HOMOLOGICAL ALGEBRA

This document is about abelian groups and R-modules, but we will later see that everything makes sense with abelian groups replaced by objects from an "abelian category."

Suppose we have a short exact sequence

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0, \tag{1}$$

so that f is an injection, g is a surjection, and $\ker g = \operatorname{im} f$.

1. Show that f is an isomorphism if and only if $C = 0$, and similarly that $A = 0$ if and only if g is an isomorphism.
2. Show that $B = 0$ if and only if A and C are zero.
3. Show that B is not always determined up-to-isomorphism by A and C. That is, show that there exist A, C, B, and B' with B not isomorphic to B' so that
$$0 \to A \to B \to C \to 0$$
and
$$0 \to A \to B' \to C \to 0$$
are both exact sequences.
4. To ponder: Given A and C, what choices are there for f, B, and g? This is called the "extension problem" for A and C.

Definition 0.1. A module P is called **projective** if the functor $\mathrm{Hom}(P, -)$ is exact, so that the sequence

$$0 \to \mathrm{Hom}(P, A) \xrightarrow{\mathrm{Hom}(P,f)} \mathrm{Hom}(P, B) \xrightarrow{\mathrm{Hom}(P,g)} \mathrm{Hom}(P, C) \to 0$$

is still exact as a sequence of abelian groups.

5. Show that X and Y are projective exactly when $X \oplus Y$ is projective.
6. Show that the abelian group $\mathbb{Z}$ is projective, and deduce the same for $\mathbb{Z}^n$.
7. Show that if P is projective, then any surjection $p\colon A \twoheadrightarrow P$ admits a map $i\colon P \to A$ with $p \circ i = 1_P$. The surjection p is then said to be "split" and i is called a "splitting."
8. Returning to (1), show that if C is projective, then $B \cong A \oplus C$, and so the extension problem is easy in this case.

The next problems indicate some of the interest in non-free projective modules, indicating a connection to K-theory. Let $S^1 = [0, 1]/(0 \sim 1)$ be the circle, and let R be the ring of continuous functions $S^1 \to \mathbb{R}$.

9. Show that $M = \{f\colon [0, 1] \to \mathbb{R} \mid f(0) = -f(1)\}$ is an R-module.
10. Using the intermediate value theorem, show that M is not free.
11. Find a continuous path $\gamma\colon [0, 1] \to \mathrm{GL}(2, \mathbb{R})$ with $\gamma(0) = I$ and $\gamma(1) = -I$.
12. Show that $M \oplus M \cong R \oplus R$, and conclude that M is projective.
13. Find a two-by-two matrix $\pi \in M_2(R)$ so that $\pi\pi = \pi$ (this condition gives us the word "projective"), π has rank one when its entries are evaluated at any point $t \in S^1$,

but π is not conjugate to

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

under the action of $\mathrm{GL}(2, R)$.

An **injective module** I is one for which the contravariant representable functor $\mathrm{Hom}(-, I)$ is exact.

14. If k is a field, show that k is injective as a k-module.
15. Is $\mathbb{Z}$ an injective abelian group?
16. Show that $\mathbb{Z}/2$ is injective as a $\mathbb{Z}/2$-module, but not as an abelian group.
17. Let $\mathbb{Q}$ denote the group rational numbers with addition. Show that the quotient group $\mathbb{Q}/\mathbb{Z}$ is an injective abelian group. Is this group indecomposable?
18. Find a nonzero injective $\mathbb{Z}[x]$-module.

Definition 0.2. A **chain complex** is a sequence of modules $C_\bullet = \{C_n : n \in \mathbb{Z}\}$ together with homomorphisms $d_n \colon C_n \to C_{n-1}$, called **differentials** or **boundary maps**, satisfying $d_n \circ d_{n+1} = 0$ for all $n \in \mathbb{Z}$. The kernel of d_n is the module of n**-cycles of** $C_\bullet$, written $Z_n(C_\bullet)$, and the image of d_{n+1} is the module of n**-boundaries of** $C_\bullet$, written $B_n(C_\bullet)$. The n**th homology of** $C_\bullet$ is the quotient $H_n(C_\bullet) = {}^{Z_n(C_\bullet)}\!/\!_{B_n(C_\bullet)}$.

