7 Polynomial functors

Polynomial functors have a long history and come in various forms. The prototypical polynomial functor $\mathsf{Ab} \to \mathsf{Ab}$ sends an abelian group A to its kth tensor power $A^{\otimes k}$. These and similar play an important role in algebraic geometry and representation theory of group schemes. In 1950, Eilenberg–Maclane defined them in terms of "cross effects" to compute the homology of Eilenberg–Maclane spaces $K(\pi, n)$. In homological stability, polynomial functors were introduced by Dwyer in 1980 to compute algebraic K-theory groups. We will use a variant of Dwyer's definition that seems to be most general.

Definition 7.1. Let M be a $\mathsf{VIC}(\mathbb{Z})$–module. Define ΣM to be the $\mathsf{VIC}(\mathbb{Z})$–module that is M precomposed with the functor $\mathbb{Z}\oplus - \colon \mathsf{VIC}(\mathbb{Z}) \to \mathsf{VIC}(\mathbb{Z})$ that sends $(f, C) \in \mathrm{Hom}_{\mathsf{VIC}(\mathbb{Z})}(A, B)$ to $(\mathrm{id}_{\mathbb{Z}} \oplus f, C) \in \mathrm{Hom}_{\mathsf{VIC}(\mathbb{Z})}(\mathbb{Z} \oplus A, \mathbb{Z} \oplus B)$.
There is a canonical $\mathsf{VIC}(\mathbb{Z})$–homomorphism $M \to \Sigma M$ given by the maps $M_A \to M_{\mathbb{Z}\oplus A}$ induced by $(A \subset \mathbb{Z} \oplus A, \mathbb{Z}) \in \mathrm{Hom}_{\mathsf{VIC}(\mathbb{Z})}(A, \mathbb{Z} \oplus A)$.
Let us denote

$$(\mathrm{co})\ker(M) := (\mathrm{co})\ker(M \to \Sigma M).$$

Let $d \in \mathbb{N}_0 \cup \{-\infty\}$. We say M has *polynomial degree* $-\infty$ *in ranks* $> d$ if $M_n \cong 0$ for all $n > d$.
We say M has *polynomial degree* ≤ 0 *in ranks* $> d$ if $(\ker M)_n \cong (\operatorname{coker} M)_{n+1} \cong 0$ for all $n > d$.
Let $r \geq 1$. We say M has *polynomial degree* $\leq r$ *in ranks* $> d$ if $(\ker M)_n \cong 0$ for all $n > d$ and $\operatorname{coker} M$ has polynomial degree $\leq r - 1$ in ranks $> d - 1$.
(For simplicity, we will define $0 - 1 = -\infty$.)

Proposition 7.2. *If a* $\mathsf{VIC}(\mathbb{Z})$*–module* M *has polynomial degree* $\leq r$ *in ranks* $> d$*, then there is a polynomial* $p \in \mathbb{Q}[X]$ *of degree* $\leq r$ *such that* $\operatorname{rk} M_n = p(n)$ *for all* $n > d$*.*

Theorem 7.3 (Dwyer 1980)**.** *If* M *has finite polynomial degree then*

$$H_i(\mathrm{GL}_{n-1}(\mathbb{Z}); M_{n-1}) \longrightarrow H_i(\mathrm{GL}_n(\mathbb{Z}); M_n)$$

is an isomorphism for $n \gg i$*.*

Exercise 7.4. Check that $H_1(\mathrm{IA})$ has polynomial degree ≤ 3 in ranks $> -\infty$.

Theorem 7.5 (Miller-P.-Petersen)**.** *Let* M *be a* $\mathsf{VIC}(\mathbb{Z})$*–module that has polynomial degree* $\leq r$ *in ranks* $> d$*. Then*

$$HS_i(M)_n \cong 0 \qquad \text{for } n > \max(d + i, 2i + r).$$

Corollary 7.6 (Miller-P.-Wilson, Miller-P.-Petersen)**.** $H_2(\mathrm{IA})$ *is presented in degrees* ≤ 9*.*