Algebraic Structures in Representation Stability

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Contents

Foreword

These are notes for my course at the 2019 MSRI Summer School on Representation Stability. While the summer school is going on (June 24–July 5), I'll be updating these notes on most days.

Lecture 1

Overview

Representation stability makes use of many kinds of novel algebraic objects. Some work in the subject uses the algebraic objects for applications, such as understanding the cohomology of configuration spaces. Other work in the subject studies these objects in the abstract; this work is often needed in applications, but is interesting in and of itself. This lecture series will focus on the latter type of work.

Before getting into any details, we give an overview of the kinds of algebraic objects that occur in the area.

A tour of the zoo

Representations of combinatorial categories

Let $\mathcal{C}$ be a category. A *$\mathcal{C}$-module* or a *representation of $\mathcal{C}$* is a functor $\mathcal{C} \to \mathbf{Vec}$, where $\mathbf{Vec}$ is the category of complex vector spaces. Of course, one could consider other coefficient fields (or rings), but we'll stick with $\mathbf{C}$ in this course to keep things simple. The $\mathcal{C}$ appearing in representation stability tend to be combinatorial in nature: the objects are often finite sets (possibly with extra structure), and the morphisms are functions, diagrams, colorings, etc. Some specific examples:

- The category **FI**: objects are finite sets, morphisms are injections. In the subject of representation stability, **FI**-modules have a special place, for two reasons: first, they have seen the most application; and second, they are the simplest interesting case, and therefore serve as a model example.
- The category $\mathbf{FI}_d$, where $d \geq 1$ is an integer: objects are finite sets, morphisms are injections together with a d-coloring on the complement of the image. (A d-coloring on a set S is simply a function $S \to [d]$, where $[d] = \{1, \ldots, d\}$) In a certain sense, **FI**-modules are like modules over a univarite polynomial ring, while $\mathbf{FI}_d$-modules are like modules over a d-variable polynomial ring.

- The category **FIM**: objects are finite sets, morphisms are injections together with a perfect matching on the complement of the image. (A perfect matching on a set S is an undirected graph with vertex set S such that each vertex belongs to precisely one edge.)
- The category **FS**: objects are finite sets, morphisms are surjections. Typically, it is $\mathbf{FS}^{\mathrm{op}}$-modules that one is actually interested in, where the $(-)^{\mathrm{op}}$ denotes the opposite category.
- The category **OI**: objects are finite totally ordered sets, morphisms are order-preserving injections.
- The category **VI**: objects are finite dimensional vector spaces over a fixed field $\mathbf{F}$, morphisms are linear injections.

Representations of "large" groups

Many "large" groups (or similar structures, like Lie algebras) figure prominently in representation stability. The most important two, by far, are the infinite symmetric group $\mathfrak{S}_\infty$ and the infinite general linear group $\mathbf{GL}_\infty$. Some other examples include:

- Wreath products $\mathfrak{S}_\infty \ltimes G^\infty$, typically with G finite.
- The infinite orthogonal and symplectic groups.
- Infinite dimensional "super" Lie groups or Lie algebras, such as the infinite periplectic Lie superalgebra.
- The Witt algebra. This is the Lie algebra generated by elements $\{L_i\}_{i \in \mathbf{Z}}$ with commutators $[L_i, L_j] = (i-j)L_{i+j}$.

Typically, in the context of representation stability, the large groups come with a "standard" representation (e.g., $\mathbf{GL}_\infty$ acting on $\mathbf{C}^\infty$), and one is only interested in representations that are "algebraic," in the sense that they can be constructed from the standard representation using direct sums and tensor products.

Equivariant rings and modules

Let R be a ring on which a group G acts by ring homomorphisms. An *equivariant R-module* is an R-module M equipped with an action of G that is compatible with action on R, in the sense that $g(ax) = (ga)(gx)$ for all $g \in G$, $a \in R$, $x \in M$. Equivariant modules appear in mathematics all the time. The novel aspect in representation stability is that the objects involved tend to be very large. Some examples:

- Take R to be the infinite variable polynomial ring $\mathbf{C}[x_1, x_2, \ldots]$ and G to be the infinite symmetric group $\mathfrak{S}_\infty$, acting by permuting the variables.
- Take $R = \mathbf{C}[x_1, x_2, \ldots]$ as above and G to be the infinite generali linear group $\mathbf{GL}_\infty$, acting by linear substitutions in the variables.
- Take $R = \mathbf{C}[x_{i,j}]_{i,j \geq 1}$ with $x_{i,j} = x_{j,i}$, and take $G = \mathbf{GL}_\infty$. We think of the variables as the entries of an infinite symmetric matrix $A = (x_{i,j})$, and $g \in G$ acts by gA^tg. (Precisely, $gx_{i,j}$ is the (i,j) entry of gA^tg.)

Algebras in tensor categories and their modules

Let $\mathcal{A}$ be an abelian category equipped with a symmetric tensor product $\otimes$. A *commutative algebra* in $\mathcal{A}$ is an object A equipped with a multiplication map $A \otimes A \to A$ satisfying the appropriate axioms (e.g., commutativity). A *module* over an algebra A is an object M equipped with a map $A \otimes M \to M$ also satisfying the appropriate axioms. One can in fact carry over many of the basic definitions and results from commutative algebra to this setting. If $\mathcal{A}$ is the category of vector spaces, equipped with the usual tensor product, one recovers classical commutative algebra. In other cases, one obtains a new theory that is similar to commutative algebra in many ways, but with its own distinct character.

We give just one example now. Let **FB** be the following category: objects are finite sets, morphisms are bijections. For two **FB**-modules M and N, define their tensor product $M \circledast N$ to be the **FB**-module given by

$$(M \circledast N)(S) = \bigoplus_{S = A \amalg B} M(A) \otimes N(B).$$

Let $\mathbf{A}$ be the **FB**-module given by $\mathbf{A}(S) = \mathbf{C}$ for all S. This is naturally an algebra for the above defined tensor product. Modules for this algebra are equivalent to **FI**-modules (Exercise 2.8). More generally, commutative algebras in this tensor category are called *twisted commutative algebras* (tca's). TCA's and their modules are an important generalization of **FI**-modules.

The main problem

The main problem we address in this lecture series is the following: given some class of algebraic objects, how does one understand their structure? For instance, what is the structure of a general **FI**-module?

This is a very broad and open-ended question. Here are some specific problems one might solve to give an answer:

- Describe the simple modules.
- Describe the injective and projective modules.
- Determine if there is a noetherian property.
- Analyze the structure of injective or projective resolutions.
- Compute the Grothendieck group.
- Compute Ext groups between important modules.
- Show that the category is equivalent to an already understood category.
- Prove a "rationality theorem" for Hilbert series (if this makes sense).
- Break the category up into pieces (using Serre quotients), and address these issues on each piece.

The goal of this course is to show how one might solve problems like this. We will focus on **FI**-modules for the first several lectures, since they are the simplest objects and the ideas are easiest to explain for them. The final two lectures will show how the methods can be applied to some other algebraic objects.

Exercises

$\mathcal{C}$-modules

Exercise 1.1 (⋆)**.** Define what a morphism of $\mathcal{C}$-modules is, and show that the category $\mathbf{Mod}_{\mathcal{C}}$ of $\mathcal{C}$-modules is abelian. How does one compute kernels, cokernels, and images in this category? □

Exercise 1.2 (⋆)**.** Define what it means for a $\mathcal{C}$-module to be finitely generated. □

Suppose S is a partially ordered set (poset). We can then define a category $\mathcal{C}$ as follows: the objects are the elements of S, and $\mathrm{Hom}_{\mathcal{C}}(x, y)$ has a single element if $x \le y$, and is empty otherwise. Composition is uniquely defined. We say that $\mathcal{C}$ is the category *associated* to S.

Exercise 1.3 (⋆)**.** Let $\mathcal{C}$ be the category associated to the poset $\mathbf{Z}$. Show that the category of $\mathcal{C}$-modules is equivalent to the category of graded $\mathbf{C}[t]$-modules, where t has degree 1. □

Exercise 1.4 (⋆⋆⋆)**.** Let $S = \{a, b, c, d\}$ and partially order the objects by $a \le b, c$ and $b, c \le d$. Let $\mathcal{C}$ be the associated category. Describe the category of $\mathcal{C}$-modules as completely as you can. For example:

(a) What are the simple, projective, and injective objects?

(b) What are the injective and projective resolutions of the simple objects?

(c) What are the Ext groups between the simple objects?

(d) What is the Grothendieck group? □

Exercise 1.5 (⋆⋆)**.** (Feel free to skip this exercise if you aren't familiar with sheaves.) Let $\mathcal{A}$ be the category of sheaves $\mathcal{M}$ of vector spaces on $\mathbf{R}^2$ such that $\mathcal{M}|_{\mathbf{R}^2 \setminus \{0\}}$ is locally constant. Construct a category $\mathcal{C}$ such that $\mathcal{A}$ is equivalent to $\mathbf{Mod}_{\mathcal{C}}$. □

Representations of $\mathfrak{S}_\infty$ and $\mathbf{GL}_\infty$

Define $\mathfrak{S}_\infty = \bigcup_{n \ge 1} \mathfrak{S}_n$, and $\mathbf{GL}_\infty = \bigcup_{n \ge 1} \mathbf{GL}_n$, and $\mathbf{C}^\infty = \bigcup_{n \ge 1} \mathbf{C}^n$.

Exercise 1.6 (⋆⋆)**.** Show that $\bigwedge^r(\mathbf{C}^\infty)$ is an irreducible representation of $\mathbf{GL}_\infty$. □

Exercise 1.7 (⋆)**.** Show that $\mathbf{C}^\infty$ is not a semi-simple representation of $\mathfrak{S}_\infty$. □

Exercise 1.8 (⋆⋆)**.** Describe the irreducible constituents of $(\mathbf{C}^\infty)^{\otimes 2}$ as an $\mathfrak{S}_\infty$-representation. □

FB-modules

Exercise 1.9 (⋆)**.** Let M and N be **FB**-modules. Show that the is a natural isomorphism $M \circledast N \cong N \circledast M$ of **FB**-modules; that is, $\circledast$ is a symmetric tensor product. □

A *representation of* $\mathfrak{S}_*$ is defined to be a sequence $M = (M_n)_{n \ge 0}$ where M_n is a representation of the symmetric group $\mathfrak{S}_n$. A *morphism* $M \to N$ of $\mathfrak{S}_*$-representations is simply a sequence $(f_n)_{n \ge 0}$ where $f_n \colon M_n \to N_n$ is a morphism of $\mathfrak{S}_n$-representations. Let $\mathbf{Rep}(\mathfrak{S}_*)$ denote the category of $\mathfrak{S}_*$-representations.

Exercise 1.10 (⋆⋆)**.** In this exercise, we compare **FB**-modules and $\mathfrak{S}_*$-representations.

(a) Construct a natural equivalence between $\mathbf{Mod}_{\mathbf{FB}}$ and $\mathbf{Rep}(\mathfrak{S}_*)$.

