Exercises

1.) We want to compare different definitions of polynomial functors.

(a) Show that there is a $\mathsf{VIC}(\mathbb{Z})$–module that sends a finitely generated free $\mathbb{Z}$–module to its underlying abelian group that has polynomial degree ≤ 1 in ranks $> -\infty$.

(b) Show that there is a $\mathsf{VIC}(\mathbb{Z})$–module that sends a finitely generated free $\mathbb{Z}$–module to the dual of its underlying abelian group that has polynomial degree ≤ 1 in ranks $> -\infty$. (Here $\mathrm{GL}_n(\mathbb{Z})$ acts via its transpose.)

(c) Let $F\colon \mathsf{Ab} \to \mathsf{Ab}$ be a functor. Show that there is a functor $\mathrm{cr}_1(F)\colon \mathsf{Ab} \to \mathsf{Ab}$ called the first cross effects of F such that $F(A) = F(0) \oplus \mathrm{cr}_1(F)(A)$ for all abelian groups A.

(d) Let $F\colon \mathsf{Ab} \to \mathsf{Ab}$ be a functor. Show that there is a functor $\mathrm{cr}_2(F)\colon \mathsf{Ab} \times \mathsf{Ab} \to \mathsf{Ab}$ called the second cross effects of F such that $F(A \oplus B) = F(0) \oplus \mathrm{cr}_1(F)(A) \oplus \mathrm{cr}_1(F)(B) \oplus \mathrm{cr}_2(F)(A, B)$ for all pairs of abelian groups A, B.

(e) Let $\mathsf{VIC}(\mathbb{Z}) \to \mathsf{Ab}$ the functor that forgets about the complement. Let $F\colon \mathsf{Ab} \to \mathsf{Ab}$ be a functor whose second cross effects vanish. Consider F as a $\mathsf{VIC}(\mathbb{Z})$–module. Show that it has polynomial degree ≤ 1 in ranks $> -\infty$.

(f) Prove that if a $\mathsf{VIC}(\mathbb{Z})$–module has polynomial degree $\leq r$ in ranks $> d$ then there is a polynomial $p \in \mathbb{Q}[X]$ such that $\operatorname{rk} M_n = p(n)$ for all $n > d$.

2.) We want to show that $H_1(\mathrm{IA})$ has polynomial degree ≤ 3 in ranks $> -\infty$.

(a) Let M, M', M'' be $\mathsf{VIC}(\mathbb{Z})$–modules and $M' \to M \to M''$ morphisms such that

$$0 \to M'_n \to M_n \to M''_n \to 0$$

is a short exact sequence for $n > d$. Prove that if N' has polynomial degree $\leq r$ in ranks $> d$ and N'' has polynomial degree $\leq r$ in ranks $> d - 1$, then N has polynomial degree $\leq r$ in ranks $> d$.

(b) Let M and N be $\mathsf{VIC}(\mathbb{Z})$–modules and assume that M has polynomial degree $\leq r$ in ranks $> d$ and N has polynomial degree $\leq s$ in ranks $> e$. Prove that $M \otimes N$ has polynomial degree $\leq r + s$ in ranks $> \max(d, e)$.

(c) Show that there is a $\mathsf{VIC}(\mathbb{Z})$–module M with $M_n \cong \mathrm{Hom}_{\mathsf{Ab}}(\mathbb{Z}^n, \bigwedge^2 \mathbb{Z}^n)$ that has polynomial degree ≤ 3 in ranks $> -\infty$.

(d) Show that M coincides with the $\mathsf{VIC}(\mathbb{Z})$–module $H_1(\mathrm{IA})$.