1 Free groups and their automorphisms

Definition 1.1. Let T be a set. A *word* from the *alphabet* T is a map $[n] := \{1, \dots, n\} \to T$ for some $n \in \mathbb{N}_0$. We denote the *empty word* $[0] \to T$ by ε. We call n the *length* of a word $[n] \to T$.

Example 1.2. For an alphabet $T = \{a, b, c\}$, words are $\varepsilon, ab, aaa, abaca$. The have the lengths $0, 2, 3, 5$, respectively.

Definition 1.3. Let S be a set. Let $T = S \times \{-1, 1\}$, where by abuse of notation we identify $S \cong S \times \{1\} \subset T$ and write simply s for $(s, 1) \in T$. We also write s^{-1} for $(s, -1) \in T$. We call s^{-1} the inverse of s.

Define an equivalence relation on the set of words from the alphabet T by adding/removing a pair of adjacent s and s^{-1}.

The *free group* F_S is the set of all equivalence classes of words from T. Group multiplication is given by concatenation.

For $S = \{x_1, \dots, x_n\}$ denote F_S by F_n.

Exercise 1.4. Show that this describes a well defined group.

Theorem 1.5. *Let G be a group. A set map $S \to G$ uniquely determines a group homomorphism $F_S \to G$ extending the set map via $S \subset F_S$.*

Proof. Exercise. □

Definition 1.6. Let S be a subset of a group G. We say that S *generates* the smallest subgroup of G that contains S.

Observe that $S \subset G$ generates the group G if and only if the induced map $F_S \to G$ is surjective. More generally, the image of $F_S \to G$ is the subgroup generated by S.

Definition 1.7. Let G be a group. An *automorphism* of G is a group homomorphism $f \colon G \to G$, i.e. $f(gh) = f(g)f(h)$, that is bijective. The set of automorphisms of G is denoted by $\mathrm{Aut}(G)$ and it forms a group.

Note that an element of $f \in \mathrm{Aut}(F_n)$ is determined by the images $f(x_1), \dots, f(x_n)$.

Example 1.8. There is an inclusion of the symmetric group S_n into $\mathrm{Aut}(F_n)$ by sending $\sigma \in S_n$ to the automorphism defined by $x_i \mapsto x_{\sigma(i)}$.

Inverting the ith generator is an automorphism:

$$\mathrm{inv}_i \colon x_j \longmapsto \begin{cases} x_i^{-1} & j = i \\ x_j & j \neq i \end{cases}$$

Multiplying the ith generator to the jth (from the left or the right) is an automorphism:

$$\mathrm{leftmul}_{ij} \colon x_k \longmapsto \begin{cases} x_i x_j & k = j \\ x_k & k \neq j \end{cases}$$

$$\mathrm{rightmul}_{ij} \colon x_k \longmapsto \begin{cases} x_j x_i & k = j \\ x_k & k \neq j \end{cases}$$

Theorem 1.9 (Nielsen, 1924)**.** $\mathrm{Aut}(F_n)$ *is generated by permutations,* inv_1*, and* $\mathrm{leftmul}_{12}$.

Definition 1.10. Let G be a group. For $a, b \in G$, the *commutator* is $[a, b] = aba^{-1}b^{-1}$. The *commutator subgroup* G' of G is generated by all commutators. More generally, let H be a subgroup of G, denote $[G, H]$ to be the subgroup generated by commutators $[g, h]$ with $g \in G$ and $h \in H$.

The *lower central series* of G is a series of subgroups $\gamma_i G$ of G, defined recursively by $\gamma_1 G = G$ and $\gamma_{i+1} G = [G, \gamma_i G]$.

We call $G^{\mathrm{ab}} := G/G'$ the *abelianization* of G.

Proposition 1.11. *There is a surjective group homomorphism*

$$\mathrm{Aut}(F_n) \longrightarrow \mathrm{GL}_n(\mathbb{Z}).$$

Proof. Note that $F_n^{\mathrm{ab}} \cong \mathbb{Z}^n$ (exercise). Because abelianizing is functorial (exercise), we get a group homomorphism

$$\mathrm{Aut}(F_n) \longrightarrow \mathrm{Aut}(\mathbb{Z}^n).$$

Observe that $\mathrm{Aut}(\mathbb{Z}^n) \cong \mathrm{GL}_n(\mathbb{Z})$ is the group of invertible integral $n \times n$ matrices (exercise). $\mathrm{GL}_n(\mathbb{Z})$ is generated by $\begin{pmatrix} -1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}$ and all matrices with ones on the diagonal and one one off the diagonal (exercise). All of these generators are in the image of the map $\mathrm{Aut}(F_n) \to \mathrm{GL}_n(\mathbb{Z})$. More precisely, they are images of inv_1 and $\mathrm{leftmul}_{ij}$. □

Definition 1.12. The *Torelli subgroup* of $\mathrm{Aut}(F_n)$ is the kernel

$$\mathrm{IA}_n := \ker(\mathrm{Aut}(F_n) \to \mathrm{GL}_n(\mathbb{Z})).$$