(b) Determine what the tensor product $\circledast$ on $\mathbf{Mod}_{\mathbf{FB}}$ corresponds to on $\mathbf{Rep}(\mathfrak{S}_*)$. We'll continue to denote this by $\circledast$.

(c) Since $\circledast$ is a symmetric tensor product on $\mathbf{Mod}_{\mathbf{FB}}$, the tensor product $\circledast$ on $\mathbf{Rep}(\mathfrak{S}_*)$ is also symmetric. Given two $\mathfrak{S}_*$-representations M and N, explicitly write down the isomorphism $M \circledast N \cong N \circledast M$.

(d) Describe, as explicitly as possible, exactly what a commutative algebra object in $\mathbf{Rep}(\mathfrak{S}_*)$ (with respect to $\circledast$) is. We will refer to these as tca's as well.

From now on, we will freely pass between **FB**-modules and $\mathfrak{S}_*$-representations. □

Exercise 1.11 (⋆⋆)**.** Let V be a vector space and let $\mathbf{V}$ be the $\mathfrak{S}_*$-representation given by $\mathbf{V}_1 = V$ and $\mathbf{V}_n = 0$ for $n \neq 1$. Describe the tca $\mathrm{Sym}(\mathbf{V})$. Here Sym denotes the symmetric algebra formed with respect to the tensor product $\circledast$. □

Exercise 1.12 (⋆⋆)**.** Let $\mathbf{V}$ be the $\mathfrak{S}_*$-representation given by $\mathbf{V}_2 = \mathbf{C}$, with trivial $\mathfrak{S}_2$-action, and $\mathbf{V}_n = 0$ for $n \neq 2$. Describe $\mathrm{Sym}(\mathbf{V})$. □

Exercise 1.13 (⋆⋆)**.** Let M be an **FB**-module such that M_n is finite dimensional for all n. Define the *Hilbert series* of M by

$$\mathrm{H}_M(t) = \sum_{n \ge 0} \dim(M_n) \frac{t^n}{n!}.$$

Suppose that N is another **FB**-module for which $\mathrm{H}_N(t)$ is defined. Show that

$$\mathrm{H}_{M \circledast N}(t) = \mathrm{H}_M(t) \cdot \mathrm{H}_N(t).$$

□

Lecture 2

The ABC's of FI-modules

In this lecture, we look at some of the most basic properties of **FI**-modules, and introduce a few important examples of **FI**-modules, namely $\mathbf{M}_\lambda$, $\mathbf{P}_d$, $\mathbf{P}_\lambda$, and $\mathbf{L}_\lambda$.

References: [CEF] is probably the best source for this material. [SS1] contains some similar material, but it's written in a different language.

First definitions

FI-modules

Recall that **FI** is the category whose objects are finite sets and whose morphisms are injections, and that an **FI**-module is a functor $\mathbf{FI} \to \mathbf{Vec}$ where **Vec** is the category of complex vector spaces.

Let M be an **FI**-module. For $n \geq 0$, we write M_n for the value of the functor M on the finite set $[n] = \{1, \dots, n\}$. Since the automorphism group of $[n]$ in the category **FI** is the symmetric group $\mathfrak{S}_n$, it follows that $\mathfrak{S}_n$ acts on M_n, that is, M_n is a representation of $\mathfrak{S}_n$. For a morphism $\varphi\colon [n] \to [m]$ in **FI**, we write $\varphi_*\colon M_n \to M_m$ for the induced map (i.e., the value of M on φ); we refer to these as *transition maps*. Let $i_n\colon [n] \to [n+1]$ be the standard inclusion, defined by $i_n(x) = x$, which is a morphism in **FI**, and let $t_n\colon M_n \to M_{n+1}$ be the map $(i_n)_*$; these are the most important transition maps. One easily verifies that t_n is $\mathfrak{S}_n$-equivariant, where here $\mathfrak{S}_n$ acts on M_{n+1} via the standard inclusion $\mathfrak{S}_n \subset \mathfrak{S}_{n+1}$.

Every object in **FI** is isomorphic to $[n]$ for some n. It follows that the values of M on arbtirary sets are determined by the M_n. Similarly, a morphism $[n] \to [m]$ in **FI**, with $m > n$, can be factored into a series of morphisms $[n] \to [n+1] \to \cdots \to [m]$, and any morphism $[n] \to [n+1]$ is in the $\mathfrak{S}_{n+1}$-orbit of the standard inclusion i_n. It follows that all of the transition maps in M are by the t_n's. Thus M is entirely determined by the data $(M_n, t_n)_{n\geq 0}$. Conversely, given data $(M_n, t_n)_{n\geq 0}$ satisfying a certain condition, one can construct a corresponding **FI**-module: see Exercise 2.6.

Let M and N be **FI**-modules. By definition, a morphism of **FI**-modules is a natural transformation of functors $M \to N$. We write $\mathrm{Hom}_{\mathbf{FI}}(M, N)$ for the space of morphisms. It is naturally a vector space.

Let $f\colon M \to N$ be a morphism of **FI**-modules. Evaluating everything on $[n]$, we obtain linear maps $f_n\colon M_n \to N_n$ for each $n \geq 0$; in fact, f_n is a map of $\mathfrak{S}_n$-representations. By the definition of natural transformation, the f_n are compatible with the transition maps; that is, the diagram

$$\begin{array}{ccc} M_n & \longrightarrow & M_{n+1} \\ {\scriptstyle f_n}\downarrow & & \downarrow{\scriptstyle f_{n+1}} \\ N_n & \longrightarrow & N_{n+1} \end{array} \tag{2.1}$$

commutes. The f_n completely determine f. Conversely, any sequence of maps $f_n\colon M_n \to N_n$ of $\mathfrak{S}_n$-representations compatible with transition maps comes from a map of **FI**-modules (Exercise 2.7).

Finite generation

Let M be an **FI**-module. By an *element* of M, we mean an element of M_n for some n. Let S be a set of elements of M. There is then a smallest **FI**-submodule of M containing S (simply take the intersection of all **FI**-submodules containing S). We call this the submodule of M *generated* by S. Explicitly, this is obtained by starting with S and then repeatedly applying transition maps and elements of symmetric groups, and forming linear combinations. We say that M is *finitely generated* if it is generated by a finite set of elements.

Torsion

Let M be an **FI**-module and let $x \in M_n$ be an element. We say that $x \in M_n$ is *torsion* if there exists some $k \geq 0$ such that $(t_{n+k-1} \cdots t_n)(x) = 0$, where the t_i's denote the transition maps; it is equivalent to ask that there is some morphism $\varphi\colon [n] \to [m]$ such that $\varphi_*(x) = 0$. The collection of all torsion elements in M forms an **FI**-submodule (Exercise 2.11). We say that M itself is *torsion* if all of its elements are, and we say that M is *torsion-free* if it has no non-zero torsion elements.

Some representation theory

Specht modules

The irreducible representations of the symmetric group $\mathfrak{S}_n$ are parametrized by partitions of n. For a partition λ, we let $\mathfrak{M}_\lambda$ denote the corresponding irreducible; this is called a *Specht module*. The exact construction of $\mathfrak{M}_\lambda$ in general is not important for now, but it's good to know a few cases:

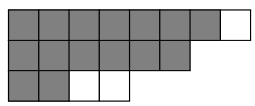
- If $\lambda = (n)$ then $\mathfrak{M}_\lambda$ is the trivial representation.
- If $\lambda = 1^n = (1, 1, \ldots, 1)$ then $\mathfrak{M}_\lambda$ is the sign representation.

- If $\lambda = (n-1,1)$ then $\mathfrak{M}_\lambda$ is the standard representation, that is, the subspace of $\mathbf{C}^n$ consisting of vectors whose coordinates sum to zero.
- If $\lambda = (n-2,1,1)$ then $\mathfrak{M}_\lambda$ is the second exterior power of the standard representation; more generally, if $\lambda = (n-k,1^k)$ then $\mathfrak{M}_\lambda$ is the kth exterior power of the standard representation.

The Pieri rule

Let λ and μ be partitions. We write $\lambda \subset \mu$ to indicate that the Young diagram of λ is contained in that for μ; concretely, this just means $\lambda_i \leq \mu_i$ for all i. In this case, we write $\mu \setminus \lambda$ for the complement of (the Young diagram of) λ in (that of) μ. This is not a Young diagram, but a so-called skew Young diagram. We say that it is a *horizontal strip* if it has no two boxes in the same column, and we then write $\mu \setminus \lambda \in \mathrm{HS}$.

For example, suppose that $\lambda = (7,6,2)$ and $\mu = (8,6,4)$. Then $\lambda \subset \mu$, and we can picture the Young diagrams as



where the shaded boxes are the diagram of λ, and all the boxes are the diagram of μ. The unshaded boxes constitute the diagram of $\mu \setminus \lambda$. Since no two of these boxes belong to the same column, $\mu \setminus \lambda$ is a horizontal strip.

The Pieri rule describes how certain inductions and restrictions of Specht modules behave. (It is a special case of the general rule, which we won't need, called the *Littlewood–Richardson* rule.) Let λ be a partition of n and let $m \geq 0$. Then the inductive form of the Pieri rule states

$$\mathrm{Ind}_{\mathfrak{S}_n \times \mathfrak{S}_m}^{\mathfrak{S}_{n+m}}(\mathfrak{M}_\lambda \otimes \mathrm{triv}) \cong \bigoplus_{\substack{|\mu|=n+m \\ \mu\setminus\lambda \in \mathrm{HS}}} \mathfrak{M}_\mu,$$

For $0 \leq m \leq n$, the restrictive form states

$$\mathrm{Res}_{\mathfrak{S}_{n-m} \times \mathfrak{S}_m}^{\mathfrak{S}_n}(\mathfrak{M}_\lambda)^{\mathfrak{S}_m} \cong \bigoplus_{\substack{|\mu|=n-m \\ \lambda\setminus\mu \in \mathrm{HS}}} \mathfrak{M}_\mu.$$

Note that $\mathfrak{S}_m$ invariants here. The two forms are equivalent via Frobenius reciprocity (Exercise 2.2).

