dots, V_n)$ equipped with the following extra structure:

(C1) For each system of linear maps $f_i \colon V_i \to V_i'$, a linear map

$$f_* \colon F_n(V_1, \dots, V_n) \to F_n(V_1', \dots, V_n').$$

(C2) For each $\sigma \in S_n$, a linear map

$$\sigma_* \colon F_n(V_1, \dots, V_n) \to F_n(V_{\sigma(1)}, \dots, V_{\sigma(n)}).$$

(C3) A linear map

$$F_n(V_1, \dots, V_{n-1}, V_n \otimes V_{n+1}) \to F_{n+1}(V_1, \dots, V_{n+1})$$

There are various compatibilities and technical conditions required, which we ignore for now.

Sources of examples

If X is a Δ-variety and $\mathbf{F}$ is a contravariant linear invariant of varieties (or closed immersions of varieties), then

$$F_n(V_1, \ldots, V_n) = \mathbf{F}(X_n(V_1, \ldots, V_n))$$

is naturally a Δ-module.

Reason: (B1)–(B3) induce (C1)–(C3) by functoriality of $\mathbf{F}$.

Sources of examples

Possibilities for $\mathbf{F}$:

- Coordinate ring.
- Defining ideal (inside of $\mathbf{V}$).
- Syzygies (relative to $\mathbf{V}$).
- Local cohomology.
- Topological cohomology.

Example: equations of the Segre

Define

$$F_n(V_1, \ldots, V_n) \subset \mathrm{Sym}^2(V_1 \otimes \cdots \otimes V_n)$$

to be the quadratic equations which vanish on the Segre $X_n(V_1, \ldots, V_n)$.

Then F is naturally a Δ-module.

Example: equations of the Segre

The usefulness of the Δ-module structure is that it allows us to produce equations of complicated Segre varieties from those of more simple ones.

Example: equations of the Segre

Start with the equation α cutting out the Segre $X_2(\mathbf{C}^2, \mathbf{C}^2)$.

Choosing $f_1 \colon \mathbf{C}^2 \to \mathbf{C}^m$ and $f_2 \colon \mathbf{C}^2 \to \mathbf{C}^n$, (C1) gives a linear map

$$f_* \colon F_2(\mathbf{C}^2, \mathbf{C}^2) \to F_2(\mathbf{C}^m, \mathbf{C}^n).$$

We can therefore build an element $f_*(\alpha)$ of $F_2(\mathbf{C}^m, \mathbf{C}^n)$.

Varying f_1 and f_2 produces many elements.

Example: equations of the Segre

Since $\mathbf{C}^{mn} = \mathbf{C}^m \otimes \mathbf{C}^n$, (C3) gives a map

$$F_2(\mathbf{C}^\ell, \mathbf{C}^{mn}) \to F_3(\mathbf{C}^\ell, \mathbf{C}^m, \mathbf{C}^n).$$

We get many elements of F_3 by taking the images of the elements in F_2 we have already constructed.

We can similarly go from 3 to 4 factors.

Example: equations of the Segre

Thus the single equation α gives many equations on every Segre.

In fact, we obtain all equations of each Segre from α!

We say that α **generates** F.

Example: equations of the Segre

Write $\{1, \ldots, n\} = A \amalg B$ and choose linear maps

$$f_1 \colon \mathbf{C}^2 \to \bigotimes_{i \in A} V_i, \qquad f_2 \colon \mathbf{C}^2 \to \bigotimes_{i \in B} V_i.$$

We obtain a map $f^* \colon \mathbf{V}_n(V_1, \ldots, V_n) \to \mathbf{V}_2(\mathbf{C}^2, \mathbf{C}^2)$. Let X_{f_1, f_2} be the inverse image of $X_2(\mathbf{C}^2, \mathbf{C}^2)$.

The statement that α generates F is equivalent to the statement that $X_n(V_1, \ldots, V_n)$ is the inersection of the X_{f_1, f_2} as we vary A, B, f_1 and f_2.

Let X be a Δ-variety and let $F^{p,d}$ be the Δ-module of p-syzygies of X of degree d.

The goal of this course is to sketch the proof of the following two results about this Δ-module.

The first theorem

Theorem

The Δ-module $F^{p,d}$ is finitely generated.

The second theorem

We will define the **Hilbert series** f associated to a Δ-module F.

This is a formal power series in several variables.

From it, one can read off the decomposition of $F_n(V_1, \ldots, V_n)$ as a representation of $\mathbf{GL}(V_1) \times \cdots \times \mathbf{GL}(V_n)$ for all $(V_1, \ldots, V_n)$.

The second theorem

Theorem

The Hilbert series of $F^{p,d}$ is a rational function.

Effectiveness

The proofs of these theorems are effective: there is an algorithm which, given X, p and d, computes the generators and Hilbert series of $F^{p,d}$ in finitely many steps.

Unfortunately, the algorithm involves linear algbera over a polynomial ring in $\sim p^p$ indeterminates, and is therefore totally impractical.

Two theorems on Δ-modules

The two theorems on syzygies are deduced from the following two abstract results about Δ-modules:

Theorem

A finitely generated Δ-module is noetherian.

Theorem

The Hilbert series of a finitely generated Δ-module is rational.

Obtaining the theorems on syzygies from these abstract results is easy:

- By definition, $F_n^{p,d}(V_1, \dots, V_n)$ is the homology of a certain Koszul complex $K_n^{\bullet,d}(V_1, \dots, V_n)$.
- It turns out that each $K^{p,d}$ is a Δ-module, and that the Koszul differentials are maps of Δ-modules. Furthermore, each $K^{p,d}$ is obviously finitely generated.
- Since $K^{p,d}$ is noetherian, the subquotient $F^{p,d}$ is finitely generated.
- Rationality of the Hilbert series of $F^{p,d}$ follows.

The ladder

To prove the two abstract results about Δ-modules, we proceed along the following "ladder:"

modules over ordinary rings

$\downarrow$

modules over twisted commutative algberas

$\downarrow$

modules over algebras in $\mathrm{Sym}(\mathcal{S})$

$\downarrow$

Δ-modules

A conjecture

Our theorems provide a lot of understanding about p-syzygies of a fixed degree, but do nothing to understand the possible degrees of p-syzygies.

For example, if one wants to understand the 5-syzygies of X, one knows that $F^{5,d}$ is finitely generated for each d, but it could be that this Δ-module is non-zero for infinitely many d.

Conjecture

If X is **bounded** then $F^{p,d} = 0$ for $d \gg p$.

Most (all?) X of interest are bounded.

Known cases of the conjecture

- If $X =$ the Segre then $F^{p,d} = 0$ for $d > 2p$. This follows from the existence of a quadratic Gröbner basis (Eisenbud–Reeves–Totaro).
- If $X =$ the tangent variety to the Segre then $F^{1,d} = 0$ for $d > 4$. Due to Oeding–Raicu (arXiv:1111.6202), improving earlier bound $d > 6$ of Landsberg–Weyman (arXiv:math/0509388).
- If $X =$ the secant variety to the Segre then $F^{1,d} = 0$ for $d > 3$. Due to Raicu (arXiv:1011.5867), confirms the GSS conjecture.
- If $X =$ a higher secant variety of the Segre, then Draisma–Kuttler (arXiv:1103.5336) establish a topological version of the conjecture for $p = 1$.

§2. Twisted commutative algberas

Twisted commutative algberas (tca's) are generalizations of graded rings.

Definition 1 (sequence model)

A tca is an associative unital graded ring $A = \bigoplus_{n \geq 0} A_n$ equipped with an action of the symmetric group S_n on A_n such that:

- The multiplication map $A_n \otimes A_m \to A_{n+m}$ is $S_n \times S_m$ equivariant.
- For $x \in A_n$ and $y \in A_m$, we have $yx = \tau(xy)$, where $\tau \in S_{n+m}$ switches $\{1, \ldots, n\}$ and $\{n+1, \ldots, n+m\}$.

The second axiom is the **twisted commutativity axiom**.

Definition 1 — example

Let U be a finite dimensional vector space, and put $A_n = U^{\otimes n}$.

This is an associative unital ring under the multiplication map $A_n \otimes A_m \to A_{n+m}$ which concatenates pure tensors. In fact, A is the tensor algebra on U.

The group S_n acts on A_n by permuting the tensor factors.

In general, A is highly non-commutative. However, it does satisfy the twisted commutativity axiom, and is therefore a tca.

Definition 2 (fs model)

Let (fs) be the category whose objects are finite sets and whose morphisms are bijections.

A tca is a functor $A\colon (\mathrm{fs}) \to \mathrm{Vec}$ equipped with a multiplication map

$$A_L \otimes A_{L'} \to A_{L \amalg L'}$$

which is associative, unital and commutative.

Commutativity means that the following diagram commutes:

$$\begin{array}{ccc} A_L \otimes A_{L'} & \longrightarrow & A_{L \amalg L'} \\ \downarrow & & \downarrow \\ A_{L'} \otimes A_L & \longrightarrow & A_{L' \amalg L} \end{array}$$

Definition 2 — example

For a vector space U and a finite set L, define $U^{\otimes L}$ to be the universal vector space equipped with a multi-linear map from $\mathrm{Fun}(L, U)$.

If L has cardinality n then $U^{\otimes L}$ is isomorphic to $U^{\otimes n}$. The advantage of the construct $U^{\otimes L}$ is that it is functorial in L.

We think of the factors of pure tensors in $U^{\otimes L}$ as being indexed by L.

Let $A_L = U^{\otimes L}$. Then A is a tca, multiplication being given by concatenation of tensors.

Definition 3 (Schur model)

A tca is a rule which assigns to each vector space V an associative commutative unital $\mathbf{C}$-algbera $A(V)$ and to each linear map of vector spaces $V \to V'$ an algebra homomorphism $A(V) \to A(V')$.

There is a technical condition required that we ignore for now.

Definition 3 — example

Let $A(V) = \mathrm{Sym}(V)$ be the symmetric algebra on V. If $x_1, \ldots, x_n$ is a basis of V then $\mathrm{Sym}(V)$ is the polynomial ring $\mathbf{C}[x_1, \ldots, x_n]$.

Given a linear map $V \to V'$ we get a ring homomorphism $A(V) \to A(V')$. It follows that A has the structure of a twisted commutative algebra.

Definition 4 (**GL** model)

A tca is an commutative associative unital $\mathbf{C}$-algebra equipped with an action of the group $\mathbf{GL}(\infty) = \bigcup_{n \geq 1} \mathbf{GL}(n)$ by algebra homomorphisms.