A **cochain complex** is a sequence of module $C^\bullet = \{C^n : n \in \mathbb{Z}\}$ together with homomorphisms $d^n \colon C^n \to C^{n+1}$ called differentials or coboundary maps, satisfying $d^{n+1} \circ d^n = 0$ for all $n \in \mathbb{Z}$.

19. Set $C_n = \mathbb{Z}/9\mathbb{Z}$ for all $n \geq 0$ and $C_n = 0$ for all $n < 0$. Moreover, define $d_n \colon C_n \to C_{n-1}$ by $d_n(x) = 3x$. Prove that $C_\bullet$ is a complex of $\mathbb{Z}/9\mathbb{Z}$-modules and compute its homology.
20. Write the definition of the n**th cohomology** of a cochain complex $C^\bullet$. Hint: translate the definition of homology to cochain complexes.

Definition 0.3. A **chain map** between two chain complexes $(C_\bullet, d_{C,\bullet})$ and $(D_\bullet, d_{D,\bullet})$ is a sequence of homomorphisms $f_n \colon C_n \to D_n$ that satisfy $d_{D,n} \circ f_n = f_{n-1} \circ d_{C,n}$ for all $n \in \mathbb{Z}$.

21. Prove that if $f \colon C_\bullet \to D_\bullet$ is a chain map, then $f_n(Z_n(C_\bullet)) \subseteq Z_n(D_\bullet)$ and $f_n(B_n(C_\bullet)) \subseteq B_n(D_\bullet)$ for all $n \in \mathbb{Z}$.
22. Prove that for each $n \in \mathbb{Z}$ H_n is a functor from Ch(Ab), the category of chain complexes of abelian groups with chain maps, to Ab.

Let S be a commutative ring. The category $\mathrm{Ch}(S - \mathrm{Mod})$ is an abelian category, and therefore contains exact sequences.

Definition 0.4. A **left resolution** of an S-module M is an exact sequence of S-modules

$$\dots \xrightarrow{d_{n+1}} E_n \xrightarrow{d_n} \dots \xrightarrow{d_2} E_1 \xrightarrow{d_1} E_0 \xrightarrow{\varepsilon} M \to 0.$$

The homomorphisms d_n are called **boundary maps**, and the homomorphism ε is called the **augementation map**. For succinctness, we often write left resolutions as

$$E_\bullet \xrightarrow{\varepsilon} M \to 0.$$

A **right resolution** of an S-module M is an exact sequence of S-modules

$$0 \to M \xrightarrow{\varepsilon} D_0 \xrightarrow{d^0} D_1 \xrightarrow{d^1} \dots \xrightarrow{d^{n-1}} D_n \xrightarrow{d^n} \dots$$

For succinctness, we often write right resolutions as

$$0 \to M \xrightarrow{\varepsilon} C_\bullet.$$

If all the E_n are free modules, then we say that $E_\bullet \xrightarrow{\varepsilon} M \to 0$ is a **free resolution**, and if all the E_n are projective modules, then we say that $E_\bullet \xrightarrow{\varepsilon} M \to 0$ is a **projective resolution**. If all the C_n are injective modules, then we say that $0 \to M \xrightarrow{\varepsilon} C_\bullet$ is an **injective resolution**.

23. Show that every S-module M has a surjection from a (possibly infinite) direct sum of copies of S. What does this say about the existence of a free resolution of M?

Definition 0.5. A **quasi-isomorphism** of chain complexes is a chain map $f_\bullet \colon C_\bullet \to D_\bullet$ so that $f_n \colon H_n(C_\bullet) \to H_n(D_\bullet)$ is an isomorphism for all $n \in \mathbb{Z}$.