In the special case when $m = 1$, the horizontal strip condition is automatic, and the rule simplifies a bit:

$$\mathrm{Ind}_{\mathfrak{S}_n}^{\mathfrak{S}_{n+1}}(\mathfrak{M}_\lambda) \cong \bigoplus_{\substack{|\mu|=n+1 \\ \lambda\subset\mu}} \mathfrak{M}_\mu$$

and

$$\mathrm{Res}_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n}(\mathfrak{M}_\lambda) \cong \bigoplus_{\substack{|\mu|=n-1 \\ \mu\subset\lambda}} \mathfrak{M}_\mu.$$

As for an example, we have

$$\mathrm{Ind}_{\mathfrak{S}_6}^{\mathfrak{S}_8}(\mathfrak{M}_{3,3}) \cong \mathfrak{M}_{5,3} \oplus \mathfrak{M}_{6,3,1} \oplus \mathfrak{M}_{3,3,2}$$

and

$$\mathrm{Res}_{\mathfrak{S}_4}^{\mathfrak{S}_6}(\mathfrak{M}_{3,3}) \cong \mathfrak{M}_{3,1}.$$

Simple modules

Let λ be a partition of n. There is a unique (up to isomorphism) **FB**-module $\mathbf{M}_\lambda$ such that $\mathbf{M}_{\lambda,n} = \mathfrak{M}_\lambda$ and $\mathbf{M}_{\lambda,k} = 0$ for $k \neq n$. We regard $\mathbf{M}_\lambda$ as an **FI**-module with transition maps equal to 0. It is clear that $\mathbf{M}_\lambda$ is a simple **FI**-module, that is, it has no **FI**-submodules other than 0 and $\mathbf{M}_\lambda$ itself: indeed, if M is a non-zero **FI**-submodule then M_n is a non-zero $\mathfrak{S}_n$-subrepresentation of $\mathfrak{M}_\lambda$, and thus all of $\mathfrak{M}_\lambda$ since $\mathfrak{M}_\lambda$ is irreducible, and so M is all of $\mathbf{M}_\lambda$. The $\mathbf{M}_\lambda$'s account for all of the simple **FI**-modules (Exercise 2.4).

Projective modules

The principal projectives

Fix an integer $d \geq 0$. Define an **FI**-module $\mathbf{P}_d$ as follows:

$$\mathbf{P}_d(S) = \mathbf{C}[\mathrm{Hom}_{\mathbf{FI}}([d], S)].$$

Here we write $\mathbf{C}[X]$ for a vector space with basis indexed by X. We thus see that the vector space $\mathbf{P}_d(S)$ has a basis e_φ indexed by the injections $\varphi\colon [d] \to S$. Morphisms in **FI** act on $\mathbf{P}_d$ simply by post-composition; that is, if $\psi\colon S \to T$ is a morphism in **FI** and e_φ is a basis element of $\mathbf{P}_d(S)$ then $\psi_*(e_\varphi) = e_{\psi \circ \varphi}$.

The space $\mathbf{P}_{d,d} = \mathbf{P}_d([d])$ has a canonical element, namely $e_{\mathrm{id}_{[d]}}$ where $\mathrm{id}_{[d]}\colon [d] \to [d]$ is the identity function. We denote this element by ϵ_d. It is clear that ϵ_d generates $\mathbf{P}_d$ as an **FI**-module. Indeed, given any injection $\varphi\colon [d] \to S$ we have $\varphi_*(\epsilon) = e_{\varphi \circ \mathrm{id}_{[d]}} = e_\varphi$. Thus the submodule generated by ϵ_d contains e_φ for all φ, and is thus all of $\mathbf{P}_d$.

Proposition 2.2. *Let M be an **FI**-module. Then the map*

$$\Phi\colon \mathrm{Hom}(\mathbf{P}_d, M) \to M_d, \qquad f \mapsto f(\epsilon_d)$$

is an isomorphism.

Proof. Any morphism $\mathbf{P}_d \to M$ is determined by where it sends ϵ_d, since ϵ_d generates $\mathbf{P}_d$. This shows that Φ is injective. To prove surjectivity, let $x \in M_d$ be given. Define $f_n\colon \mathbf{P}_{d,n} \to M_n$ to be the linear map taking e_φ to $\varphi_*(x)$. Given a morphism $\psi\colon [n] \to [m]$ in **FI**, we have

$$\psi_*(f_n(e_\varphi)) = \psi_*(\varphi_*(x)) = (\psi \circ \varphi)_*(x) = f_m(e_{\psi \circ \varphi}) = f_m(\psi_*(e_\varphi)).$$

It follows (Exercise 2.7) that the f_n's define a map of **FI**-modules $f\colon \mathbf{P}_d \to M$. We have

$$\Phi(f) = f(\epsilon_d) = f(e_{\mathrm{id}_{[d]}}) = (\mathrm{id}_{[d]})_*(x) = x,$$

which completes the proof. □

Corollary 2.3. *The* **FI***-module* $\mathbf{P}_d$ *is projective.*

Proof. Let $p\colon M \to N$ be a surjection of **FI**-modules and let $f\colon \mathbf{P}_d \to N$ be an arbitrary map of **FI**-modules. Let $x = f(\epsilon_d) \in N_d$. Since p is surjective, we can find an element $y \in M_d$ with $p(y) = x$. Let $g\colon \mathbf{P}_d \to M$ be the unique map satisfying $g(\epsilon_d) = y$, which exists by the proposition. Then $(p \circ g)(\epsilon_d) = x = f(\epsilon_d)$, and so $p \circ g = f$, again via the proposition. We have thus lifted f through p, which verifies the lifting criterion for $\mathbf{P}_d$ to be projective. □

We refer to $\mathbf{P}_d$ as the *principal projective* **FI**-module of degree d. The principal projective modules play a role analogous to free modules. Indeed, if R is a graded ring and $R[d]$ denotes the rank one free module with a generator of degree d, then for any graded R-module we have $\mathrm{Hom}_R(R[d], M) = M_d$. This is completely analogous to the mapping property of $\mathbf{P}_d$.

Interlude: group actions on FI-modules

Let M be an **FI**-module. An *action* of a group G on M is simply a group homomorphism $G \to \mathrm{Aut}_{\mathbf{FI}}(M)$. Concretely, this means that G acts linearly on $M(S)$ for all S, and that for any morphism $\varphi\colon S \to T$ in **FI** the induced map $\varphi_*\colon M(S) \to M(T)$ is G-equivariant; in particular, the action of G on M_n commutes with the action of $\mathfrak{S}_n$. By an *($\mathbf{FI} \times G$)-module*, we mean an **FI**-module equipped with an action of G.

Suppose now that G is finite, and let $L_1, \ldots, L_r$ be its irreducible (complex) representations. Given any representation V of G, the canonical map

$$\bigoplus_{i=1}^{r} \mathrm{Hom}_G(L_i, V) \otimes L_i \to V$$

is an isomorphism. (For $f \in \mathrm{Hom}_G(L_i, V)$ and $x \in L_i$, the above map takes $f \otimes x$ to $f(x)$.) The ith summand in this decomposition is called the *isotypic piece* of V corresponding to L_i, while the "coefficient" $\mathrm{Hom}_G(L_i, V)$ is called the *multiplicity space* of L_i in V.

Let M be an $(\mathbf{FI} \times G)$-module. Define $\mathrm{Hom}_G(L_i, M)$ to be the **FI**-module given by $S \mapsto \mathrm{Hom}_G(L_i, M(S))$. By the previous paragraph, the natural map

$$\bigoplus_{i=1}^{r} \mathrm{Hom}_G(L_i, M) \otimes L_i \to M$$

is an isomorphism of $(\mathbf{FI} \times G)$-modules. This explains how to decompose M under the action of G.

The $\mathbf{P}_\lambda$ modules

Recall that the principal projective $\mathbf{P}_d$ is defined by $\mathbf{P}_d(S) = \mathbf{C}[\mathrm{Hom}_{\mathbf{FI}}([d], S)]$. The group $\mathfrak{S}_d$ acts on this space by pre-composition: precisely, for $\sigma \in \mathfrak{S}_d$ we have $\sigma \cdot e_\varphi = e_{\varphi \circ \sigma^{-1}}$. This defines an action of $\mathfrak{S}_d$ on $\mathbf{P}_d$ by morphisms of **FI**-modules. Indeed, if $\psi \colon S \to T$ is a morphism in **FI** then

$$\sigma\psi_*(e_\varphi) = e_{\psi\circ\varphi\circ\sigma^{-1}} = \psi_*(\sigma e_\varphi).$$

We thus see that $\mathbf{P}_d$ is in fact an $(\mathbf{FI} \times \mathfrak{S}_d)$-module. We can therefore decompose $\mathbf{P}_d$ under the action of $\mathfrak{S}_d$. For a partition λ of d, we define

$$\mathbf{P}_\lambda = \mathrm{Hom}_{\mathfrak{S}_d}(\mathfrak{M}_\lambda, \mathbf{P}_d).$$

By the general theory, we have an isomorphism $(\mathbf{FI} \times \mathfrak{S}_d)$-modules

$$\mathbf{P}_d \cong \bigoplus_{|\lambda|=d} \mathfrak{M}_d \otimes \mathbf{P}_\lambda.$$

In particular, ignoring the $\mathfrak{S}_d$ action, we see that $\mathbf{P}_\lambda$ is a summand of $\mathbf{P}_d$, and thus a projective **FI**-module.

Proposition 2.4. *For an* **FI***-module M, we have a natural isomorphism*

$$\mathrm{Hom}_{\mathbf{FI}}(\mathbf{P}_\lambda, M) = \mathrm{Hom}_{\mathfrak{S}_d}(\mathfrak{M}_\lambda, M_d).$$

Proof. We have

$$\begin{aligned} \mathrm{Hom}_{\mathbf{FI}}(\mathbf{P}_\lambda, M) &= \mathrm{Hom}_{\mathbf{FI}}(\mathrm{Hom}_{\mathfrak{S}_d}(\mathfrak{M}_\lambda, \mathbf{P}_d), M) \\ &= \mathrm{Hom}_{\mathfrak{S}_d}(\mathfrak{M}_\lambda, \mathrm{Hom}_{\mathbf{FI}}(\mathbf{P}_d, M)) \\ &= \mathrm{Hom}_{\mathfrak{S}_d}(\mathfrak{M}_\lambda, M_d). \end{aligned}$$

□

Proposition 2.5. *For $n \geq d$ we have an isomorphism*

$$\mathbf{P}_{\lambda,n} \cong \mathrm{Ind}_{\mathfrak{S}_d \times \mathfrak{S}_{n-d}}^{\mathfrak{S}_n}(\mathfrak{M}_\lambda \otimes \mathrm{triv}).$$

Proof. The group $\mathfrak{S}_n$ acts transitively on the set $\mathrm{Hom}_{\mathbf{FI}}([d], [n])$, and the stabilizer of the standard injection is $\mathfrak{S}_{n-d}$. We thus see that this set is in bijection with the set of cosets $\mathfrak{S}_n/\mathfrak{S}_{d-n}$, and so $\mathbf{P}_{d,n} \cong \mathbf{C}[\mathfrak{S}_n/\mathfrak{S}_{n-d}]$. We thus have

$$\begin{aligned} \mathrm{Hom}_{\mathfrak{S}_d}(\mathfrak{M}_\lambda, \mathbf{P}_{d,n}) &\cong (\mathfrak{M}_\lambda^* \otimes \mathbf{P}_{d,n})^{\mathfrak{S}_d} \\ &\cong (\mathfrak{M}_\lambda^* \otimes \mathbf{P}_{d,n})_{\mathfrak{S}_d} \\ &\cong \mathbf{C}[\mathfrak{S}_n/\mathfrak{S}_{n-d}] \otimes_{\mathfrak{S}_d} \mathfrak{M}_\lambda^* \\ &\cong \mathbf{C}[\mathfrak{S}_n] \otimes_{\mathfrak{S}_d \times \mathfrak{S}_{n-d}} (\mathfrak{M}_\lambda^* \otimes \mathrm{triv}) \\ &= \mathrm{Ind}_{\mathfrak{S}_d \times \mathfrak{S}_{n-d}}^{\mathfrak{S}_n}(\mathfrak{M}_\lambda^* \otimes \mathrm{triv}). \end{aligned}$$

Since $\mathfrak{M}_\lambda$ is self-dual (as is every finite dimensional representation of $\mathfrak{S}_d$), the result follows. □

For a partition λ, let $\lambda[n] = (n - |\lambda|, \lambda_1, \lambda_2, \ldots)$. Provided $n \geq |\lambda| + \lambda_1$, this is a parition of n. The Young diagram of $\lambda[n]$ is simply the Young diagram of λ, with a row of length $n - |\lambda|$ stuck on top. For notational convenience, we define $\mathfrak{M}_{\lambda[n]}$ be zero if $\lambda[n]$ is not a partition.