There is a technical condition required that we ignore for now.

Definition 4 — example

The symmetric algebra $\mathrm{Sym}(\mathbf{C}^\infty) = \mathbf{C}[x_1, x_2, \ldots]$ is a tca.

Other examples can be obtained by taking the symmetric algebra on other representations of $\mathbf{GL}(\infty)$, for instance $\mathrm{Sym}(\bigwedge^2 \mathbf{C}^\infty)$ or $\mathrm{Sym}(\mathrm{Sym}^2 \mathbf{C}^\infty)$.

Comparisons

Each definition has its advantages and shortcomings:

- Tca's in the sequence model are concrete (a single ring) and usually small (the graded pieces are finite dimensional). However, the lack of commutativity is an annoyance.
- The fs model is like the sequence model, but tends to be more natural, i.e., many constructions are simpler. The price is that it is more abstract.
- The Schur model relates tca's directly to usual commutative algebra. The rings $A(V)$ tend to be finitely generated. However, one has to deal with the system of all the rings $A(V)$.
- Tca's in the **GL** model are concrete (a single ring) and commutative in the usual sense. However, they're often huge!

Equivalences

The equivalences between the four definitions of tca's are induced by more fundamental equivalences of certain kinds of linear data:

- Sequences of representations of the symmetric groups.
- Functors (fs) $\to$ Vec.
- Functors Vec $\to$ Vec.
- Representations of $\mathbf{GL}(\infty)$.

We will discuss each of these categories and the equivalences between them.

Representation theory of the symmetric group

Irreducible representations of S_n are indexed by partitions of n.

We denote by $\mathbf{M}_\lambda$ the irreducible associated to λ.

Our conventions are such that $\mathbf{M}_{(n)}$ is the trivial representation and $\mathbf{M}_{(1^n)}$ is the sign representation.

Every representation of S_n is a direct sum of irreducible representations (complete reducibility).

In other words, the category $\mathrm{Rep}(S_n)$ is semi-simple and its simple objects are the $\mathbf{M}_\lambda$ with $|\lambda| = n$.

The category $\mathrm{Rep}(S_*)$

We let $\mathrm{Rep}(S_*)$ be the following category:

- Objects are sequences $(V_n)_{n \geq 0}$, where V_n is a representation of S_n.
- A morphism $f \colon (V_n) \to (V_n')$ consists of morphisms of representations $f_n \colon V_n \to V_n'$ for each $n \geq 0$.

Structure of Rep(S_*)

For a partition λ of n, we regard $\mathbf{M}_\lambda$ as the object (V_k) of Rep(S_*) with $V_k = \mathbf{M}_\lambda$ for $k = n$ and $V_k = 0$ otherwise.

Every object of Rep(S_*) is a direct sum of $\mathbf{M}_\lambda$'s.

In other words, Rep(S_*) is semi-simple, and the simple objects are the $\mathbf{M}_\lambda$.

The tensor product

The tensor product of graded vector spaces V and V' is defined by

$$(V \otimes V')_n = \bigoplus_{i+j=n} V_i \otimes V'_j.$$

Let $V = (V_n)$ and $V' = (V'_n)$ be two objects of $\mathrm{Rep}(S_*)$. Motivated by the above, we define their tensor product by

$$(V \otimes V')_n = \bigoplus_{i+j=n} \mathrm{Ind}_{S_i \times S_j}^{S_n}(V_i \otimes V'_j).$$

There is a natural isomorphism $V \otimes V' = V' \otimes V$. This makes use of the element τ which interchanges $\{1, \ldots, n\}$ with $\{n+1, \ldots, n+m\}$.

Tensor products of simple objects

If λ is a partition of n and μ a partition of m then

$$\mathbf{M}_\lambda \otimes \mathbf{M}_\mu = \mathrm{Ind}_{S_n \times S_m}^{S_{n+m}}(\mathbf{M}_\lambda \otimes \mathbf{M}_\mu).$$

The decomposition of this representation into irreducibles is given by the **Littlewood–Richardson rule**.

We let $c_{\lambda,\mu}^{\nu}$ denote the multiplicity of $\mathbf{M}_\nu$ in $\mathbf{M}_\lambda \otimes \mathbf{M}_\mu$. This is the **Littlewood–Richardson coefficient**.

Tca's

Let $A \in \mathrm{Rep}(S_*)$. Giving a map $m \colon A \otimes A \to A$ is the same as giving a map of S_n-representations

$$\mathrm{Ind}_{S_i \times S_j}^{S_n}(A_i \otimes A_j) \to A_n$$

for all $i + j = n$.

By Frobenius reciprocity, this is the same as giving a map of $S_i \times S_j$ representations $A_i \otimes A_j \to A_{i+j}$.

Tca's

The map m is called **commutative** if the diagram

$$\begin{array}{ccc} A \otimes A & \xrightarrow{m} & A \\ \sigma \downarrow & & \| \\ A \otimes A & \xrightarrow{m} & A \end{array}$$

commutes, where σ is the switching-of-factors map.

Exercise

Show that m is commutative if and only if the maps $A_i \otimes A_j \to A_{i+j}$ satisfy the twisted commutativity axiom.

Tca's

A tca in the sequence model is therefore an object A of $\mathrm{Rep}(S_*)$ equipped with a multiplication map $A \otimes A \to A$ which is commutative, associative and unital.

The category $\mathrm{Vec}^{(\mathrm{fs})}$

Let $\mathrm{Vec}^{(\mathrm{fs})}$ denote the following category:

- Objects are functors $(\mathrm{fs}) \to \mathrm{Vec}$.
- Morphisms are natural transformations of functors.

The tensor product and tca's

We define the tensor product of F and G in $\mathrm{Vec}^{(\mathrm{fs})}$ by

$$(F \otimes G)_L = \bigoplus_{L = A \amalg B} F_A \otimes G_B$$

Giving a map $F \otimes F \to F$ is the same as giving a map $F_A \otimes F_B \to F_{A \amalg B}$.

Thus a tca in the fs model is an object A of $\mathrm{Vec}^{(\mathrm{fs})}$ equipped with a map $A \otimes A \to A$ satisfying the required axioms.

Equivalence with $\mathrm{Rep}(S_*)$

Let $[n]$ denote the finite set $\{1, \ldots, n\}$.

If F is an object of $\mathrm{Vec}^{(\mathrm{fs})}$ then $F_{[n]}$ carries a representation of $\mathrm{Aut}([n]) = S_n$, and so $(F_{[n]})_{n \geq 0}$ is an object of $\mathrm{Rep}(S_*)$.

Exercise

Show that the above construction defines an equivalence of categories $\mathrm{Vec}^{(\mathrm{fs})} \to \mathrm{Rep}(S_*)$ which is compatible with the tensor products.

Polynomial functors

A functor $F\colon \mathrm{Vec} \to \mathrm{Vec}$ is **polynomial** if for every pair of vector spaces V and W, the map

$$F\colon \operatorname{Hom}(V, W) \to \operatorname{Hom}(F(V), F(W))$$

is a polynomial map of vector spaces.

Concretely, this means that the matrix entries of $F(f)$ are polynomial functions of those of f, for $f \in \operatorname{Hom}(V, W)$.

The symmetric and exterior power functors are the basic examples.

Schur functors

For a vector space V, let S_n act on $V^{\otimes n}$ by permuting tensor factors.

Define $\mathbf{S}_\lambda(V) = \mathrm{Hom}_{S_n}(\mathbf{M}_\lambda, V^{\otimes n})$.

Exercise

Show that $\mathbf{S}_\lambda$ is a polynomial functor.

We call $\mathbf{S}_\lambda$ the **Schur functor** associated to λ.

We have $\mathbf{S}_{(n)} = \mathrm{Sym}^n$ and $\mathbf{S}_{(1^n)} = \bigwedge^n$.

Structure of polynomial functors

Let F and G be polynomial functors. We define a functor $F \oplus G$ by $(F \oplus G)(V) = F(V) \oplus G(V)$. It is a polynomial functor.

Theorem

Every polynomial functor is a direct sum of Schur functors.

Tensor products

Let F and G be polynomial functors. We define a functor $F \otimes G$ by $(F \otimes G)(V) = F(V) \otimes G(V)$. It is a polynomial functor.

Exercise

Show that the decomposition of a tensor product of Schur functors is given by the Littlewood–Richardson rule, i.e., that the multiplicity of $\mathbf{S}_\nu$ in $\mathbf{S}_\lambda \otimes \mathbf{S}_\mu$ is $c^{\nu}_{\lambda,\mu}$.

Tca's

A tca in the Schur model consists of a polynomial functor A equipped with a map $A \otimes A \to A$ such that $A(V)$ is a commutative associative unital ring for each V.

The category $\mathcal{S}$

Let $\mathcal{S}$ be the category of polynomial functors Vec $\to$ Vec.

We have an equivalence of categories $\mathrm{Rep}(S_*) \to \mathcal{S}$ which takes $\mathbf{M}_\lambda$ to $\mathbf{S}_\lambda$. This equivalence preserves the tensor products.

A tca in the Schur model is an object A of $\mathcal{S}$ equipped with a multiplication map $A \otimes A \to A$ satisfying the required axioms.

Representations of $\mathbf{GL}(n)$

Let V be a representation of $\mathbf{GL}(n)$. Denote by ρ the homomorphism $\mathbf{GL}(n) \to \mathbf{GL}(V)$ giving the action and choose a basis of V.

- V is **algebraic** if the matrix entries of $\rho(g)$ are rational functions of the matrix entries of g.
- V is **polynomial** if the matrix entries of $\rho(g)$ are polynomials in the matrix entries of g.

The category $\mathrm{Rep}(\mathbf{GL}(n))$ of algebraic representations of $\mathbf{GL}(n)$ is semi-simple: every algebraic representation is a direct sum of irreducible algebraic representations.

Weights

Let $T(n) \subset \mathbf{GL}(n)$ be the subgroup of diagonal matrices. It is isomorphic to $(\mathbf{C}^\times)^n$. Let $U(n) \subset \mathbf{GL}(n)$ be the group of strictly upper triangular matrices.

A **weight** is an algebraic homomorphism $T(n) \to \mathbf{C}^\times$. Every weight is of the form

$$[z_1, \ldots, z_n] \mapsto z_1^{a_1} \cdots z_n^{a_n}$$

where the a_i are integers. The group of weights is isomorphic to $\mathbf{Z}^n$.