24. For any S-module M, find a chain complex of free S-modules $C_\bullet$ so that $H_0 C_\bullet \cong M$ and $H_p C_\bullet = 0$ otherwise. Probably you will want to pick $C_p = 0$ for $p < 0$.
25. Find a quasi-isomorphism from $C_\bullet$ to the complex $\ldots \to 0 \to M \to 0 \to \ldots$.
26. If $X_\bullet$ is a projective resolution of an S-module A and $Z_\bullet$ is a projective resolution of an S-module C, and if these objects sit in a short exact sequence $0 \to A \to B \to C \to 0$, find a projective resolution $Y_\bullet$ of B so that $0 \to X_\bullet \to Y_\bullet \to Z_\bullet \to 0$ is a short exact sequence of chain complexes whose degree-zero homology recovers the original sequence.
27. Let
$$0 \to X_\bullet \xrightarrow{f} Y_\bullet \xrightarrow{g} Z_\bullet \to 0$$
be a short exact sequence of chain complexes. Show that $X_{n-1} \oplus Y_n$ with differential $d(x, y) = d(x) + (-1)^n f(x) + d(y)$ is quasi-isomorphic to $Z_\bullet$. Call this complex $M(f)_\bullet$; it is the **mapping cone** of f.

Given a chain complex $X_\bullet$, for any $k \in \mathbb{Z}$ we have the shifted complex $X[k]_\bullet$, with $X[k]_n = X_{n+k}$ for all $n \in \mathbb{Z}$.

28. Show that $Y_\bullet$ is a subcomplex of $M(f)$ and $X[-1]_\bullet$ is a quotient complex.
29. Using the maps $X_\bullet \xrightarrow{f} Y_\bullet \subseteq M(f)_\bullet \twoheadrightarrow X[-1]_\bullet \xrightarrow{-f[-1]} Y[-1]_\bullet$, construct an exact sequence
$$H_n X \to H_n Y \to H_n Z \to H_{n-1} X \to H_{n-1} Y.$$

Theorem 0.6. *If $0 \to X_\bullet \xrightarrow{f} Y_\bullet \xrightarrow{g} Z_\bullet \to 0$ is a short exact sequence of chain complexes, then there are natural maps $\partial \colon H_n(Z_\bullet) \to H_{n-1}(X_\bullet)$ and a long exact sequence*

$$\ldots \xrightarrow{g} H_{n+1}(Z_\bullet) \xrightarrow{\partial} H_n(X_\bullet) \xrightarrow{f} H_n(Y_\bullet) \xrightarrow{g} H_n(Z_\bullet) \xrightarrow{\partial} H_{n-1}(X_\bullet) \xrightarrow{f} \ldots$$

Definition 0.7. Let $\mathcal{F}$ be a right exact functor from S-Mod to R-Mod[1] (where R and S are both commutative unital rings). We can construct the **left derived functors of $\mathcal{F}$**, $L_i\mathcal{F}$ ($i \geq 0$), as follows. For each S-module, M, choose a projective resolution $P_\bullet \to M \to 0$ and define

$$L_i\mathcal{F}(M) = H_i(\mathcal{F}(P_\bullet)).$$

[1] A right exact functor is a functor that preserves right exact sequences.

Let $\mathcal{G}$ be a left exact functor from S-Mod to R-Mod. We can construct the **right derived functors** of $\mathcal{G}$, $R^j\mathcal{G}$ ($j \geq 0$), as follows. For each S-module M, choose an injective resolution $0 \to M \to I_\bullet$ and define

$$R^j\mathcal{G}(M) = H^j(\mathcal{G}(I_\bullet)).$$

30. Prove that if $P_\bullet$ and $Q_\bullet$ are two different projective resolutions of an S-module M, then $H_i(\mathcal{F}(P_\bullet)) \cong H_i(\mathcal{F}(Q_\bullet))$, thus proving that our definition of left-derived functors is well-defined. (This exercise is a little challenging and requires the use of a "chain homotopy".)
31. Prove that for any S-module M, $L_0\mathcal{F}(M) \cong \mathcal{F}(M)$.
32. Prove the exercises analogous to Problems 30 and 31 for right derived functors.