Corollary 2.6. *For all $n \geq 0$ we have*

$$\mathbf{P}_{\lambda,n} \cong \bigoplus_{\substack{\lambda \setminus \mu \in \mathrm{HS}, \\ \lambda_1 + |\mu| \leq n}} \mathfrak{M}_{\mu[n]}.$$

Note that for $n \gg 0$, the inequality $\lambda_1 + |\mu| \leq n$ is automatic.

Proof. This simply comes from combining the proposition with Pieri's rule. We leave the details to Exercise 2.3. □

Spechtral modules

Fix a partition λ, and put $n_0 = |\lambda| + \lambda_1$. By Corollary 2.6, $\mathbf{P}_{\lambda,n_0}$ contains a unique copy of $\mathfrak{M}_{\lambda[n_0]}$. Define $\mathbf{L}_\lambda$ to be the **FI**-submodule of $\mathbf{P}_{\lambda,n_0}$ generated by this representation. We refer to the $\mathbf{L}_\lambda$'s as *Spechtral* ***FI****-modules* due to their close connection to Specht modules:

Proposition 2.7. *We have $\mathbf{L}_{\lambda[n]} \cong \mathfrak{M}_{\lambda[n]}$ for all $n \geq 0$.*

Proof. Let $n \geq n_0$. By Corollary 2.6, $\mathbf{P}_{\lambda,n}$ contains one copy of $\mathfrak{M}_{\lambda[n]}$, and that all other irreducibles in it have the form $\mathfrak{M}_{\mu[n]}$ with $|\mu| < |\lambda|$. Now, if $|\mu| < |\lambda|$ then $\mathrm{Res}^{\mathfrak{S}_{n+1}}_{\mathfrak{S}_n}(\mathfrak{M}_{\mu[n+1]})$ only contains irreducibles of the form $\mathfrak{M}_{\nu[n]}$ with $|\nu| \leq |\mu|$, again by the Pieri rule; thus there are no non-zero $\mathfrak{S}_n$-equivariant maps $\mathfrak{M}_{\lambda[n]} \to \mathfrak{M}_{\mu[n+1]}$. It follows that under the transition map $\mathbf{P}_{\lambda,n} \to \mathbf{P}_{\lambda,n+1}$, the representation $\mathfrak{M}_{\lambda[n]}$ must map into $\mathfrak{M}_{\lambda[n+1]}$. Since $\mathbf{P}_{\lambda,n}$ is torsion-free (Exercise 2.18), it follows that the image of $\mathfrak{M}_{\lambda[n]}$ in $\mathfrak{M}_{\lambda[n+1]}$ is non-zero, and thus generates all of $\mathfrak{M}_{\lambda[n+1]}$ as an $\mathfrak{S}_{n+1}$-representation, since it is irreducible. The result follows. □

In fact, the module $\mathbf{L}_\lambda$ is essentially uniquely characterized by the above proposition:

Proposition 2.8. *Let M be a torsion-free* **FI***-module such that $M_n \cong \mathfrak{M}_{\lambda[n]}$ for all $n \geq 0$. Then M is isomorphic to $\mathbf{L}_\lambda$ as an* **FI***-module.*

Proof. Left to Exercise 2.16. □

Example 2.9. The **FI**-module $\mathbf{L}_1$ has $\mathbf{L}_{1,0} = 0$ and $\mathbf{L}_{1,n} = \mathfrak{M}_{n-1,1}$ for all $n \geq 1$. (Recall that $\mathfrak{M}_{n-1,1}$ is the standard representation of $\mathfrak{S}_n$.) This **FI**-module is the first example of a torsion-free non-projective **FI**-module, so commonly appears in examples and exercises throughout the course. □

Exercises

Pieri's rule

Exercise 2.1 ($\star$)**.** Let $\lambda = (5, 3, 3, 1)$, a partition of 12. Compute the induction of $\mathbf{M}_\lambda$ to $\mathfrak{S}_{13}$ using Pieri's rule. Then compute the induction of $\mathbf{M}_\lambda$ to $\mathfrak{S}_{14}$ by inducing the result of the first computation, using Pieri's rule on each irreducible component. ☐

Exercise 2.2 ($\star$)**.** Show that the two versions of Pieri's rule (the one with induction and the one with restriction) are equivalent to each other. ☐

Exercise 2.3 ($\star$)**.** Carefully carry out the Pieri rule computation in Corollary 2.6. ☐

General properties of FI-modules

Exercise 2.4 ($\star$)**.** Show that every simple **FI**-module is isomorphic to some $\mathbf{M}_\lambda$. ☐

Exercise 2.5 ($\star$)**.** Give an example of an **FI**-module M that is not finitely generated, but for which M_n is finite dimensional for all n. Can you find an example with M torsion-free? ☐

Exercise 2.6 ($\star\star$)**.** Let $M = (M_n)_{n\geq 0}$ be an $\mathfrak{S}_*$-representation and for each n let $t_n \colon M_n \to M_{n+1}$ be a map of $\mathfrak{S}_n$-representations. Show that this data comes from an **FI**-module if and only if for each n and k the map $t_{n+k-1} \cdots t_n \colon M_n \to M_{n+k}$ has image contained in $M_{n+k}^{\mathfrak{S}_k}$. (When we say "comes from an **FI**-module" we mean that there is an **FI**-module M' such that $M_n = M'([n])$ and $t_n = M'(i_n)$ where $i_n \colon [n] \to [n+1]$ is the standard injection.) ☐

Exercise 2.7 ($\star$)**.** Let M and N be **FI**-modules. Show that morphisms $M \to N$ exactly correspond to sequences $(f_n)_{n\geq 0}$, where $f_n \colon M_n \to N_n$ is a map of $\mathfrak{S}_n$-representations and the f's are compatible with the transition maps (in the sense that the diagram (2.1) commutes). ☐

Exercise 2.8 ($\star\star$)**.** Recall from the first lecture the algebra $\mathbf{A}$ in the tensor category $\mathbf{Mod_{FB}}$. Show that the category of **FI**-modules is equivalent to the category of $\mathbf{A}$-modules. ☐

Exercise 2.9 ($\star\star$)**.** For an **FI**-module M and an integer $n \geq 0$ define $\tau_{\geq n} M$ to be the following **FI**-module:

$$(\tau_{\geq n} M)(S) = \begin{cases} M(S) & \text{if } \#S \geq n \\ 0 & \text{if } \#S < n \end{cases}$$

(a) Verify that $\tau_{\geq n}(M)$ is a well-defined **FI**-submodule of M.

(b) Show that $\tau_{\geq n}(M)$ is finitely generated if M is.

(c) Show that $\tau_{\geq n}(\mathbf{P}_\lambda)$ is not projective if $n > |\lambda|$. This gives a second source of torsion-free non-projective **FI**-modules (the first being the $\mathbf{L}_\lambda$'s). However, these examples are less interesting since they differ from projectives in only finitely many degrees. (You may want to do Exercise 2.19 before this). ☐

Exercise 2.10 (⋆⋆)**.** Let M and N be **FB**-modules. Define their *pointwise tensor product* $M \boxtimes N$ by $(M \boxtimes N)(S) = M(S) \otimes N(S)$. Show that if M and N are **FI**-modules then $M \boxtimes N$ is naturally an **FI**-module, and that if M and N are finitely generated then so is $M \boxtimes N$. □

Torsion modules

Exercise 2.11 (⋆)**.** Let M be an **FI**-module. Show that the torsion elements of M form an **FI**-submodule T, and that M/T is torsion-free. □

Exercise 2.12 (⋆)**.** Show that an **FI**-module has finite length if and only if it is finitely generated and torsion. □

Exercise 2.13 (⋆⋆)**.** Compute the Ext^1 groups between simple **FI**-modules. □

Spechtral modules

Exercise 2.14 (⋆)**.** Show that $\mathbf{L}_1$ is the kernel of a map $\mathbf{P}_1 \to \mathbf{P}_0$. □

Exercise 2.15 (⋆)**.** Determine the Hilbert series (Exercise 1.13) of $\mathbf{L}_\lambda$ using the hook length formula (Theorem B.3). □

Exercise 2.16 (⋆⋆)**.** Let M be a torsion-free **FI**-module. Suppose there exists a partition λ such that M_n is either 0 or $\mathfrak{M}_{\lambda[n]}$ for all n. Show that M is isomorphic to a submodule of $\mathbf{L}_\lambda$. As a corollary, obtain the uniqueness statement in Proposition 2.7. □

Projective modules

Exercise 2.17 (⋆)**.** Let M be an **FI**-module.

(a) Show that M can be written as a quotient of a direct sum of principal projectives.