A weight $(a_1, \ldots, a_n)$ is **dominant** if $a_1 \geq a_2 \geq \cdots \geq a_n$ and **positive** if $a_i \geq 0$ for each i.

Highest weight theory

Theorem

- *If V is an irreducible algebraic representation of* $\mathbf{GL}(n)$ *then* $V^{U(n)}$ *is one dimensional and* $T(n)$ *acts on it through a dominant weight. This weight is called the* **highest weight** *of* V*.*
- *Two irreducible representations with the same highest weight are isomorphic.*
- *Every dominant weight occurs as the highest weight of some irreducible algebraic representation.*
- *An irreducible algebraic representation is polynomial if and only if its highest weight is positive.*

Relation to Schur functors

Let $\lambda = (\lambda_1, \lambda_2, \ldots)$ be a partition. The **length** of λ, denoted $\ell(\lambda)$, is the largest n such that λ_n is non-zero.

Positive dominant weights are the same thing as partitions of length at most n.

Theorem

Let λ be a partition. If $\ell(\lambda) \leq n$ then $\mathbf{S}_\lambda(\mathbf{C}^n)$ is the irreducible representation of $\mathbf{GL}(n)$ with highest weight λ. If $\ell(\lambda) > n$ then $\mathbf{S}_\lambda(\mathbf{C}^n) = 0$.

Corollary

Every polynomial representation of $\mathbf{GL}(n)$ is a direct sum of $\mathbf{S}_\lambda(\mathbf{C}^n)$'s.

Representations of $\mathbf{GL}(\infty)$

The above theory implies that $\mathbf{S}_\lambda(\mathbf{C}^\infty)$ is a non-zero irreducible representation of $\mathbf{GL}(\infty)$ for any λ, and that $\mathbf{S}_\lambda(\mathbf{C}^\infty)$ and $\mathbf{S}_\mu(\mathbf{C}^\infty)$ are isomorphic if and only if $\lambda = \mu$.

A representation of $\mathbf{GL}(\infty)$ is **polynomial** if it is a direct sum of the $\mathbf{S}_\lambda(\mathbf{C}^\infty)$'s. We let $\mathrm{Rep}^{\mathrm{pol}}(\mathbf{GL})$ denote the category of polynomial representations.

The functor $\mathcal{S} \to \mathrm{Rep}^{\mathrm{pol}}(\mathbf{GL})$ given by $F \mapsto F(\mathbf{C}^\infty)$ is an equivalence, and preserves tensor products.

Exercise

Give a direct equivalence $\mathrm{Rep}^{\mathrm{pol}}(\mathbf{GL}) \to \mathrm{Rep}(S_*)$.

Tca's

A tca in the **GL** model is a commutative associative unital $\mathbf{C}$-algebra A on which $\mathbf{GL}(\infty)$ acts by algebra homomorphisms such that A forms a polynomial representation of $\mathbf{GL}(\infty)$.

The category $\mathcal{V}$

To summarize, we have seen that the following four categories are equivalent:

- $\mathrm{Rep}(S_*)$ — sequences of representations of symmetric groups.
- $\mathrm{Vec}^{(\mathrm{fs})}$ — functors from (fs) to Vec.
- $\mathcal{S}$ — polynomial functors of Vec.
- $\mathrm{Rep}^{\mathrm{pol}}(\mathbf{GL})$ — polynomial representations of $\mathbf{GL}(\infty)$.

Furthermore, each of these categories has a tensor product and the equivalences preserve the tensor product.

We let $\mathcal{V}$ denote an abstract tensor category equivalent to any of the above four. We use this category when we don't want to think about the details of the underlying model.

Tca's in $\mathcal{V}$

We can define tca's independent of the choice of model as an algebra in $\mathcal{V}$: a tca is an object A of $\mathcal{V}$ equipped with a commutative associative unital multiplication map $A \otimes A \to A$.

We can also define modules over a given tca: if A is a tca then an A-module is an object M of $\mathcal{V}$ equipped with a multiplication map $A \otimes M \to M$ satisfying the usual axioms.

Exercise

Unravel the definition of "module" in the four models.

The object $U\langle 1\rangle$

Let $\mathbf{C}\langle 1\rangle$ be the following object of $\mathcal{V}$:

- $\mathrm{Rep}(S_*)$: the sequence (V_n) with $V_1 = \mathbf{C}$ and $V_n = 0$ for $n \neq 1$.
- $\mathrm{Vec}^{(\mathrm{fs})}$: the functor assigning $\mathbf{C}$ to sets of cardinality 1 and 0 to all other sets.
- $\mathcal{S}$: the identity functor.
- $\mathrm{Rep}^{\mathrm{pol}}(\mathbf{GL})$: the standard representation $\mathbf{C}^\infty$.

For a vector space U we let $U\langle 1\rangle$ be $U \otimes \mathbf{C}\langle 1\rangle$.

The tca $\mathrm{Sym}(U\langle 1\rangle)$

The tca $A = \mathrm{Sym}(U\langle 1\rangle)$ is the most important tca for us. It is given in the various models as follows:

- $\mathrm{Rep}(S_*)$: the tensor algebra on U.
- $\mathrm{Vec}^{(\mathrm{fs})}$: $A_L = U^{\otimes L}$.
- $\mathcal{S}$: $A(V) = \mathrm{Sym}(U \otimes V)$.
- $\mathrm{Rep}^{\mathrm{pol}}(\mathbf{GL})$: $\mathrm{Sym}(U \otimes \mathbf{C}^\infty)$.

Other polynomial tca's

Define $\mathbf{C}\langle n\rangle$ to be $\mathbf{C}\langle 1\rangle^{\otimes n}$ and $U\langle n\rangle = U \otimes \mathbf{C}\langle n\rangle$.

Let $A = \mathrm{Sym}(\mathbf{C}\langle n\rangle)$. In the **GL**-model, $\mathbf{C}\langle n\rangle$ is $(\mathbf{C}^\infty)^{\otimes n}$, and A is the symmetric algebra on this representation.

Exercise

Work in the fs model and suppose $n = 2$. Show that A_L has a natural basis consisting of the directed graphs on L. What happens for $n > 2$?

Finite generation of tca's

A tca A is **finitely generated** if it is a quotient of $\mathrm{Sym}(F)$ for some finite length object F of $\mathcal{V}$.

A tca A is **finitely generated in degree** n if it is a quotient of $\mathrm{Sym}(U\langle n\rangle)$ for some finite dimensional vector space U.

Finite generation of tca's

In the **GL**-model, A is finitely generated if and only if there exist finitely many elements $x_1, \ldots, x_n$ such that A is generated as an algebra by the elements gx_i for $1 \leq i \leq n$ and $g \in \mathbf{GL}(\infty)$.

In the Schur model, if A is finitely generated as a tca then $A(V)$ is finitely generated as a $\mathbf{C}$-algebra for all finite dimensional V.

Exercise

Give an example of a tca A which is not finitely generated but for which $A(V)$ is finitely generated as a $\mathbf{C}$-algebra for all finite dimensional V.

Finite generation of modules

An A-module is **finitely generated** if it is a quotient of $A \otimes F$ for some finite length object F of $\mathcal{V}$.

In the **GL**-model, the A-module M is finitely generated if there exist finitely many elements $x_1, \ldots, x_n$ such that M is generated as an A-module by the gx_i for $1 \leq i \leq n$ and $g \in \mathbf{GL}(\infty)$.

In the Schur model, if M is a finitely generated A-module then $M(V)$ is a finitely generated $A(V)$-module for all V. The converse does not hold, as before.

Noetherianity

An A-module M is **noetherian** if every ascending chain of submodules stabilizes. Equivalently, every submodule of M is finitely generated.

The tca A is **noetherian** if every finitely generated A-module is noetherian.

Note: most A-modules are not quotients of a direct sum of A's. Thus noetherianity of A as a tca does not necessarily follow from noetherianity of A as an A-module.

Question

If A is noetherian as an A-module is A noetherian as a tca?

Boundedness

Recall that $\ell(\lambda)$ denotes the length of the partition λ.

For an object M of $\mathcal{V}$, we define

$$\ell(M) = \sup\{\ell(\lambda) \mid \mathbf{M}_\lambda \text{ is a constituent of } M\}.$$

We say that M is **bounded** if $\ell(M) < \infty$.

Any sub or quotient of a bounded object is bounded.

Boundedness

An important consequence of the Littlewood–Richardson rule is the identity $\ell(M \otimes N) = \ell(M) + \ell(N)$. Therefore:

Proposition

The tensor product of bounded objects is bounded.

Corollary

A finitely generated module over a bounded tca is bounded.

Boundedness principle

If M is a bounded object, with any kind of extra structure, then one can recover M completely from $M(\mathbf{C}^n)$ if n is sufficiently large.

This principle is very useful, since $M(\mathbf{C}^n)$ tends to lie in the realm of familiar commutative algebra.

Here is one instance of the boundedness principle:

Proposition

Suppose $\ell(M) \le n$. Then $N \mapsto N(\mathbf{C}^n)$ defines a bijection

$$\{\text{subobjects of } M\} \to \{\mathbf{GL}(n)\text{-subrepresentations of } M(\mathbf{C}^n)\}.$$

Proof.

Write $M = \bigoplus_{\ell(\lambda) \le n} V_\lambda \otimes \mathbf{S}_\lambda$ where V_λ is a multiplicity space. To give a subobject of M amounts to giving a subspace of V_λ for each λ.

We have $M(\mathbf{C}^n) = \bigoplus_{\ell(\lambda) \le n} V_\lambda \otimes \mathbf{S}_\lambda(\mathbf{C}^n)$. By the length condition, the representations $\mathbf{S}_\lambda(\mathbf{C}^n)$ are irreducible and pairwise non-isomorphic. It follows that giving a $\mathbf{GL}(n)$-subrepresentation of $M(\mathbf{C}^n)$ is also the same as giving a subspace of V_λ for each λ. □

Here is another, closely related, instance:

Proposition

Suppose M is an A-module and $\ell(M) \le n$. Then $N \mapsto N(\mathbf{C}^n)$ defines a bijection

$$\{A\text{-submodules of } M\} \to \{\mathbf{GL}(n)\text{-stable } A(\mathbf{C}^n)\text{-submodules of } M(\mathbf{C}^n)\}.$$

Exercise

Prove this. (The proof is similar to that of the previous proposition.)

Theorem

A finitely generated bounded tca is noetherian.

Proof.