Definition 0.8. If R is a ring, and if M_R, ${}_RN$ are modules with opposite-sided actions, the **tensor product** is an abelian group defined by the formula

$$M \otimes_R N = \mathbb{Z} \cdot (M \times N)/\sim,$$

where $\sim$ is generated by the relations

$$(mr, n) \sim (m, rn)$$
$$(m + m', n) \sim (m, n) + (m', n) \qquad (m, n + n') \sim (m, n) + (m, n')$$
$$(0, n) \sim 0 \qquad (m, 0) \sim 0$$

for all $m, m' \in M$, $n, n' \in N$, and $r \in R$. The equivalence class of $1 \cdot (m, n) \in \mathbb{Z} \cdot (M \times N)$ is written $m \otimes_R n$, or even just $m \otimes n$.

We mention that, in the definition, $\mathbb{Z} \cdot (M \times N)$ denotes the free abelian group on the set $M \times N$, even though this set carries the structure of an abelian group. This is intentional. In the tensor product, we definitely do not want $(m + m') \otimes (n + n') = (m \otimes n) + (m' \otimes n')$, even though $(m + m', n + n') = (m, n) + (m', n')$ does hold in $M \times N$. Rather, in the tensor product,

$$\begin{aligned}(m + m') \otimes (n + n') &= m \otimes (n + n') + m' \otimes (n + n') \\ &= (m \otimes n) + (m \otimes n') + (m' \otimes n) + (m' \otimes n').\end{aligned}$$

In other words, the symbol $\otimes$ is meant to be **bilinear**, like an inner product, or like matrix multiplication.

33. Prove that $R \otimes_R N \cong N$.
34. By symmetry, conclude that $M \otimes_R R \cong M$ as well.
35. Given another left R-module ${}_RN'$, show that $M \otimes_R (N \oplus N') \cong (M \otimes_R N) \oplus (M \otimes_R N')$.
36. Show that $\mathbb{Z}^m \otimes_{\mathbb{Z}} \mathbb{Z}^n \cong \mathbb{Z}^{mn}$.
37. Given a map $\phi\colon {}_RN \to {}_RN'$, produce a map $M \otimes_R \phi\colon M \otimes_R N \to M \otimes_R N'$.
38. With ϕ as in the previous problem, show that $\operatorname{coker}(M \otimes_R \phi) \cong M \otimes_R \operatorname{coker}(\phi)$.

Let G be a cyclic group with 6 elements generated by $g \in G$, and write $\otimes_G$ for $\otimes_{\mathbb{Z}G}$. Let $M_G = \mathbb{Z}^2$ where g acts by the matrix

$$\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}.$$

In the next three problems, we take ${}_GN = \mathbb{Z}/7\mathbb{Z}$, but we vary the action of G.

39. Setting $gn = n$ for all $n \in N$, compute $M \otimes_G N$.
40. Setting $gn = n + n$ for all $n \in N$, compute $M \otimes_G N$.

41. Setting $gn = n + n + n$ for all $n \in N$, compute $M \otimes_G N$.

Let G be a group with two elements, and let $R = \mathbb{Z}G$ be the group ring of G. (This is the ring whose elements are formal $\mathbb{Z}$-linear combinations of elements of G, and where multiplication of elements is given by composition in the group.) Recall that left G-modules become left R-modules and *vice versa*.