(b) Show that M is finitely generated if and only if it can be written as a quotient of a finite direct sum of principal projectives. □

Exercise 2.18 (⋆)**.** Show that the modules $\mathbf{P}_n$ and $\mathbf{P}_\lambda$ are torsion-free. □

Exercise 2.19 (⋆⋆)**.** Show that every projective **FI**-module is a direct sum of $\mathbf{P}_\lambda$'s. □

Exercise 2.20 (⋆⋆)**.** Recall (Exercise 2.8) that **FI**-modules are equivalent to **A**-modules. Show that, under this equivalence, $\mathbf{P}_\lambda$ corresponds to $\mathbf{M}_\lambda \circledast \mathbf{A}$. □

Exercise 2.21 (⋆)**.** Compute the Hilbert series (Exercise 1.13) of $\mathbf{P}_n$ and $\mathbf{P}_\lambda$. □

Exercise 2.22 (⋆⋆⋆)**.** Let $P \subset Q$ be projective **FI**-modules. Show that P is a direct summand of Q. (Suggestion: first treat the case where $P = \mathbf{P}_\lambda$ and $Q = \mathbf{P}_\mu$.) □

Exercise 2.23 (⋆⋆)**.** Let M be an **FI**-module that is not projective. Using the previous exercise, show that M has infinite projective dimension. □

Finite length injectives

Exercise 2.24 (⋆⋆)**.** Fix $d \geq 0$. Define an **FI**-module $\mathbf{I}_d$ by

$$\mathbf{I}_d(S) = \mathbf{C}[\mathrm{Hom}_{\mathbf{FI}}(S, [d])]^*.$$

For an arbitrary **FI**-module M, construct a natural isomorphism

$$\mathrm{Hom}(M, \mathbf{I}_d) \cong M_d^*,$$

where here $(-)^*$ denotes the dual vector space. Conclude that $\mathbf{I}_d$ is an injective **FI**-module. □

Exercise 2.25 (⋆⋆)**.** Show that the symmetric group $\mathfrak{S}_d$ acts on $\mathbf{I}_d$ (by automorphisms of **FI**-modules). For a partition λ of d, let $\mathbf{I}_\lambda$ be the λ-piece of $\mathbf{I}_d$, i.e.,

$$\mathbf{I}_\lambda = \mathrm{Hom}_{\mathfrak{S}_d}(\mathfrak{M}_\lambda, \mathbf{I}_d).$$

Describe the functor $\mathrm{Hom}(-, \mathbf{I}_\lambda)$ on **FI**-modules, show that $\mathbf{I}_\lambda$ is indecomposable, and show the socle (=maximal semi-simple submodule) of $\mathbf{I}_\lambda$ is the simple $\mathbf{M}_\lambda$. □

Exercise 2.26 (⋆⋆)**.** Show that every finite length **FI**-module has finite injective dimension. □

Exercise 2.27 (⋆⋆⋆)**.** Let K be the Grothendieck group of the category of finite length **FI**-modules.

(a) Show that the classes $[\mathbf{M}_\lambda]$ form a $\mathbf{Z}$-basis of K.

(b) Show that the classes $[\mathbf{I}_\lambda]$ form a $\mathbf{Z}$-basis of K.

(c) Determine the change of basis matrices between these two bases.

(d) Compute the Ext pairings (Exercise A.10) of these elements. □

The amplitude filtration

We define the *amplitude*[1] of a partition λ to be $\lambda_2 + \lambda_3 + \cdots$; in other words, the amplitude is the number of boxes in the Young diagram below the first row. We define the amplitude of an $\mathfrak{S}_n$-representation V to be the maximum amplitude of a partition λ for which $\mathfrak{M}_\lambda$ appears in V. Finally, we define the amplitude of an **FB**-module (or **FI**-module) M to be the supremum of the amplitudes of the M_n's over $n \geq 0$.

Exercise 2.28 (⋆)**.** Show that any finitely generated **FI**-module has finite amplitude. □

Exercise 2.29 (⋆⋆)**.** For an **FB**-module M, define $F^n M$ to be the **FB**-submodule of M spanned by the $\mathfrak{M}_\lambda$-isotypic pieces with λ of amplitude at least n. Show that if M is an **FI**-module then $F^n M$ is an **FI**-submodule. □

Exercise 2.30 (⋆⋆)**.** Let M be an **FI**-module, and suppose that every partition appearing in M has amplitude exactly r. Show that M decomposes as a direct sum of **FI**-modules $\bigoplus_{|\lambda|=r} M_\lambda$, where every $\mathfrak{S}_n$-representation appearing in M_λ has the form $\mathfrak{M}_{\lambda[n]}$. □

[1] This is not standard terminology.

Lecture 3

The noetherian property

Consider the graded ring $\mathbf{C}[t]$, where t has degree 1. A (non-negatively) graded module over this ring is a sequence of vector spaces $(V_n)_{n\geq 0}$ equipped with linear maps $V_n \to V_{n+1}$. This closely resembles an **FI**-module, the difference being that in the **FI**-case the V_n's also have an action of $\mathfrak{S}_n$. We can therefore regard **FI**-modules as analogous to $\mathbf{C}[t]$-modules. This analogy turns out to be surprisingly good.

One of the most fundamental theorems about polynomial rings, especially from the point of view of modules, is the Hilbert basis theorem, which states that these rings are noetherian. In this lecture, we establish the analogous property for **FI**-modules, which is equally fundamental.

References: the material in this section mostly comes from [SS4]. For a different account of the noetherian property in the **FI**-language, see [CEF].

The main theorem

Let $\mathcal{C}$ be a category. We say that a $\mathcal{C}$-module M is *noetherian* if the ascending chain condition holds on submodules of M; equivalently, M is noetherian if every $\mathcal{C}$-submodule of M is finitely generated. We say that $\mathcal{C}$ has the *noetherian property* if every finitely generated $\mathcal{C}$-module is noetherian. It is not difficult to show that a direct sum of two noetherian modules is noetherian, and that any subquotient of a noetherian module is noetherian. It follows that $\mathcal{C}$ has the noetherian property if and only if every principal projective $\mathcal{C}$-module is noetherian.

The following is the main theorem of this lecture:

Theorem 3.1. *The category* **FI** *has the noetherian property.*

Remark 3.2. Theorem 3.1 has a long history. Snowden [Sn] first proved it (in a different language) in characteristic 0; then Church, Ellenberg, and Farb [CEF] reproved it in characteristic 0; then Church, Ellenberg, Farb, and Nagpal [CEFN] proved it over arbitrary noetherian rings; and finally, Sam and Snowden [SS4] deduced it over arbitrary noetherian rings as a special case of a general Gröbner theory for $\mathcal{C}$-modules. The proof in this lecture follows that of [SS4], as it generalizes to other categories $\mathcal{C}$. In Exercise 6.12, we will see the proof from [Sn]. $\square$

The category OI

Let **OI** be the category whose objects are finite totally ordered sets, and whose morphisms are order-preserving injections. There is a natural functor

$$\Phi\colon \mathbf{OI} \to \mathbf{FI}$$

that simply forgets the order. If $M\colon \mathbf{FI} \to \mathbf{Vec}$ is an **FI**-module then we can pre-compse with Φ to obtain an **OI**-module, which we denote by $\Phi^*(M)$. This construction has the following important property:

Proposition 3.3. *Let M be an* **FI***-module. Then M is finitely generated if and only if $\Phi^*(M)$ is.*

Proof. Suppose that $\Phi^*(M)$ is finitely generated, and let $x_1, \ldots, x_n$ be generators. Thus every element of M can be generated from the x_i's use morphisms in **OI**. Since these morphisms are all in **FI**, it follows that the x_i's generate M as an **FI**-module. Thus M is finitely generated.

For the converse, we first treat the case of principal projectives. Thus suppose that $M = \mathbf{P}_d$, so that $M_n = \mathbf{C}[\mathrm{Hom}_{\mathbf{FI}}([d], [n])]$. Every morphism $\varphi\colon [d] \to [n]$ in **FI** can be factored as $[d] \xrightarrow{\sigma} [d] \xrightarrow{\psi} [n]$ where σ is a permutation and ψ is order-preserving (i.e., a morphism in **OI**). We have $e_\varphi = \psi_*(e_\sigma)$, and thus e_φ belongs to the **OI**-module generated by e_σ. It follows that the e_σ's, for $\sigma \in \mathfrak{S}_d$, generate M as an **OI**-module. Thus $\Phi^*(M)$ is finitely generated.

We now treat the general case. Thus suppose M is finitely generated. Choose a surjection $\mathbf{P}_{d_1} \oplus \cdots \oplus \mathbf{P}_{d_r} \to M$. Since Φ^* is clearly exact, we thus have a surjection $\Phi^*(\mathbf{P}_{d_1}) \oplus \cdots \oplus \Phi^*(\mathbf{P}_{d_r}) \to \Phi^*(M)$. Since each $\Phi^*(\mathbf{P}_{d_i})$ is finitely generated, it follows that $\Phi^*(M)$ is finitely generated. $\square$

Corollary 3.4. *Let M be an* **FI***-module. If $\Phi^*(M)$ is noetherian then M is noetherian.*

Proof. Suppose $\Phi^*(M)$ is noetherian. Let N be an **FI**-submodule of M. Then $\Phi^*(N)$ is an **OI**-submodule of $\Phi^*(M)$, and therefore finitely generated by the noetherian assumption. The proposition now implies that N is finitely generated. Thus every **FI**-submodule of M is finitely generated, and so M is noetherian. $\square$

The above results show that the noetherian property for **OI** would imply it for **FI**. In the following section, we establish the noetherian property for **OI**.

The noetherian property for OI

We now prove the noetherian property for **OI**. The proof has two components: the first, establishes the ascending chain condition for so-called monomial submodules, which is an essentially combinatorial result; the second uses Gröbner theory to deduce the general case from the monomial case.

Monomials

Fix an integer $d \geq 0$, and let Q be the principal projective **OI**-module generated in degree d. Thus $Q(S) = \mathbf{C}[\mathrm{Hom}_{\mathbf{OI}}([d], S)]$. Let $\mathcal{M}_n = \mathrm{Hom}_{\mathbf{OI}}([d], [n])$, and put $\mathcal{M} = \coprod_{n \geq 0} \mathcal{M}_n$. Thus $\mathcal{M}_n$ is (or indexes) a basis for the vector space Q_n: for $f \in \mathcal{M}_n$, we write e_f for the corresponding basis vector of Q_n. We think of the e_f's (and sometimes the f's themselves) as analogous to monomials. We say that an **OI**-submodule M of Q is a *monomial submodule* if it is generated by those e_f's which it contains. Our immediate goal is to establish the ascending chain condition for these submodules.

Let $f, g \in \mathcal{M}$. We say that g *divides* f, written $g \mid f$, if $f = h \circ g$ for some morphism h in the category **OI**. The divisibility relation $\mid$ endows $\mathcal{M}$ with a partial order. We also put a partial order $\preceq$ on $\mathbf{N}^d$ by $(n_1, \ldots, n_d) \preceq (m_1, \ldots, m_d)$ if $n_i \leq m_i$ for all i.

Proposition 3.5. *We have an isomorphism of posets $\Phi \colon \mathcal{M} \to \mathbf{N}^{d+1}$ given by*

$$\Phi(f) = (f(1) - 1, f(2) - f(1) - 1, \ldots, f(d) - f(d-1) - 1, n - f(d)),$$

for $f \in \mathcal{M}_n$.

Proof. It is clear that Φ is a bijection. Suppose now that $g \mid f$, and write $f = h \circ g$; let $g \colon [d] \to [n]$ and $f \colon [d] \to [m]$, so that $h \colon [n] \to [m]$. There are $g(1)$ element of $[n]$ that are $\leq g(1)$. Since h maps these to distinct elements of $[m]$, and preserves order, it follows that there are at least $g(1)$ elements of $[m]$ that are $\leq h(g(1)) = f(1)$. Thus $g(1) \leq f(1)$. Similarly, that are $g(2) - g(1)$ elements of $[n]$ between $g(1)$ (exclusive) and $g(2)$ (inclusive), and thus at least this many elements of $[m]$ between $h(g(1)) = f(1)$ and $h(g(2)) = f(2)$. Thus $g(2) - g(1) \leq f(2) - f(1)$. Continuing in this way, we see that $\Phi(g) \leq \Phi(f)$. To complete the proof, we must show that $\Phi(g) \leq \Phi(f)$ implies $g \mid f$. We leave this to the reader. □

We now introduce some terminology from order theory. Let X be a poset. An *ideal* of X is a subset Y of X with the property that $y \in Y$ and $y \leq z$ implies $z \in Y$. We say that X is a *well-partial-order* (wpo) if the ascending chain condition holds for ideals. We say that X is *well-founded* if it satisfies the descending chain condition, that is, there is no infinite strictly decreasing chain in X. An *anti-chain* in X is a sequence $x_1, x_2, \ldots$ in X that are all mutually incomparable, i.e., $x_i \not\leq x_j$ for all $i \neq j$.