Suppose A is finitely generated and bounded. Let M be a finitely generated A-module and put $n = \ell(M)$. Then $N \mapsto N(\mathbf{C}^n)$ defines an injection

$$\{A\text{-submodules of } M\} \to \{A(\mathbf{C}^n)\text{-submodules of } M(\mathbf{C}^n)\}.$$

Since $A(\mathbf{C}^n)$ is a finitely generated $\mathbf{C}$-algebra, it is noetherian. Since M is a finitely generated A-module, $M(\mathbf{C}^n)$ is a finitely generated $A(\mathbf{C}^n)$-module, and therefore noetherian. It follows that the right side above satisfies ACC, and so the left side does as well. □

Theorem

The tca $A = \mathrm{Sym}(U\langle 1\rangle)$ *is bounded; in fact,* $\ell(A) = \dim(U)$.

Proof.

We have

$$A(V) = \mathrm{Sym}(U \otimes V) = \bigoplus_{\lambda} \mathbf{S}_\lambda(U) \otimes \mathbf{S}_\lambda(V),$$

where the sum is over all partitions. This is the **Cauchy formula**. Since $\mathbf{S}_\lambda(U) = 0$ if $\ell(\lambda) > \dim(U)$, only those $\mathbf{S}_\lambda(V)$ with $\ell(\lambda) \leq \dim(U)$ are constituents of A. □

Exercise

Prove the Cauchy formula.

Since a tca finitely generated in degree 1 is a quotient of $\mathrm{Sym}(U\langle 1\rangle)$, we find:

Corollary

A tca finitely generated in degree 1 is noetherian.

The boundedness principle is the primary approach to studying bounded objects, but it does not trivialize all problems.

For example, consider the problem of determining the free resolution of an A-module M, where $A = \mathrm{Sym}(\mathbf{C}\langle 1 \rangle)$.

The free resolution of $M(\mathbf{C}^n)$ is finite since $A(\mathbf{C}^n) = \mathbf{C}[x_1, \ldots, x_n]$, but the resolution of M itself is typically infinite.

Thus, even though the resolution of M can be recovered from $M(\mathbf{C}^n)$ in principle, it is not the case that the resolution of $M(\mathbf{C}^n)$ immediately gives the resolution of M.

See [SS2] for a detailed study of resolutions of A-modules.

Hilbert series

Let M be an object of $\mathcal{V}$, taken in the sequence model. We define the **Hilbert series** of M by

$$H_M(t) = \sum_{n=0}^{\infty} \dim(M_n) \frac{t^n}{n!}.$$

Obviously, this is only defined when each M_n is finite dimensional.

Exercise

Show that $H_{M \otimes N}(t) = H_M(t) H_N(t)$.

An example of Hilbert series

Let $A = \mathrm{Sym}(U\langle 1\rangle)$, where U has dimension d. In the sequence model, $A_n = U^{\otimes n}$ and so $\dim(A_n) = d^n$.

We therefore have

$$H_A(t) = \sum_{n \geq 0} d^n \frac{t^n}{n!} = e^{dt}.$$

Another example of Hilbert series

Let $A = \mathrm{Sym}(U\langle 1\rangle)$, where U has dimension d. Let B be the quotient of A by the ideal generated by $(n+1) \times (n+1)$ minors. Thus $B(\mathbf{C}^\infty)$ is the coordinate ring of the rank n determinantal variety in $\mathrm{Hom}(U, \mathbf{C}^\infty)$.

We have a decomposition

$$B(\mathbf{C}^\infty) = \bigoplus_{\ell(\lambda) \le n} \mathbf{S}_\lambda(U) \otimes \mathbf{S}_\lambda(\mathbf{C}^\infty).$$

It follows that

$$H_B(t) = \sum_{\ell(\lambda) \le n} \dim(\mathbf{S}_\lambda(U)) \dim(\mathbf{M}_\lambda) \frac{t^{|\lambda|}}{|\lambda|!}.$$

One can attempt to compute this sum using the hook length and hook content formulas. We will give a better way.

The main theorem on Hilbert series

Theorem

Let M be a finitely generated module over a tca finitely generated in degree 1. Then $H_M(t)$ is a polynomial in t and e^t.

- Define $H^*_M(t)$ like $H_M(t)$ but without the factorials. The theorem is equivalent to the statement that $H^*_M(t)$ is a rational function whose poles are of the form $1/k$ with k a positive integer.
- The series $H_M(t)$ forgets a lot of information about M, namely the S_n action on each piece. It is possible to define an **enhanced Hilbert series** which records this information. There is a corresponding rationality result for it. See [SS2].

Equivariant Hilbert series

Let G be a group, and let $\mathrm{K}(G)$ denote the representation ring of G.

Suppose M is a non-negatively graded representation of G. We define the G-**equivariant Hilbert series** of M by

$$H_{M,G}(t) = \sum_{n=0}^{\infty} [M_n] t^n,$$

where $[M_n]$ denotes the class of M_n in $\mathrm{K}(G)$. This series belongs to $\mathrm{K}(G)[\![t]\!]$.

Similarly, if M is an object of $\mathcal{V}$ with an action of G, we have the Hilbert series $H_{M,G}(t)$ (with factorials) and $H^*_{M,G}(t)$ (without factorials).

Notation

- Let $T = T(n)$ be the diagonal torus in $\mathbf{GL}(n)$.
- Let $\alpha_i \colon T \to \mathbf{C}^\times$, for $1 \le i \le n$, be the standard projectors.
- We identify $\mathrm{K}(T)$ with $\mathbf{Z}[\alpha_i^{\pm 1}]$.
- We let $f \mapsto \overline{f}$ be the involution of $\mathrm{K}(T)$ which sends α_i to α_i^{-1}.
- We put $|f|^2 = f\overline{f}$.
- We let $\int_T d\alpha \colon \mathrm{K}(T) \to \mathbf{Z}$ be the map which sends 1 to 1 and all other monomials to 0.
- We put $\Delta(\alpha) = \prod_{i<j}(\alpha_i - \alpha_j)$.

Weyl's integration formula

Suppose χ_1 and χ_2 are the characters of irreducible algebraic representations of $\mathbf{GL}(n)$, regarded as elements of $\mathrm{K}(T)$.

We have the following formula of Weyl:

$$\frac{1}{n!}\int_T \chi_1 \overline{\chi}_2 |\Delta|^2 d\alpha = \begin{cases} 1 & \text{if } \chi_1 = \chi_2 \\ 0 & \text{otherwise} \end{cases}$$

The key formula

By the boundedness principle, we can recover $H_M(t)$ from $M(\mathbf{C}^n)$ for n sufficiently large, assuming M is bounded. The following result makes this explicit:

Proposition

Let $M \in \mathcal{V}$ satisfy $\ell(M) \leq n$. Then

$$H_M(t) = \frac{1}{n!} \int_T H_{M(\mathbf{C}^n),T}(t; \alpha) \exp\left(\sum \overline{\alpha}_i\right) |\Delta|^2 d\alpha.$$

Proof of the key formula

- Write $M = \bigoplus V_\lambda \otimes \mathbf{S}_\lambda$, where V_λ is a multiplicity space.
- $H_M(t) = \sum \dim(V_\lambda) \dim(\mathbf{M}_\lambda) \frac{t^{|\lambda|}}{|\lambda|!}$.
- $H_{M(\mathbf{C}^n),T}(t;\alpha) = \sum \dim(V_\lambda)(\text{the character of } \mathbf{S}_\lambda(\mathbf{C}^n)) t^{|\lambda|}$.
- Put $f(\alpha) = \sum \frac{1}{|\lambda|!} \dim(\mathbf{M}_\lambda) \cdot (\text{the character of } \mathbf{S}_\lambda(\mathbf{C}^n))$.
- Weyl's integration formula gives

$$H_M(t) = \frac{1}{n!} \int_T H_{M(\mathbf{C}^n),T}(t;\alpha) f(\overline{\alpha}) |\Delta|^2 d\alpha.$$

Proof of the key formula (cont'd)

- Schur–Weyl gives a decomposition

$$(\mathbf{C}^n)^{\otimes k} = \bigoplus_{|\lambda|=k} \mathbf{M}_\lambda \otimes \mathbf{S}_\lambda(\mathbf{C}^n).$$

- The character of the left side is $(\sum \alpha_i)^k$.
- " right side is $\sum \dim(\mathbf{M}_\lambda) \cdot (\text{the character of } \mathbf{S}_\lambda(\mathbf{C}^n))$.
- Dividing by $k!$ and summing over k gives $f(\alpha) = \exp(\sum \alpha_i)$.

Rationality of equivariant Hilbert series

Let A_0 be the polynomial ring $\mathrm{Sym}(U \otimes \mathbf{C}^n)$. The group T acts on A_0 through its action on $\mathbf{C}^n$.

Lemma

Let M_0 be a finitely generated A_0-module with a compatible action of T. Then

$$H_{M_0,T}(t;\alpha) = \frac{p(t;\alpha)}{\prod_{i=1}^{n}(1-\alpha_i t)^d}$$

where p is a polynomial and $d = \dim(U)$.

Exercise

Prove the lemma.

Proof of main theorem

Let $A = \mathrm{Sym}(U\langle 1\rangle)$ and let M be a finitely generated A-module. Put $n = \ell(M)$ and $d = \dim(U)$.

Combining the previous lemma and the key formula, we obtain

$$H_M(t) = \int_T \frac{p(t;\alpha)}{\prod_{i=1}^n (1-\alpha_i t)^d} \exp\left(\sum \overline{\alpha}_i\right) d\alpha$$

for some polynomial p. (We have absorbed the $n!$ and Δ into p.)

It is now an elementary computation to show that this integral is a polynomial in t and e^t.

Revisiting the second example

Recall B is the quotient of $A = \mathrm{Sym}(U\langle 1\rangle)$ by $(n+1)\times(n+1)$ minors. We have $\ell(B) = n$ and

$$B(\mathbf{C}^n) = \bigoplus_{\ell(\lambda)\le n} \mathbf{S}_\lambda(U)\otimes \mathbf{S}_\lambda(\mathbf{C}^n) = \mathrm{Sym}(U\otimes \mathbf{C}^n).$$

We have $H_{B(\mathbf{C}^n),T}(t;\alpha) = \prod (1-\alpha_i t)^{-d}$, where $d = \dim(U)$. The key formula gives

$$H_B(t) = \frac{1}{n!}\int_T \frac{|\Delta(\alpha)|^2}{\prod_{i=1}^n (1-\alpha_i t)^d} \exp\left(\sum \overline{\alpha}_i\right) d\alpha.$$

An equivariant form of the main theorem

Let G be a reductive group, let U be a representation of G and put $A = \mathrm{Sym}(U\langle 1\rangle)$.