42. Find a free resolution of ${}_G\mathbb{Z}$, the integers with trivial left G-action.
43. Compute $\mathbb{Z} \otimes^{L_i}_{\mathbb{Z}G} \mathbb{Z}$ for all $i \geq 0$ where $\mathbb{Z}$ carries the trivial left action of G, and $- \otimes^{L_i}_{\mathbb{Z}G} \mathbb{Z}$ denotes the left derived tensor product of the right exact functor $- \otimes_{\mathbb{Z}G} \mathbb{Z}$.
44. Let ${}_GM = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ where the nontrivial element of G acts by $(1,0) \mapsto (1,1)$ and $(0,1) \mapsto (0,1)$. Find a free resolution of ${}_GM$.

For the last problem, let $G = S_3$ be the symmetric group.

45. Find a free resolution of ${}_G\mathbb{Z}$, the integers with trivial left G-action.

Definition 0.9. Let R be a ring. For a fixed left R-module N,

$$\mathrm{Tor}_i^R(M, N) = M \otimes_R^{L_i} N$$

and

$$\mathrm{Ext}_R^i(N, M) = R^j\mathrm{Hom}_R(N, M).$$

46. Prove that for any abelian groups A and B, $\mathrm{Tor}_1^{\mathbb{Z}}(A, B)$ is a torsion abelian group and that $\mathrm{Tor}_n^{\mathbb{Z}}(A, B) = 0$ for all $n \geq 2$.
47. Let $R = \mathbb{Z}/4\mathbb{Z}$ and consider $M = \mathbb{Z}/2\mathbb{Z}$ as an R-module.
    a. Find projective and injective resolutions for M.
    b. Compute $\mathrm{Tor}_i^R(M, M)$.
    c. Compute $\mathrm{Ext}_R^i(M, M)$.
48. Using the fact that every abelian group B has an injective resolution of the form $B \to I_0 \to I_1 \to 0 \to \ldots \to 0 \to \ldots$, prove that $\mathrm{Ext}^n_{\mathbb{Z}}(A, B) = 0$ for all abelian groups A and B and $n \geq 2$.

Definition 0.10. A **filtered chain complex $F_\bullet C_\bullet$** is a sequence of chain complexes

$$\cdots \subseteq F_0C_\bullet \subseteq F_1C_\bullet \subseteq F_2C_\bullet \subseteq \cdots$$

where each $F_nC_\bullet$ is a subcomplex of $F_{n+1}C_\bullet$.

49. If X is a topological space that is filtered by a sequence of subspaces $X_0 \subseteq X_1 \subseteq X_1 \subseteq \cdots$, show that the complex of singular chains on X is filtered by the rule $F_nC_\bullet^{sing}X = C_\bullet^{sing}X_n$.

Definition 0.11. Let M be a G-module. Then the **nth cohomology group of G with coefficients in** M is $H^n(G; M) := \mathrm{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, M)$ where G acts trivially on $\mathbb{Z}$. Similarly, the **nth homology group of G with coefficients in M is** $H_n(G; M) := \mathrm{Tor}_n^{\mathbb{Z}G}(\mathbb{Z}, M)$ where G acts trivially on $\mathbb{Z}$.

If you are familiar with invariants and coinvariants, it is helpful to think of the cohomology groups $H^n(G; M)$ as the right derived functors of $M \mapsto M^G$ and the homology groups $H_n(G; M)$ as the left derived functors of $M \mapsto M_G$.

50. Let G be the trivial group and let A be any abelian group. Compute $H_n(G; A)$ and $H^n(G; A)$ for all $n \geq 0$.

51. Let G be the infinite cyclic group (written multiplicatively) with generator x. Then we can identify $\mathbb{Z}G$ with $\mathbb{Z}[x, x^{-1}]$. Prove that $0 \to \mathbb{Z}G \xrightarrow{x-1} \mathbb{Z}G \to \mathbb{Z} \to 0$ is exact, and then show that $H^n(G; A) = 0$ and $H_n(G; A) = 0$ for all $n > 1$ and G-modules A.