Proposition 3.6. *Let X be a poset. Then X is a well-partial-order if and only if it is well-founded and has no anti-chain.*

Proof. Left to Exercise 3.14. ☐

Proposition 3.7 (Dickson's Lemma). *The poset* $\mathbf{N}^d$ *is a well-partial-order.*

Proof. It is obviously well-founded, and it is not difficult to see that it has no anti-chains. ☐

Corollary 3.8. *The poset* $\mathcal{M}$ *is a well-partial-order.*

Proposition 3.9. *The ascending chain condition holds for monomial submodules of* Q.

Proof. Let M be a monomial submodule of Q. Consider the set $S \subset \mathcal{M}$ of f's such that e_f is contained in M. This is an ideal of $\mathcal{M}$. Indeed, suppose $g \in S$ and $g \mid f$. Write $f = h \circ g$. Since $g \in S$, we have $e_g \in M$, and so $h_*(e_g) = e_{h \circ g} = e_f$ belongs to M, and so $f \in S$.

We can thus consider the map

$$\Phi \colon \{\text{monomial submodules of } Q\} \to \{\text{ideals of } \mathcal{M}\}, \qquad M \mapsto S.$$

This map is injective, essentially by definition: since M is a monomial submodule, it is generated by S, and so one can recover M from S. It is also surjective: given an ideal S of $\mathcal{M}$, let M be the **OI**-submodule of Q it generates. Then M is a monomial submodule, by definition, and one can show that it contains no new monomials; thus $\Phi(M) = S$.

We thus see that Φ is an isomorphism of posets. Since $\mathcal{M}$ is well-founded, its ideals satisfy ACC. Thus monomial submodules of Q satisfy ACC as well. ☐

Gröbner theory

Define a total order $<$ on the set $\mathcal{M}_n$, as follows: we declare $f < g$ if the tuple $(f(d), \ldots, f(1))$ is less than the tuple $(g(d), \ldots, g(1))$ lexicographically. Thus, explicitly, $f < g$ if $f \neq g$ and $f(i) < g(i)$ where i is the largest index such that $f(i) \neq g(i)$. This order is obviously compatible with post composition, in the sense that if $f < g$ and $h \colon [n] \to [m]$ is any morphism in **OI** then $hf < hg$.

Let $x \in Q_n$ be non-zero. Write $x = \sum_{f \in \mathcal{M}_n} c(f) e_f$, where the $c(f)$ are coefficients. We define the *initial term* of x, denoted $\mathrm{in}(x)$, to be $c(f)e_f$ where f is maximal (in the order $<$ defined above) subject to $c(f) \neq 0$. Note that formation of initial terms is compatible with "multiplication" by elements of **OI**, that is, for a morphism $h \colon [n] \to [m]$ in **OI** we have $\mathrm{in}(h_*(x)) = h_*(\mathrm{in}(x))$.

Now let $M \subset Q$ be an **OI**-submodule. We define the *initial submodule* of M, denoted $\mathrm{in}(M)$, to be the **OI**-submodule that in degree n is spanned by the elements $\mathrm{in}(x)$ with $x \in M_n$ non-zero. This is indeed an **OI**-submodule, and it clearly monomial. We have the following important result:

Proposition 3.10 (Gröbner lemma). *Let* $M \subset N$ *be submodules of* Q *such that* $\mathrm{in}(M) = \mathrm{in}(N)$. *Then* $M = N$.

Proof. We show that $M_n = N_n$ by induction on leading terms. Thus let $x \in N_n \setminus \{0\}$ be given, and suppose that all $y \in N_n \setminus \{0\}$ with $\mathrm{in}(y) < \mathrm{in}(x)$ belong to M_n. (This inductive procedure is valid since there are only finitely many possible initial terms, as the set $\mathcal{M}_n$ is finite.) By assumption, $\mathrm{in}(M) = \mathrm{in}(N)$, and so there exists $y \in M_n \setminus \{0\}$ such that $\mathrm{in}(x) = \mathrm{in}(y)$. Put $z = x - y$; note that this belongs to N_n since both x and y do. Since the initial terms of x and y cancel, we have $\mathrm{in}(z) < \mathrm{in}(x)$. Thus, by the inductive hypothesis, we have $z \in M$. Since $x = y + z$, it too belongs to M. □

The noetherian property for **OI** now follows: indeed, suppose that $M_1 \subset M_2 \subset \cdots$ is an ascending chain of **OI**-submodules of Q. We get an ascending chain $\mathrm{in}(M_1) \subset \mathrm{in}(M_2) \subset \cdots$ of monomial submodules of Q. Thus, by Proposition 3.9, this stabilizes, i.e., $\mathrm{in}(M_i) = \mathrm{in}(M_{i+1})$ for $i \gg 0$. By the above proposition, it follows that $M_i = M_{i+1}$ for $i \gg 0$. We thus see that Q is a noetherian **OI**-module. Since this holds for all principal projective **OI**-modules, the noetherian property follows.

The noetherian property for other categories

Gröbner theory for $\mathcal{C}$-modules

The proof of the noetherian property for **FI** can be adapted to treat many other categories $\mathcal{C}$, as follows. For an object x of $\mathcal{C}$, define a $\mathcal{C}$-module P_x by $P_x(y) = \mathbf{C}[\mathrm{Hom}_{\mathcal{C}}(x, y)]$. Just as in the **FI**-case, these are projective and called the principal projectives, and it suffices to prove that these are noetherian modules. There are two steps to the proof: first, reduce to the case of monomial submodules; and second, handle them.

Before getting started, we introduce a bit more notation and terminology. We let S be a set of objects of $\mathcal{C}$ such that every object of $\mathcal{C}$ is isomorphic to a unique object of S: for **FI**, we'd take S to be the set of $[n]$'s. We put $\mathcal{M}_x = \coprod_{y \in S} \mathrm{Hom}_{\mathcal{C}}(x, y)$. This is the set of "monomials" in P_x. For $f, g \in \mathcal{M}_x$, we say that g *divides* f, written $g \mid f$, if there exists a morphism h in $\mathcal{C}$ such that $f = h \circ g$. This defines a natural partial order on $\mathcal{M}_x$. We say that a submodule of P_x is *monomial* if it is generated by the monomials it contains.

To reduce to monomial submodules, we want to be able to form the initial terms, and for this we need a well-order on $\mathcal{M}_x$. We thus make the following assumption:

(G1) The set $\mathcal{M}_x$ admits a well-order $\leq$ that is compatible with post-composition, that is, $f \leq g$ implies $h \circ f \leq h \circ g$.

Given this, we define the initial term of an element of P_x as in the **FI**-case: just take the maximal monomial with non-zero coefficient. And for a $\mathcal{C}$-submodule $M \subset P_x$, we define its initial submodule $\mathrm{in}(M)$ to be the one generated by the initial terms of elements of M. This is a monomial submodule. Proposition 3.10 carries over, and so to prove ACC for submodules of P_x it suffices to prove ACC for monomial submodules of P_x.

Just as in the **FI**-case, monomial submodules correspond bijectively to poset ideals of $\mathcal{M}_x$, with respect to the divisibility order. We therefore make the following assumption:

(G2) Every ideal of the poset $\mathcal{M}_x$ is finitely generated.

This hypothesis implies that every monomial ideal is finitely generated, and implies ACC for monomial submodules.

Define the category $\mathcal{C}$ to be *Gröbner* if (G1) and (G2) hold, for all $x \in \mathcal{C}$. The above discussion can be summarized as follows: every Gröbner category has the noetherian property.

This is a great theorem, since it reduces checking the noetherian property to a combinatorial problem that is typically easier. However, it suffers from a glaring deficiency: no finite group admits a total order compatible with the group operation, and so condition (G1) precludes our category from having any finite order automorphisms. In particular, most of the categories we care about, such as **FI**, $\mathbf{FI}_d$, **VI**, and so on, are not Gröbner.

We can ameliorate this flaw just as in the **FI**-case. We say that a functor $\Phi \colon \mathcal{D} \to \mathcal{C}$ satisfies *property (F)* if $\Phi^*(M)$ is finitely generated whenever M is. One can then show that if Φ is essentially surjective and has property (F) and $\mathcal{D}$ has the noetherian property, then $\mathcal{C}$ has the noetherian property. Motivated by this, we say that $\mathcal{C}$ is *quasi-Gröbner* if there exists a Gröbner category $\mathcal{D}$ and a functor $\Phi \colon \mathcal{D} \to \mathcal{C}$ that is essentially surjective and has property (F). We can thus conclude that any quasi-Gröbner category has the noetherian property.

The proof of the noetherian property we gave for **FI** in fact shows that **OI** is a Gröbner category and **FI** is a quasi-Gröbner category. Many of the categories we care about, such as $\mathbf{FI}_d$ and $\mathbf{FS}^{\mathrm{op}}$ are also quasi-Gröbner. This is covered in the exercises.

Remark 3.11. If $\mathcal{C}$ is a quasi-Gröbner category then the noetherian property holds for $\mathcal{C}$-modules over any coefficient field (or, more generally, any noetherian coefficient ring). □

Nash-Williams theory

To show that a category is Gröbner, we must establish conditions (G1) and (G2). The first of these, the existence of a well-order on $\mathcal{M}_x$ is typically very easy. The second, showing that $\mathcal{M}_x$ is a well-partial-order under divisibility, can be difficult. Nash-Williams theory provides a general and powerful method for showing that a partial order is a well-partial-order.

Fix a poset X. Define a sequence $x_1, x_2, \ldots$ of elements in X to be *bad* if $x_i \nleq x_j$ for all $i < j$. We say that a bad sequence is *minimal* if for every $i \geq 1$ there is no bad sequence $x_1, \ldots, x_{i-1}, y_i, y_{i+1}, \ldots$ with $y_i < x_i$. The following simple observation is the key to Nash-Williams approach:

Proposition 3.12 (Nash-Williams)**.** *Suppose that X is well-founded but not a well-partial-order. Then X contains some minimal bad sequence $x_1, x_2, \ldots$. Furthermore, if Y is the set of all elements $y \in X$ such that $y < x_i$ for some i then Y is a well-partial-order.*

Proof. Left as Exercise 3.16. □

Higman's lemma

We now demonstrate the utility of Nash-Williams theory by giving an elegant proof of Higman's lemma.