Theorem

Let M be a finitely generated A-module with a compatible action of G. Then $H^*_{M,G}(t)$ is a rational function.

- Definition of rational: can multiply by a polynomial $q \in \mathrm{K}(G)[t]$ with $q(0) = 1$ and get a polynomial.
- The proof of this theorem is similar to that of the non-equivariant version, but more complicated.
- Rationality of $H^*_{M,G}(t)$ does not imply anything nice about $H_{M,G}(t)$.

Open problems

Question

Are finitely generated tca's noetherian?

- The tca $A = \mathrm{Sym}(\mathrm{Sym}^2(\mathbf{C}^\infty))$ satisfies ACC for ideals, and is almost certainly noetherian (though this is not proved). Note: $A = \mathbf{C}[x_{ij}]$ with $i \leq j$. This ring is **not** noetherian as an S_∞-ring.
- To show that, e.g., $A = \mathrm{Sym}(\bigwedge^3(\mathbf{C}^\infty))$ is noetherian, one might first try to show that $\mathrm{Spec}(A)$ is noetherian as a topological space. This would involve understanding the structure of $\mathbf{GL}(\infty)$ orbits on the variety $\bigwedge^3(\mathbf{C}^\infty)$.

Open problems (cont'd)

Some other problems:

- How does the Hilbert series of an A-module relate to the structure of the module?
- Does a noetherian tca have finitely many minimal prime ideals? Known in the bounded case.
- To what extent does primary decomposition hold for tca's?
- Is there a good dimension theory for tca's?
- What can one say about Hilbert series in the unbounded case? The Hilbert series of $\mathrm{Sym}(\mathbf{C}\langle n\rangle)$ is e^{t^n}, so one might hope for positive results.
- Relationship between $\mathbf{GL}(\infty)$ noetherianity and S_∞ noetherianity?

FI-modules

Church–Ellenberg–Farb (arXiv:1204.4533) introduce algebraic objects which they call "FI-modules." They give many examples of these modules: for instance, the cohomology of certain configuration spaces (as the number of points varies) forms an FI-module.

In fact, an FI-module is just a module over the tca $\mathrm{Sym}(\mathbf{C}\langle 1\rangle)$, viewed in the sequence model. See [SS1] for details.

EFW resolutions

Eisenbud–Fløystan–Weyman (arXiv:0709.1529v5) constructed pure resolutions, which was a key step in the proof of the Boij–Söderberg conjecture. Their construction can actually be seen as the computation of the projective resolutions of certain finite length modules over the tca $\mathrm{Sym}(\mathbf{C}\langle 1\rangle)$. See [SS1] for details.

Representation theory of infinite rank groups

We have been working with the category of polynomial representations of $\mathbf{GL}(\infty)$. One can define a larger category of algebraic representations of $\mathbf{GL}(\infty)$, or of other groups such as $\mathbf{O}(\infty)$. These categories are not semi-simple, in general.

In forthcoming work, S. Sam and I relate these categories to tca's. For instance, we show that $\mathrm{Rep}(\mathbf{O}(\infty))$ is equivalent to the category of finite length modules over the tca $\mathrm{Sym}(\mathrm{Sym}^2(\mathbf{C}^\infty))$. This allows us to use tools from commutative algebra, such as the Koszul complex, to study representations.

§3. Algebras in Sym($\mathcal{S}$)

Multivariate polynomial functors

A functor $F\colon \mathrm{Vec}^n \to \mathrm{Vec}$ is called **polynomial** if for any $(V_1, \dots, V_n)$ and $(V_1', \dots, V_n')$, the induced map

$$\mathrm{Hom}(V_1, V_1') \times \cdots \mathrm{Hom}(V_n, V_n') \to \mathrm{Hom}(F(V_1, \dots, V_n), F(V_1', \dots, V_n'))$$

induced by F is a polynomial map of vector spaces.

If $\lambda_1, \dots, \lambda_n$ are partitions then

$$(V_1, \dots, V_n) \mapsto \mathbf{S}_{\lambda_1}(V_1) \otimes \cdots \otimes \mathbf{S}_{\lambda_n}(V_n)$$

is a polynomial functor.

Proposition

Any polynomial functors is a direct sums of these.

Equivariant functors

Let $F \colon \mathrm{Vec}^n \to \mathrm{Vec}$ be a functor. An S_n-**equivariant structure** on F consists of giving for each $\sigma \in S_n$ an isomorphism of functors

$$\sigma_* \colon F(V_1, \ldots, V_n) \to F(V_{\sigma(1)}, \ldots, V_{\sigma(n)})$$

which satisfy an obvious compatibility condition (roughly $(\sigma\tau)_* = \sigma_*\tau_*$).

Not all functors admit an S_n-equivariant structure. For instance, $(V_1, V_2) \mapsto \mathrm{Sym}^2(V_1) \otimes \bigwedge^2(V_2)$ does not, since the roles of V_1 and V_2 are asymmetrical.

A functor can admit multiple equivariant structures. For instance, if $F(V_1, \ldots, V_n)$ is a constant functor, equal to some fixed vector space W regardless of its input, then giving an S_n-equivariant structure on F is the same as giving a representation of S_n on W.

The sequence model for $\mathrm{Sym}(\mathcal{S})$

The sequence model for $\mathrm{Sym}(\mathcal{S})$ is the following category:

- Objects are sequences $(F_n)_{n \geq 0}$, where $F_n \colon \mathrm{Vec}^n \to \mathrm{Vec}$ is an S_n-equivariant polynomial functor.
- A morphism $f \colon (F_n) \to (F'_n)$ consists of morphisms of S_n-equivariant functors $f_n \colon F_n \to F'_n$ for each n.

This is a souped-up version of the category $\mathrm{Rep}(S_*)$.

Recall that a Δ-module consists of a a rule assigning to each $(V_1, \ldots, V_n)$ a vector space $F_n(V_1, \ldots, V_n)$ with the additional structure (C1)–(C3). (C1) is simply the structure of a functor on F_n, while (C2) is an S_n-equivariant structure on S_n. Thus a Δ-module defines an object of $\mathrm{Sym}(\mathcal{S})$ (though it has even more structure, namely (C3)).

The category Vec^f

Let Vec^f be the category of finite families of vector spaces:

- Objects are pairs (V, L) where L is a finite set and V assigns to each $x \in L$ a vector space V_x.
- A morphism $(V, L) \to (V', L')$ consists of a bijection $\varphi \colon L' \to L$ and for each $x \in L$ a linear map $V_x \to V_{\varphi^{-1}(x)}$.

The category Vec^n is identified with the subcategory of Vec^f where $L = [n] = \{1, \ldots, n\}$.

The fs model for Sym($\mathcal{S}$)

A functor $F\colon \mathrm{Vec}^f \to \mathrm{Vec}$ is **polynomial** if its restriction to each Vec^n is polynomial.

The fs model for Sym($\mathcal{S}$) is the category of all polynomial functors $\mathrm{Vec}^f \to \mathrm{Vec}$. Morphsism are natural transformations of functors. This is a souped-up version of $\mathrm{Vec}^{(\mathrm{fs})}$.

Exercise

Let $F\colon \mathrm{Vec}^f \to \mathrm{Vec}$ be a polynomial functor, and define F_n to be the restriction of F to Vec^n. Show that F_n is naturally an S_n-equivariant functor, and $F \mapsto (F_n)_{n\geq 0}$ defines an equivalence between the fs and sequence models of Sym($\mathcal{S}$).

The tensor product on Sym($\mathcal{S}$)

Let $F, G \colon \mathrm{Vec}^f \to \mathrm{Vec}$ be polynomial functors. Define

$$(F \otimes G)(V, L) = \bigoplus_{L = A \amalg B} F(V|_A, A) \otimes G(V|_B, B).$$

This is a direct generalization of the tensor product on $\mathrm{Vec}^{(\mathrm{fs})}$.

Example

Suppose $F_1 = \mathrm{Sym}^2$ and $F_n = 0$ for $n \neq 1$ and $G_1 = \bigwedge^2$ and $G_n = 0$ for $n \neq 1$. Then

$$(F \otimes G)(V, [2]) = \mathrm{Sym}^2(V_1) \otimes \textstyle\bigwedge^2(V_2) \oplus \bigwedge^2(V_1) \otimes \mathrm{Sym}^2(V_2),$$

and $(F \otimes G)(V, L) = 0$ if $\#L \neq 2$. Note: the Littlewood–Richardson rule never comes in to play!

Algebras in Sym($\mathcal{S}$)

Since Sym($\mathcal{S}$) has a tensor product, we have a notion of (commutative, associative, unital) algebras in Sym($\mathcal{S}$). Explicitly, an algebra is a polynomial functor $A \colon \mathrm{Vec}^f \to \mathrm{Vec}$ equipped with a multiplication map

$$A(V, L) \otimes A(V', L') \to A(V \amalg V', L \amalg L').$$

for all (V, L) and (V', L') in Vec^f. Such algebras are souped-up versions of tca's.

An example of an algebra

Let $F \in \mathcal{S}$ be a polynomial functor, regarded as an object of Sym($\mathcal{S}$) in degree 1. Let $A = \mathrm{Sym}(F)$ be the symmetric algebra on F.

Exercise

Show that

$$A(V, L) = \bigotimes_{x \in L} F(V_x)$$

The multiplication map $A(V, L) \otimes A(V', L') \to A(V \amalg V', L \amalg L')$ is just concatenation of tensors.

This algebra is the analogue in Sym($\mathcal{S}$) of the tca $\mathrm{Sym}(U\langle 1 \rangle)$. In fact, if F is the constant functor $F(V) = U$ then A is the constant functor $(V, L) \mapsto U^{\otimes L}$, and so $A = \mathrm{Sym}(U\langle 1 \rangle)$.

Evaluation on constant families

Let U be a vector space. Denote by U_L the constant family (V, L) where $V_x = U$ for all $x \in L$. We denote by $i\colon (\mathrm{fs}) \to \mathrm{Vec}^f$ the functor $L \mapsto U_L$.

If $F\colon \mathrm{Vec}^f \to \mathrm{Vec}$ is a polynomial functor then $L \mapsto F(U_L)$ is an object of $\mathrm{Vec}^{(\mathrm{fs})}$. We denote this by $i^*(F)$. We thus have a functor $i^*\colon \mathrm{Sym}(\mathcal{S}) \to \mathcal{V}$.