Let Σ be a finite set, and let $X = \Sigma^\star$ be the set of all words in the alphabet Σ. Define a partial order on X by $w \le w'$ if one can delete letters from w' to get w. In other words, if $w = w_1 \cdots w_n$ and $w' = w'_1 \cdots w'_m$ then $w \le w'$ if and only if there exists an order-preserving injection $\varphi\colon [n] \to [m]$ such that $w_i = w'_{\varphi(i)}$. This order is called the *Higman order*.

Proposition 3.13 (Higman's lemma)**.** *The poset X is a well-partial-order.*

Proof. We proceed by contradiction; thus suppose X is not a well-partial-order. Since X is clearly well-founded, by Proposition 3.12 there exists a minimal bad sequence $x_1, x_2, \ldots$ in X. Furthermore, letting Y be the set of words y such that $y < x_i$ for some i, the proposition shows that Y is a well-partial-order. Let y_i be x_i with the final letter removed; note that no x_i is the empty word, or else the sequence wouldn't be bad. Since $y_i < x_i$, we have $y_i \in Y$. By Exercise 3.15, there is a sequence of indices $i_1 < i_2 < \cdots$ such that $y_{i_1} \le y_{i_2} \le \cdots$. Now, among the infinitely many words $x_{i_1}, x_{i_2}, \ldots$, there must be two that have the same final letter, say x_i and x_j (with $i < j$): this is where we use that our alphabet is finite. Since $y_i \le y_j$ and x_i and x_j have the same final letter, it follows that $x_i \le x_j$. This is contradicts the assumption that the x's are a bad sequence, and thus completes the proof. □

Exercises

Consequences of the noetherian property

Exercise 3.1 ($\star$)**.** Let M be a finitely generated **FI**-module and let T be the torsion submodule of M. Show that $T_n = 0$ for all n sufficiently large. □

Exercise 3.2 ($\star\star$)**.** Let M be a finitely generated **FI**-module.

(a) Show that M admits a finite length filtration $0 = F^0 \subset \cdots \subset F^n = M$ such that F^i/F^{i-1} is either simple or a submodule of a Spechtral module. (Hint: make use of the amplitude filtration from the exercises in Lecture 2).

(b) Show that there is a polynomial p such that $\dim(M_n) = p(n)$ for all $n \gg 0$. Conclude that $\mathrm{H}_M(t)$ has the form $p(t)e^t + q(t)$, where $p(t)$ and $q(t)$ are polynomials.

(c) Show that there are partitions $\lambda_1, \ldots, \lambda_r$ such that

$$M_n \cong \bigoplus_{i=1}^{r} \mathfrak{M}_{\lambda_i[n]}$$

for all $n \gg 0$.

Part (c) is one of the most important properties of **FI**-modules, and explains the name "representation stability:" the representations M_n "stabilize" in the sense that their irreducible decomposition is essentially the same for all large n. In fact, historically, people (really Tom Church and Benson Farb [CF]) had observed sequences of symmetric group representations having this property, and then invented **FI**-modules to provide a theoretical framework for them. □

Exercise 3.3 (⋆⋆)**.** Let μ and ν be partitions. Show that there exist partitions $\lambda_1, \ldots, \lambda_r$ such that

$$\mathfrak{M}_{\mu[n]} \otimes \mathfrak{M}_{\nu[n]} \cong \bigoplus_{i=1}^{r} \mathfrak{M}_{\lambda_i[n]}$$

for all $n \gg 0$. (Hint: Consider the pointwise tensor product of Spechtral **FI**-modules.) In other words, the decomposition of the tensor product above stabilizes. This result is *Murnaghan's stability theorem*. The relatively simple proof suggested here via **FI**-modules is due to Church, Ellenberg, and Farb [CEF, §3.4]. □

Exercise 3.4 (⋆⋆)**.** Let $\mathcal{C}$ be a category with the noetherian property, and let I be a $\mathcal{C}$-module. Prove that the following are equivalent:

(a) I is injective.

(b) $\mathrm{Ext}^1(M, I) = 0$ for all finitely generated $\mathcal{C}$-modules M.

(c) Given an injection $M \to N$ of finitely generated $\mathcal{C}$-modules, the induced map $\mathrm{Hom}(N, I) \to \mathrm{Hom}(M, I)$ is surjective.

Using this, show that an arbitrary direct sum of injective $\mathcal{C}$-modules is injective. □

Projective resolutions and Tor

Let M be an **FI**-module. Define $\mathrm{Tor}_0(M)$ to be the **FB**-module given by

$$\mathrm{Tor}_0(M)_n = M_n / \{\text{the } \mathfrak{S}_n\text{-subrepresentation generated by } \mathrm{im}(M_{n-1} \to M_n)\},$$

where the denominator is taken to be 0 if $n = 0$. One easily checks that Tor_0 is right-exact; we let Tor_i be the ith left-derived functor of Tor_0.[1]

Exercise 3.5 (⋆)**.** Let M be an **FI**-module. Show that M is finitely generated if and only if $\mathrm{Tor}_0(M)$ is finite length. □

Exercise 3.6 (⋆)**.** Let M be a finitely generated **FI**-module. Show that M has a projective resolution

$$\cdots \to P_2 \to P_1 \to P_0 \to M \to 0$$

where each P_i is finitely generated. Conclude that $\mathrm{Tor}_i(M)$ is finite length for all $i \geq 0$. □

Exercise 3.7 (⋆⋆)**.** Let M be an **FI**-module. We say that a projective resolution $P_\bullet \to M$ is *minimal* if the map $\mathrm{Tor}_0(P_{n+1}) \to \mathrm{Tor}_0(P_n)$ is zero for all $n \geq 0$. Show that M has a minimal projective resolution, and that it is unique up to isomorphism. □

Exercise 3.8 (⋆⋆⋆)**.** Determine the minimal projective resolution of the simple **FI**-module $\mathbf{M}_0$, and compute $\mathrm{Tor}_i(\mathbf{M}_0)$ for all $i \geq 0$. □

Exercise 3.9 (⋆⋆)**.** Recall that **FI**-modules are equivalent to **A**-modules (Exercise 2.8). Show that $\mathrm{Tor}_0(M)$ is identified with $M \circledast_{\mathbf{A}} \mathbf{M}_0$, where here $\circledast_{\mathbf{A}}$ is the tensor product over **A**. This explains the notation Tor_i: it literally is $\mathrm{Tor}_i^{\mathbf{A}}(-, \mathbf{M}_0)$. □

[1] What we call $\mathrm{Tor}_\bullet$ is sometimes called "**FI**-module homology" in the literature.

The noetherian property for some other categories

Exercise 3.10 (⋆⋆)**.** Show that the noetherian property holds for $\mathbf{FI}_d$-modules. This is a fairly straightforward adaptation of the proof for **FI**-modules. □

Exercise 3.11 (⋆⋆⋆)**.** Show that the noetherian property holds for $\mathbf{FS}^{\mathrm{op}}$-modules. This follows the same approach we used for **FI**-modules. □

Exercise 3.12 (⋆⋆)**.** Let $\mathbf{F}$ be a finite field. Recall that **VI** is the category of finite dimensional $\mathbf{F}$-vector spaces and linear injections. Deduce the noetherian property for **VI**-modules from that for $\mathbf{FS}^{\mathrm{op}}$-modules, by making use of a natural functor $\mathbf{FS}^{\mathrm{op}} \to \mathbf{VI}$. □

Remark. The noetherian property for **VI** was essentially the content of the Lannes–Schwartz artinian conjecture. This conjecture was proved by Sam–Snowden [SS4] using the approach outlined above, and simultaneously by Putman–Sam [PS] using a distinct method. □

Order theory

Exercise 3.13 (⋆)**.** Let X and Y be posets. Define a partial order on $X \times Y$ by $(x, y) \le (x', y')$ if $x \le x'$ and $y \le y'$. Show that if X and Y are well-partial-orders then so is $X \times Y$. As a corollary, give a rigorous proof of Dickson's lemma (Proposition 3.7). □

Exercise 3.14 (⋆⋆)**.** Prove Proposition 3.6: a poset is a well-partial-order if and only if it is well-founded and has no anti-chains. □

Exercise 3.15 (⋆⋆)**.** Let X be a wpo and let $x_1, x_2, \ldots$ a sequence in X. Show that there exists $i_1 < i_2 < \cdots$ such that $x_{i_1} \le x_{i_2} \le \cdots$ □

Exercise 3.16 (⋆⋆)**.** Prove Nash-Williams' result Proposition 3.12. □

Exercise 3.17 (⋆⋆)**.** Let X be a poset, and let $X^\star$ be the set of all words in the alphabet X. Define a partial order on $X^\star$ as follows: $x_1 \cdots x_n \le y_1 \ldots y_m$ if there exists an order-preserving injection $\varphi \colon [n] \to [m]$ such that $x_i \le y_{\varphi(i)}$ for all $i \in [n]$. If X is discrete (i.e., $x \le y$ if and only if $x = y$) then this is simply the Higman order. Show that if X is a well-partial-order then so is $X^\star$. □

Exercise 3.18 (⋆⋆⋆)**.** For the purposes of this exercise, a *tree* is a connected undirected finite graph without cycles with a distinguished root node. A *homeomorphic embedding* $f \colon T \to T'$ between two trees is a map on vertices with the following properties:

(a) If y is a descendent of x in T then $f(y)$ is a descendent of $f(x)$;

(b) If y_1 and y_2 are immediate descendents of x in T then the shortest path from $f(y_1)$ to $f(y_2)$ in T' passes through $f(x)$.

Let $\mathcal{T}$ be the set[2] of all trees. Define an order $\le$ on $\mathcal{T}$ by $T \le T'$ if there exists a homeomorphic embedding $T \to T'$. This is not quite a partial order since $T \le T'$ and $T' \le T$ does

[2] To avoid set-theoretic issues, one can impose that the vertices are chosen from a fixed infinite set.

not imply $T = T'$; however, if we say that such T and T' are equivalent then $\leq$ defines a partial order on the set $\overline{\mathcal{T}}$ of equivalence classes. Show that this is a well-partial-order on $\overline{\mathcal{T}}$. This result is known as *Kruskal's tree theorem*. $\square$

Lecture 4

Nagpal's shift theorem

There is a natural way that one can "shift" an **FI**-module. It's easy to see that shifting an **FI**-module makes it "nicer" in certain ways. Rohit Nagpal proved the ultimate theorem in this direction: repeatedly shifting any finitely generated **FI**-module will eventually produce a projective **FI**-module. The purpose of this lecture is to state this theorem and deduce some important consequences. The proof, which is entirely elementary is also given, but that is not our focus.

References: Nagpal's theorem first appeared in [Nag, Theorem A]. However, that paper is a difficult read. I believe these notes contain the only other account of the theorem.