The functor i^* is compatible with tensor products, and so takes algebras to algebras. The algebra $\mathrm{Sym}(F)$ goes to the tca $\mathrm{Sym}(F(U)\langle 1 \rangle)$.

Note that $i^*(F)$ always carries an action of $\mathbf{GL}(U)$.

Vertical boundedness

Let $F \colon \mathrm{Vec}^n \to \mathrm{Vec}$ be a polynomial functor. We can decompose F as a direct sum of tensor products of Schur functors $\mathbf{S}_\lambda$. Define $L(F)$ as the supremum of $\ell(\lambda)$ over those λ for which $\mathbf{S}_\lambda$ occurs in this decomposition.

For an object $F = (F_n)$ of Sym($\mathcal{S}$), define $L(F)$ as the supremum of the $L(F_n)$. We say F is **vertically bounded** if $L(F) < \infty$.

Example

Let $F \in \mathcal{S}$ and let $A = \mathrm{Sym}(F)$. We saw that $A(V, L) = \bigotimes F(V_x)$. Thus $L(A) = \ell(F)$. In particular, if F has finite length then A is vertically bounded.

Failure of the boundedness principle

Let $F \in \mathrm{Sym}(\mathcal{S})$ and let U be a vector space with $\dim(U) \geq L(F)$. One might hope for a "boundedness principle" where one does not lose information by evaluating on U_L. However, this is not the case.

For example, suppose $F, G \in \mathcal{S}$ and let $A \colon \mathrm{Vec}^2 \to \mathrm{Vec}$ be given by

$$A(V_1, V_2) = F(V_1) \otimes G(V_2) \oplus G(V_1) \otimes F(V_2).$$

We regard $A \in \mathrm{Sym}(\mathcal{S})$.

We have $A(U_L) = (F(U) \otimes G(U))^{\oplus 2}$ if $\#L = 2$ and $A(U_L) = 0$ otherwise. Thus one can only $F \otimes G \in \mathcal{S}$ from $A(U_L)$, and not F and G individually.

Despite the failure of the boundedness principle in general, one does have the following result:

Proposition

Let M be an object of Sym($\mathcal{S}$) *and let U be a vector space with* $\dim(U) \geq L(M)$. *If N and N' are subobjects of M such that* $i^*(N) = i^*(N')$ *then $N = N'$.*

Proof.

Decompose M as $\bigoplus V_{\lambda_1,\ldots,\lambda_n} \otimes \mathbf{S}_{\lambda_1} \otimes \cdots \otimes \mathbf{S}_{\lambda_n}$ where the V's are multiplicity spaces. The subobjects N and N' correspond to subspaces of the multiplicity spaces. The point is simply that none of the Schur functors appearing in M vanish on U, and so one can check for equality of subspaces of multiplicity spaces after evalauting on U. □

Theorem

An algebra in Sym($\mathcal{S}$) *finitely generated in degree 1 is noetherian.*

Proof.

Let $A = \mathrm{Sym}(F)$ where $F \in \mathcal{S}$ has finite length. It suffices to show A is noetherian. Let M be a finitely generated A-module. Choose a vector space U with $\dim(U) \geq L(M)$ and put $A' = i^*(A)$ and $M' = i^*(M)$. Then A' is a tca and M' is an A'-module. We have a map

$$\{A\text{-submodules of } M\} \to \{A'\text{-submodules of } M'\}$$

given by $N \mapsto i^*(N)$. This is injective by the previous proposition. The right side satisfies ACC since A' is noetherian. Thus the left side satisfies ACC and A is noetherian. □

Analysis of proof

The tca A' in the above proof is $\mathrm{Sym}(F(U)\langle 1\rangle)$. We deduced noetherianity of A from that of A'.

Recall that we deduced noetherianity of A' from that of an ordinary polynomial ring by a similar argument. We have $\ell(A') = \dim(F(U))$, and so by the boundedness principle one does not lose information by evaluating A' on a vector space V of this dimension. Noetherianity of $A'(V) = \mathrm{Sym}(V \otimes F(U))$ implies that of A'.

So ultimately, we work with a polynomial ring in $\dim(F(U))^2$ variables. If, e.g., F is the pth tensor power functor then $L(A) = \ell(F) = p$ and thus $\dim(U) = p$. So $F(U)$ has dimension p^p, and we require p^{2p} variables!

Note also that the tca A' appearing in the proof is naturally given in the fs model, but our proof that A' is noetherian naturally uses the Schur model. So it is important to be able to switch between these models.

Definition of the Hilbert series

Let $F\colon \mathrm{Vec}^n \to \mathrm{Vec}$ be a polynomial functor. Decompose F as

$$F(V_1, \dots, V_n) = \bigoplus_{i \in I} \mathbf{S}_{\lambda_{1,i}}(V_1) \otimes \cdots \otimes \mathbf{S}_{\lambda_{n,i}}(V_n)$$

over some index set I. Define polynomials in variables s_λ by

$$H_F^* = \sum_{i \in I} s_{\lambda_{1,i}} \cdots s_{\lambda_{n,i}}, \qquad H_F = \frac{1}{n!} H_F^*.$$

In general, F cannot be recovered from H_F. For example, if $H_F^* = s_\lambda s_\mu$ then $F(V_1, V_2)$ can either be $\mathbf{S}_\lambda(V_1) \otimes \mathbf{S}_\mu(V_2)$ or $\mathbf{S}_\mu(V_1) \otimes \mathbf{S}_\lambda(V_1)$.

However, if F admits an S_n-equivariant structure, then it can be recovered from H_F.

Definition of the Hilbert series (cont'd)

Let $F = (F_n)$ be an object of Sym($\mathcal{S}$), taken in the sequence model. We define the **Hilbert series** of F by

$$H_F = \sum_{n \geq 0} H_{F_n}, \qquad H_F^* = \sum_{n \geq 0} H_{F_n}^*.$$

These are formal power series in the variables s_λ.

One can recover each F_n, as a functor $\mathrm{Vec}^n \to \mathrm{Vec}$, from H_F. However, the data of the S_n-equivariance is lost.

In general, H_F can involve infinitely many variables. However, in cases of interest, all the partitions appearing in F will have the same size, and so H_F will only involve finitely many of the s_λ.

An example of Hilbert series

Let $A = \mathrm{Sym}(\mathbf{S}_\lambda)$. Then $A(V, L) = \bigotimes_{x \in L} \mathbf{S}_\lambda(V_x)$ and so

$$A_n(V_1, \ldots, V_n) = \mathbf{S}_\lambda(V_1) \otimes \cdots \otimes \mathbf{S}_\lambda(V_n).$$

We therefore have $H^*_{A_n} = s_\lambda^n$ and so

$$H^*_A = \frac{1}{1 - s_\lambda}, \qquad H_A = e^{s_\lambda}.$$

Main theorem on Hilbert series

Theorem

Let M be a finitely generated module over an algebra A in Sym($\mathcal{S}$) *which is finitely generated in degree 1. Then H_M^* is a rational function in the s_λ.*

Question

Is it the case that H_M is a polynomial in the s_λ and the e^{s_λ}? This is not implied by the theorem, but holds for all examples I know.

Sketch of proof

Let A and M be as in the statement of the theorem. Choose U with $\dim(U) \geq L(M)$ and define $A' = i^*(A)$ and $M' = i^*(M)$. Then A' is a tca finitely generated in degree 1 and M' is a finitely generated A'-module.

The group $G = \mathbf{GL}(U)$ acts on A' and M'. We can therefore consider the G-equivariant Hilbert series $H^*_{M',G}$, which is a power series with coefficients in $\mathrm{K}(G)$. This is rational by earlier results.

Unfortunately, we cannot recover H^*_M from $H^*_{M',\mathbf{GL}(U)}$. We have already seen the reason: the Schur functors appearing in M are multiplied together in M'.

Sketch of proof (cont'd)

Fortunately, a modification of this idea does work. Let $U_1, \ldots, U_n$ be copies of U and let $G = \mathbf{GL}(U_1) \times \cdots \times \mathbf{GL}(U_n)$. Define a tca A' by

$$A'_L = \bigoplus_{L = L_1 \amalg \cdots \amalg L_n} A(U_{L_1}) \otimes \cdots \otimes A(U_{L_n})$$

and define M' similarly.

As before, G acts on A' and M' and the equivariant Hilbert series $H^*_{M',G}$ is rational.

One can show that H^*_M **can** be recovered from $H^*_{M',G}$ is n is taken to be sufficiently large. This gives rationality of H^*_M.

§4. Δ-modules

The sequence model of Δ-modules

Using the language we now have, we can rephrase our original definition as follows: a Δ-module is a sequence $(F_n)_{n \geq 0}$, where $F_n \colon \mathrm{Vec}^n \to \mathrm{Vec}$ is an S_n-equivariant polynomial functor, equipped with natural transformations

$$F_n(V_1, \ldots, V_{n-1}, V_n \otimes V_{n+1}) \to F_{n+1}(V_1, \ldots, V_{n+1}).$$

This natural transformation is the data originally called (C3).

There are still compatibility conditions required between various pieces of structure. We prefer not to state these conditions explicitly; they will be automatically handled in a fs model of Δ-modules.

The category Vec^Δ

Let Vec^Δ be the following category:

- Objects are families of vector spaces (V, L) as in Vec^Δ.
- A morphism $(V, L) \to (V', L')$ consists of a surjection $\varphi \colon L' \to L$ and for each $x \in L$ a linear map $V_x \to \bigotimes_{\varphi(y)=x} V'_y$.

There is a map

$$((V_1, \ldots, V_{n-1}, V_n \otimes V_{n+1}), [n]) \to ((V_1, \ldots, V_{n+1}), [n+1])$$

in Vec^Δ, where the surjection $[n+1] \to [n]$ collapses n and $n+1$ to n.

The fs model of Δ-modules

A Δ-**module** is a polynomial functor $F\colon \mathrm{Vec}^{\Delta} \to \mathrm{Vec}$. (Polynomial means that the restriction to Vec^f is polynomial.)

The map

$$((V_1, \ldots, V_{n-1}, V_n \otimes V_{n+1}), [n]) \to ((V_1, \ldots, V_{n+1}), [n+1])$$

induces the structure (C3) on Δ-modules.

The Δ-module Q_n

Define Q_n to be the Δ-module given by

$$Q_n(V, L) = \bigotimes_{x \in L} V_x^{\otimes n}.$$

A map $(V, L) \to (V', L')$ in Vec^Δ consists of a surjection $\varphi \colon L' \to L$ and linear maps $V_x \to \bigotimes_{\varphi(y)=x} V_y$ for $x \in L$. Taking the nth tensor power of this map and then tensoring over $x \in L$ gives a map $Q_n(V, L) \to Q_n(V', L')$. This explains how Q_n is a functor on Vec^Δ.

Q_n as an algebra in $\mathrm{Sym}(\mathcal{S})$

The Δ-module Q_n also has the structure of an algebra in $\mathrm{Sym}(\mathcal{S})$. This algebra structure is simply the map

$$Q_n(V, L) \otimes Q_n(V', L') \to Q_n(V \amalg V', L \amalg L')$$

given by concatenation of tensors. In fact, Q_n is the tensor algebra on the nth tensor power functor.

As Q_n is finitely generated in degree 1, our results on $\mathrm{Sym}(\mathcal{S})$ algebras (noetherianity, Hilbert series) apply to it.

We note that the symmetric group S_n acts on Q_n. This action is compatible with the algebra and Δ-module structure.

The key result on Δ-modules

Theorem

Any Δ-submodule of Q_n is automatically a $Q_n^{S_n}$-submodule.

Proof.

We must show that if $a \in Q_n(V, L)^{S_n}$ and $m \in Q_n(V', L')$ then am belongs to the Δ-submodule of Q_n generated by m. Since $Q_n(V, L)^{S_n}$ is spanned by nth powers, it suffices to treat the case where $a = a_0^{\otimes n}$ with $a_0 \in Q_1(V, L)$.

Pick an element $x \in L'$. Define a map $(V', L') \to (V \amalg V', L \amalg L')$ as follows. The surjection $L \amalg L' \to L'$ is the identity on L' and collapses L to x. The map $V'_x \to V'_x \otimes \bigotimes_{y \in L} V_y$ is $\mathrm{id} \otimes a_0$. This map in Vec^{Δ} induces a map $Q_n(V', L') \to Q_n(V \amalg V', L \amalg L')$ by the Δ-module structure on Q_n, under which m maps to am. □

Noetherianity of Δ-modules

Theorem

The Δ-module Q_n is noetherian.

Proof.

An ascending chain of Δ-submodules is an ascending chain of $Q_n^{S_n}$-submodules of Q_n. Since Q_n is noetherian and S_n is a finite group, Q_n is noetherian as a module over $Q_n^{S_n}$, and so any such ascending chain stabilizes. □

Hilbert series of Δ-modules

The Hilbert series of a Δ-module is defined to be the Hilbert series of the underlying object in $\mathrm{Sym}(\mathcal{S})$.

Theorem

The Hilbert series of any subquotient of Q_n is rational.

Proof.

Any such subquotient is naturally a finitely generated module over $Q_n^{S_n}$. Rationality follows from rationality of Hilbert series for finitely generated Q_n-modules. (The S_n doesn't affect much.) □

§5. Applications to syzygies

Syzygies

Let $S = \mathrm{Sym}(V)$ be a polynomial ring and let R be a quotient ring. The space of p-syzygies of R is $\mathrm{Tor}_p^S(R, \mathbf{C})$. If $F_\bullet \to R$ is a minimal free resolution of R as an S-module then this Tor is just F_p/S_+F_p.

This Tor can also be calculated using the free resolution of $\mathbf{C}$ as an S-module. This resolution, the **Koszul resolution**, is given by $S \otimes \bigwedge^\bullet(V)$. Tensoring with R over S, we see that the complex $K = R \otimes \bigwedge^\bullet(V)$ computes $\mathrm{Tor}_p^S(R, \mathbf{C})$.

Suppose V' is another vector space, $S' = \mathrm{Sym}(V')$ and R' is a quotient of S'. Suppose $V \to V'$ is a linear map which carries R to R'. Then there is an induced morphism $K \to K'$ and thus $\mathrm{Tor}_p^S(R, \mathbf{C}) \to \mathrm{Tor}_p^{S'}(R', \mathbf{C})$.

Δ-varieties

For $(V, L) \in \mathrm{Vec}^{\Delta}$, let $\mathbf{V}(V, L) = \bigotimes_{x \in L} V_x^*$. The structure (A1)–(A3) shows that $\mathbf{V}$ defines a contravariant functor from Vec^{Δ} to the category of varieties.

A **Δ-variety** is a contravariant functor X from Vec^{Δ} to varieties equipped with a closed immersion $X \to \mathbf{V}$.

Syzygies of Δ-varieties

Let $S(V, L)$ be the the coordinate ring of $\mathbf{V}(V, L)$ and let $S_d(V, L)$ be its degree d piece. Explicitly, $S_d(V, L) = \mathrm{Sym}^d(Q_1(V, L))$ where $Q_1(V, L) = \bigotimes_{x \in L} V_x$. This is a Δ-module, and a quotient of Q_d.

Let $R(V, L)$ be the coordinate ring of $X(V, L)$ and let $R_d(V, L)$ be its degree d piece. This is a Δ-module, and a quotient of $S_d(V, L)$.

Let $K^p(V, L) = R(V, L) \otimes \bigwedge^p(Q_1(V, L))$. Let $K^{p,d}(V, L) = R_{p-d}(V, L) \otimes \bigwedge^p(Q_1(V, L))$ be its degree d piece. This is a Δ-module, and a quotient of Q_d.

Syzygies of Δ-varieties (cont'd)

The Koszul differentials give $K^{\bullet,d}$ the structure of a complex. Let $F^{p,d}$ be its pth homology. This is the space of p-syzygies of degree d for X, and forms a Δ-module.

Since $F^{p,d}$ is a subquotient of $K^{p,d}$, and thus of Q_d, it is finitely generated and has rational Hilbert series. This proves our main results on syzygies.

Syzygies of the Segre embedding

Let X be the Δ-variety given by the Segre embedding, and let $F^{p,d}$ be as above. Here are three results on these syzygies:

Theorem (Eisenbud–Reeves–Totaro)

We have $F^{p,d} = 0$ for $d > 2p$.

Theorem (Rubei)

The Segre variety satisfies the Green–Lazersfeld property N_3 but not N_4. This means that $F^{p,d} = 0$ for $d \neq p+1$ if $p = 1, 2, 3$ but not for $p = 4$.

Theorem (Lascoux, Pragacz–Weyman)

[The decomposition of $F_2^{p,d}(V_1, V_2)$.]

An Euler characteristic

Let $f_{p,d}$ be the Hilbert series of $F^{p,d}$ (with factorials), and define $\chi_d = \sum_{p \geq 0} (-1)^p f_{p,d}$.

Theorem

$$\chi_d = \sum_{p=0}^{d} \left[\frac{(-1)^p}{p!} \sum_{|\lambda|=p} (\# c_\lambda) \operatorname{sgn}(c_\lambda) \exp(s_{(d-p)} \boxtimes s'_\lambda) \right]$$

where:

- *c_λ is the conjugacy class in S_p corresponding to λ.*
- *$s'_\lambda = \sum_{|\mu|=p} \chi_\mu(c_\lambda) s_\mu$, where χ_μ is the character of $\mathbf{M}_\mu$.*
- *$\boxtimes$ is the usual product of Schur functors, computed with the Littlewood–Richardson rule.*

Key calculation in proof of theorem

Proposition

Let λ *be a partition of* p *and let* F *be the object of* $\mathrm{Sym}(\mathcal{S})$ *given by* $F(V, L) = \mathbf{S}_\lambda(\bigotimes_{x \in L} V_x)$. *Then*

$$H_F = \frac{1}{p!} \sum_{|\mu|=p} (\# c_\mu) \chi_\lambda(c_\mu) \exp(s'_\mu)$$

The nth term in the power series expansion on the right precisely records the decomposition of $\mathbf{S}_\lambda(V_1 \otimes \cdots \otimes V_n)$ into Schur functors.

Example of key calculation

Suppose $\lambda = (1,1)$. Put $s = s_{(2)}$ and $w = s_{(1,1)}$. We have $s'_{(2)} = s + w$ and $s'_{(1,1)} = s - w$. Therefore $H_F = \frac{1}{2}(e^{s+w} - e^{s-w})$.

We have the following power series expansion:

$$H_F = w + sw + \tfrac{1}{6}(w^3 + 3s^2w) + \tfrac{1}{6}(sw^3 + s^3w) + \cdots$$

The degree 3 term means exactly that there is a decomposition

$$\begin{aligned}\textstyle\bigwedge^2(V_1 \otimes V_2 \otimes V_3) = &\textstyle\bigwedge^2(V_1) \otimes \bigwedge^2(V_2) \otimes \bigwedge^2(V_3) \oplus \\ &\textstyle\mathrm{Sym}^2(V_1) \otimes \mathrm{Sym}^2(V_2) \otimes \bigwedge^2(V_3) \oplus \\ &\textstyle\mathrm{Sym}^2(V_1) \otimes \bigwedge^2(V_2) \otimes \mathrm{Sym}^2(V_3) \oplus \\ &\textstyle\bigwedge^2(V_1) \otimes \mathrm{Sym}^2(V_2) \otimes \mathrm{Sym}^2(V_3)\end{aligned}$$

Formulas for $f_{p,d}$

We have $f_{p,p+1} = (-1)^p \chi_{p+1}$ for $p = 1, 2, 3, 4$ since N_3 is satisfied.

Put $s = s_{(2)}$, $w = s_{(1,1)}$.

$$f_{1,2} = \tfrac{1}{2} e^{s+w} + \tfrac{1}{2} e^{s-w} - e^s$$

Put $s = s_{(3)}$, $w = s_{(1,1,1)}$, $t = s_{(2,1)}$.

$$f_{2,3} = \tfrac{1}{3} e^{s+w+2t} - \tfrac{1}{3} e^{s+w-t} - e^{s+t} + e^s$$

Put $s = s_{(4)}$, $w = s_{(1,1,1,1)}$, $a = s_{(3,1)}$, $b = s_{(2,2)}$, $c = s_{(2,1,1)}$.

$$f_{3,4} = \tfrac{1}{8} e^{s+w+3a+2b+3c} - \tfrac{1}{8} e^{s+w-a+2b-c} + \tfrac{1}{4} e^{s-w-a+c} - \tfrac{1}{4} e^{s-w+a-c} + \tfrac{1}{2} e^{s+b-c} - \tfrac{1}{2} e^{s+2a+b+c} + e^{s+a} - e^s$$

Meaning of formulas

Expanding in a power series,

$$f_{1,2} = \frac{1}{2}w^2 + \frac{1}{2}sw^2 + \frac{1}{24}(6w^2s^2 + w^4) + \cdots$$

The nth term describes the decomposition of $F_n^{1,2}(V_1, \ldots, V_n)$ (i.e., the quadratic relations) under the action of $\mathbf{GL}(V_1) \times \cdots \times \mathbf{GL}(V_n)$. For example,

$$\begin{aligned} F_3^{1,2}(V_1, V_2, V_3) = & \operatorname{Sym}^2(V_1) \otimes \textstyle\bigwedge^2(V_2) \otimes \bigwedge^2(V_3) \oplus \\ & \textstyle\bigwedge^2(V_1) \otimes \operatorname{Sym}^2(V_2) \otimes \bigwedge^2(V_3) \oplus \\ & \textstyle\bigwedge^2(V_1) \otimes \bigwedge^2(V_2) \otimes \operatorname{Sym}^2(V_3) \end{aligned}$$

We have thus given the complete decomposition of the spaces of p-syzygies for $p = 1, 2, 3$.

A problem

Problem

Compute $f_{4,6}$.

We have $\chi_6 = f_{4,6} - f_{5,6}$, so the Euler characteristic calculation does not give the value of $f_{4,6}$. However, that calculation shows that computing $f_{4,6}$ is equivalent to computing $f_{5,6}$.

Lascoux's resolution gives $f_{4,6} = \frac{1}{2}s^2_{(2,2,2)} + \cdots$, i.e., it computes the leading term of $f_{4,6}$.

Our proof of rationality of $f_{p,d}$ shows that $f_{4,6}$ can be computed by a finite linear algebra computation over the ring $\mathbf{C}[x_1, \ldots, x_{2,176,782,336}]$. This is totally impractical, so another method must be found!

§6. Additional topics

Alternate definition of Δ-modules

A Δ-module is an object of $\mathrm{Sym}(\mathcal{S})$ with extra structure, namely the maps (C3). We now give a different way of encoding this extra structure.

There is a comultiplication map $\Delta \colon \mathcal{S} \to \mathcal{S}^{\otimes 2}$, which takes a polynomial functor F to the polynomial functor $(V, W) \mapsto F(V \otimes W)$. Obviously this new polynomial functor is S_2-equivariant, and so Δ takes values in $\mathrm{Sym}^2(\mathcal{S})$.

There is a unique extension of Δ to a derivation of $\mathrm{Sym}(\mathcal{S})$. A Δ-module can be defined as an object M of $\mathrm{Sym}(\mathcal{S})$ equipped with a map $\Delta M \to M$ satisfying an associativity axiom. This map precisely corresponds to the map (C3).

Free Δ-modules

Given an object F of $\mathrm{Sym}(\mathcal{S})$, there is a universal Δ-module it generates, which we denote by $\Phi(F)$. In fact, Φ is the left adjoint of the forgetful functor $\mathrm{Mod}_\Delta \to \mathrm{Sym}(\mathcal{S})$.

We call a Δ-module of the form $\Phi(F)$ **free**, and **finite free** if F has finite length. An arbitrary Δ-module is finitely generated if and only if it is a quotient of a finite free Δ-module.

The functor Ψ

Given a Δ-module M, denote by $M^{\mathrm{old}}(V, L)$ the subspace of $M(V, L)$ generated by elements of $M(V', L')$ with $\#L' < \#L$. Equivalently, M^{old} is the image of $\Delta M \to M$. Then M^{old} is a Δ-submodule of M. We let $\Psi(M) = M/M^{\mathrm{old}}$. This is a Δ-module, but the maps (C3) are always 0, so we regard $\Psi(M)$ as an object of $\mathrm{Sym}(\mathcal{S})$.

A version of Nakayama's lemma holds: a Δ-module M is finitely generated if and only if $\Psi(M)$ is of finite length. In fact, M is always a quotient of $\Phi(\Psi(M))$.

Analogy with $\mathbf{C}[t]$-modules

Graded vector spaces	$\mathrm{Sym}(\mathcal{S})$
Graded $\mathbf{C}[t]$-modules	Δ-modules
$V \otimes_{\mathbf{C}} \mathbf{C}[t]$	$\Phi(F)$
$M \otimes_{\mathbf{C}[t]} \mathbf{C}$	$\Psi(M)$
multiplication by t	the map $\Delta M \to M$
tM	M^{old}

We proved two main theorems about Δ-modules: one about noetherianity and one about rationality of Hilbert series.

These two results are not the end of the story, however: there are many other results one might want to establish about Δ-modules.

Our method provides a systematic procedure for proving results about Δ-modules.

Resolutions of Δ-modules

One can attempt to resolve a Δ-module by free Δ-modules. As usual, the first step in the resolution gives the generators and the second step can be intepreted as relations between these generators.

For instance, the syzygy module $F^{1,2}$ of the Segre is generated by the defining equation of $\mathbf{P}^1 \times \mathbf{P}^1$. However, $F^{1,2}$ is not free: different sequences of the operations (C1)–(C3) can yield the same equations.

The Poincaré series

The terms of the resolution of M are $\Phi(L_i \Psi M)$. This is in analogy with how Tor's give the resolutions of modules over polynomial rings; note that $L_i\Psi$ is analogous to $\mathrm{Tor}_i^{\mathbf{C}[t]}(-, \mathbf{C})$.

We can record this information in a series:

$$P_M(q) = \sum_{i \geq 0} (-1)^i H_{(L_i \Psi M)} q^i.$$

We call $P_m(q)$ the **Poincaré series** of M. The Hilbert series is recovered by evaluating at $q = 1$ and aplying Φ. Where the Hilbert series of M depends only on the underlying object of $\mathrm{Sym}(\mathcal{S})$, the Poincaré series uses the Δ-module structure.

The main question, obviously, is if $P_M(q)$ is rational.

Poincaré series for tca's

Let A be the tca $\mathrm{Sym}(U\langle 1\rangle)$ and let M be a finitely generated A-module. The resolution of M by projective A-modules is typically infinite. S. Sam and I show that:

- Regularity is finite, i.e., the resolution of M has only finitely many linear strands.
- The ith linear strand $\mathcal{F}_i(M)$ admits the structure of a finitely generated module over $A' = \mathrm{Sym}(U^*\langle 1\rangle)$.

In fact, $\mathcal{F}$ gives an equivalence $D^b(A) \to D^b(A')$ which we call the **Fourier transform**.

An elementary manipulation gives $P_M(q) = \sum_{i\geq 0} H_{\mathcal{F}_i(M)}(qt)q^{-i}$. This shows that $P_M(q)$ belongs to $\mathbf{Q}[t, e^t, q^{\pm 1}]$.

Back to Poincaré series for Δ-modules

To obtain rationality of Poincaré series for Δ-modules is now just a matter of transferring the result for tca's to algebras in Sym($\mathcal{S}$), and then to Δ-modules. We have not done this yet, but expect to be able to.

Problem

Compute the Poincaré series of any non-free Δ-module, e.g., $F^{1,2}$ of the Segre.

Bounded Δ-varieties

Let X be a Δ-variety. Write $R(V, L)$ for the coordinate ring of $X(V, L)$. Then R is an object of $\mathrm{Sym}(\mathcal{S})$ (in fact, a Δ-module). We say that X is **bounded** if $L(R) < \infty$.

Example

Suppose X is the Segre. Then $R(V, L) = \bigoplus_{n \geq 0} \bigotimes_{x \in L} \mathrm{Sym}^n(V_x)$. It follows that $L(R) = 1$ and so X is bounded.

Boundedness is preserved under many operations on Δ-varieties. In particular, the secant varieties of the Segre are bounded. Recall:

Conjecture

If X is bounded then $F^{p,d} = 0$ for $d \gg p$.

The Δ-variety $\Delta\mathrm{Sub}_d$

Define $\mathrm{Sub}_d(V_1, \ldots, V_n) \subset V_1^* \otimes \cdots \otimes V_n^*$ to be the union of spaces of the form $U_1 \otimes \cdots \otimes U_n$ where the U_i vary over the dimension d subsapces of the V_i^*. Thus Sub_1 is the Segre.

For $d > 1$, Sub_d is not a Δ-variety but contains a maximal Δ-subvariety, called $\Delta\mathrm{Sub}_d$, which can be obtained by intersecting the Sub_d's of flattenings. The Δ-variety $\Delta\mathrm{Sub}_d$ can be characterized as the maximal Δ-variety whose coordinate ring satisfies $L \leq d$.

Question

Is $\Delta\mathrm{Sub}_d$ noetherian? That is, does any descending chain of Δ-subvarieties of $\Delta\mathrm{Sub}_d$ stabilize?

This question is weaker than the conjecture, but stronger than the result of Draisma–Kuttler.

The Segre–Veronese variety

Let $V_1, \ldots, V_n$ be vector spaces and $w_1, \ldots, w_n$ positive integers. The **Segre–Veronese variety** is the subvariety of

$$\mathrm{Sym}^{w_1}(V_1^*) \otimes \cdots \otimes \mathrm{Sym}^{w_n}(V_n^*)$$

consisting of pure tensors of pure powers.

$\mathrm{m}\Delta$-modules

Define a category $\mathrm{Vec}^{\mathrm{m}\Delta}$ as follows:

- The objects are pairs (V, L) where L is a **weighted set** and V assigns to each $x \in L$ a vector space V_x.
- A morphism $(V, L) \to (V, L')$ consists of a **weighted correspondence** $L' \to L$ and certain linear maps on the vector spaces.

The Segre–Veronese variety is a functor from $\mathrm{Vec}^{\mathrm{m}\Delta}$ to varieties.

An $\mathrm{m}\Delta$-**module** is a polynomial functor $\mathrm{Vec}^{\mathrm{m}\Delta} \to \mathrm{Vec}$. The syzygies of the Segre–Veronese are examples.

Results on syzygies

S. Sam and I have carried over the results on syzygies of Segre varieties to the Segre–Veronese case. Remarks:

- Whereas the results in the Segre case depended on the fact that $\mathrm{Sym}(U \otimes \mathbf{C}^\infty)$ is noetherian as a $\mathbf{GL}(\infty)$-algebra (which goes back to Weyl), these new results use noetherianity as an S_∞-algebra (theorem of Cohen, Aschenbrenner, Hillar, Sullivant).
- The result on Hilbert series in the Segre–Veronese case is weaker than the result in the Segre case: it does not completely determine the decompositions of the syzygy modules.
- The result on Hilbert series is also conditional at this point: it depends on an elementary statement concerning certain quivers which we have not been able to prove (but suspect to be true).

Thank you for listening!