The shift functor

The definition

Let M be an **FI**-module. We define the *shift* of M, denoted ΣM, to be the **FI**-module given by $(\Sigma M)(S) = M(S \amalg \{*\})$. Here $\{*\}$ denotes a one-point set. The transition maps are defined as follows: suppose that $\varphi \colon S \to T$ is a morphism in **FI**, and let $\varphi_+ \colon S \amalg \{*\} \to T \amalg \{*\}$ be the map that is φ on S and takes $*$ to $*$. Then $(\Sigma M)(\varphi)$ is defined to be $M(\varphi_+)$. To be more concrete, we have $(\Sigma M)_n = M_{n+1}$, where the $\mathfrak{S}_n$ action is simply obtained by restricting the $\mathfrak{S}_{n+1}$-action. We write Σ^k for the k-fold iterate of the shift functor.

Shifting and torsion

Suppose M is a finitely generated torsion **FI**-module. Then $M_n = 0$ for $n \gg 0$, say for $n \geq n_0$. We thus see that $\Sigma^{n_0+1}(M) = 0$. Thus any finitely generated torsion module is killed by a sufficiently high shift.

More generally, suppose that M is a finitely generated **FI**-module, and let T be its torsion submodule. Then T is finitely generated by the noetherian property, and thus killed by a sufficiently high shift. It follows that a sufficiently high shift of M is torsion-free. This is one way in which shifting an **FI**-module a lot makes it nicer.

Shifts of projectives

Let's now examine the effect of shift on projectives. For this, it will be convenient to introduce a slight variant of the principal projectives. For a finite set D, let $\mathbf{P}_D$ be the **FI**-module defined by

$$\mathbf{P}_D(S) = \mathbf{C}[\mathrm{Hom}_{\mathbf{FI}}(D, S)].$$

Then $\mathbf{P}_D$ is isomorphic to $\mathbf{P}_d$ where $d = \#D$, and $\mathbf{P}_d = \mathbf{P}_{[d]}$ by definition.

Now, we have

$$(\Sigma\mathbf{P}_d)(S) = \mathbf{P}_d(S \amalg \{*\}) = \mathbf{C}[\mathrm{Hom}_{\mathbf{FI}}([d], S \amalg \{*\})].$$

Consider the e_φ's where $\varphi\colon [d] \to S \amalg \{*\}$ is a map not containing $*$ in its image. These clearly span an **FI**-submodule of $\Sigma\mathbf{P}_d$, and none nothing other than $\mathbf{P}_d$, since we're essentially ignoring $*$. Next, for $i \in [d]$ fixed, consider span of the e_φ's where $\varphi\colon [d] \to S \amalg \{*\}$ is a map with $\varphi(i) = *$. This too is an **FI**-submodule. Since such a φ simply corresponds to a map $[d] \setminus \{i\} \to S$, we see that this submodule is none other than $\mathbf{P}_{[d]\setminus\{i\}}$.

We have therefore shown that

$$\Sigma\mathbf{P}_d \cong \mathbf{P}_d \oplus \bigoplus_{i=1}^{d} \mathbf{P}_{[d]\setminus\{i\}} \cong \mathbf{P}_d \oplus \mathbf{P}_{d-1}^{\oplus d}$$

Thus the shift of a principal projective is a finite sum of principal projectives. As a corollary, we see that if M is any finitely generated **FI**-module then $\Sigma(M)$ is also finitely generated: indeed, if $P \to M$ is a surjection, with P a finite sum of principal projectives, then $\Sigma(P) \to \Sigma(M)$ is a surjection, and $\Sigma(P)$ is a finite sum of principal projectives.

We can also determine the shift of $\mathbf{P}_\lambda$. Going back to the above formula, we have

$$\Sigma\mathbf{P}_d \cong \mathbf{P}_d \oplus \bigoplus_{i=1}^{d} \mathbf{P}_{[d]\setminus\{i\}}.$$

This isomorphism is $\mathfrak{S}_d$-equivariant, using the obvious action of $\mathfrak{S}_d$ on the right: on the direct sum, it permutes the factors, while $\mathfrak{S}_{d-1}$ acts on $\mathbf{P}_{[d]\setminus\{d\}}$ in the usual way. We thus see that

$$\Sigma\mathbf{P}_d \cong \mathbf{P}_d \oplus \mathrm{Ind}_{\mathfrak{S}_{d-1}}^{\mathfrak{S}_d}(\mathbf{P}_{d-1})$$

as $(\mathbf{FI} \times \mathfrak{S}_d)$-modules. We now apply $\mathrm{Hom}_{\mathfrak{S}_d}(\mathfrak{M}_\lambda, -)$, and use Frobenius reciprocity and the fact that this commutes with Σ to deduce

$$\Sigma\mathbf{P}_\lambda \cong \mathbf{P}_\lambda \oplus \mathrm{Hom}_{\mathfrak{S}_{d-1}}(\mathrm{Res}_{\mathfrak{S}_{d-1}}^{\mathfrak{S}_d}(\mathfrak{M}_\lambda), \mathbf{P}_{d-1}).$$

Finally, applying Pieri's rule to compute this restriction, we find

$$\Sigma\mathbf{P}_\lambda \cong \mathbf{P}_\lambda \oplus \bigoplus_{\substack{|\mu|=d-1 \\ \mu\subset\lambda}} \mathbf{P}_\mu.$$

The canonical map to the shift

Let M be an **FI**-module. For any finite set S, we have a natural map $M(S) \to M(S \amalg \{*\})$ coming from the canonical morphism $S \to S \amalg \{*\}$ in **FI**. One easily verifies that this is a natural transformation of functors; that is, we have a map of **FI**-modules $M \to \Sigma(M)$. The kernel of this map consists of torsion elements, essentially by definition: this map is simply built out of the transition maps appearing in M.

The degree of an FI-module

Let M be a finitely generated **FI**-module. We have seen (Exercise 3.2) that there is a polynomial p such that $\dim(M_n) = p(n)$ for $n \gg 0$. We define the *degree* of M, denoted $\deg(M)$, to be the degree of the polynomial p, using the convention that the zero polynomial has degree $-\infty$. We note that $\deg(M) = -\infty$ if and only if M is torsion.

Proposition 4.1. *Let M be a finitely generated* **FI***-module of degree $d \geq 0$, and let C be the cokernel of the canonical map $M \to \Sigma(M)$. If $d = 0$ then C is torsion, and thus of degree $-\infty$; otherwise, C has degree $d - 1$.*

Proof. Consider the exact sequence

$$0 \to T \to M \to \Sigma(M) \to C \to 0.$$

As we have said, T is torsion, and thus $T_n = 0$ for $n \gg 0$. Let p and q be such that $\dim(M_n) = p(n)$ and $\dim(C_n) = q(n)$ for $n \gg 0$; note that $\deg(p) = d$ by definition. Now, we have

$$\dim(C_n) - \dim(T_n) = \dim((\Sigma M)_n) - \dim(M_n)$$

for all $n \geq 0$. Thus, for $n \gg 0$, we find

$$q(n) = p(n+1) - p(n).$$

If $d = 0$ then p is a constant polynomial, and so $q(n) = 0$; thus C is torsion. If $d > 0$ then $\deg(q) = \deg(p) - 1$, and so $\deg(C) = d - 1$. □

Nagpal's theorem

Theorem 4.2 (Nagpal)**.** *Let M be a finitely generated* **FI***-module. Then there exists k such that $\Sigma^k M$ is projective.*

A complete proof can be found towards the end of this chapter. For now, we simply show how the theorem works in a specific example. Consider the **FI**-module $\mathbf{L}_1$, which is a submodule of the projective $\mathbf{P}_1$. Precisely, for a finite set S we have $\mathbf{P}_1(S) = \mathbf{C}[S]$ and $\mathbf{L}_1(S)$ is the subspace spanned by elements of the form $e_i - e_j$ for $i, j \in S$. Consider the map $f \colon \mathbf{P}_1 \to \Sigma(\mathbf{L}_1)$ that on a finite set S is the map $\mathbf{P}_1(S) \to \mathbf{L}_1(S \amalg \{*\})$ given by $e_i \mapsto e_i - e_*$. This is clearly an isomorphism of vector spaces, and compatible with the transition maps. It is thus an isomorphism of **FI**-modules. We conclude that $\Sigma(\mathbf{L}_1)$ is projective.

Consequences of the shift theorem

Theorem 4.3. *Let M be a finitely generated* **FI**-*module. There exists a complex*

$$0 \to M \to P_0 \to P_1 \to \cdots \to P_n \to 0$$

where each P_i is a finitely generated projective **FI**-*module and the homology groups are torsion* **FI**-*modules.*

Proof. We proceed by induction on the degree of M. If M is torsion then there is nothing to prove. Thus suppose M has degree $d \geq 0$. Let n be such that $P_0 = \Sigma^n(M)$ is projective, and let C be the cokernel of the map $M \to P_0$. Then C has degree $< d$, and thus, by the inductive hypothesis, we can find a complex $C \to P_1 \to P_2 \to \cdots \to P_n \to 0$ of the stated form. The complex $M \to P_0 \to P_1 \to \cdots \to P_n \to 0$ then has the requisite properties. □

Remark 4.4. Let P be the complex of projectives in the above theorem. Regarging M as a complex concentrated in degree 0, we thus have a map of complexes $M \to P$, and the theorem states that the cone of this map has torsion cohomology. From this, one can deduce that, in the derived category of **FI**-modules, there is an exact triangle

$$T \to M \to P \to$$

where T is a complex of torsion modules. In fact, this continues to hold if M is allowed to be a finite length complex of finitely generated **FI**-modules. This theorem is reminiscent of the structure theorem for modules over a PID: indeed, if M is a module over a PID then there is an exact sequence

$$0 \to T \to M \to P \to 0$$

with T torsion and P projective. □

Theorem 4.5. *Any projective* **FI**-*module is injective.*

Proof. Let P be a finitely generated projective and consider an injection $i \colon P \to M$ where M is a finitely generated **FI**-module. Let n be such that $\Sigma^n(M)$ is projective (Theorem 4.2), and let $j \colon M \to \Sigma^n(M)$ be the canonical map. The kernel of j consists of torsion elements, and so $\ker(j) \cap \operatorname{im}(i) = 0$, since $i(P)$ is torsion-free; it follows that the composition $j \circ i$ is injective. Since any injection of projective **FI**-modules splits (Exercise 2.22), it follows that we have a map $p \colon \Sigma^n(M) \to P$ such that $p \circ j \circ i = \mathrm{id}_P$. But now $p \circ j$ provides a splitting to i.

We have thus shown that if P is any finitely generated projective, then any inclusion $P \to M$ with M finitely generated splits. It follows that $\mathrm{Ext}^1(N, P) = 0$ for all finitely generated **FI**-modules N, and so P is injective (Exercise 3.4). Moreover, since arbitrary direct sums of injectives are injective (Exercise 3.4 again), we see that arbitrary projectives are injective. □

Theorem 4.6. *Every finitely generated* **FI**-*module has finite injective dimension.*

Proof. We proceed by induction on degree. For torsion modules (degree $-\infty$), we already know the result (Exercise 2.26). Now suppose M has degree $d \geq 0$. Let n be such that $\Sigma^n(M)$ is projective. We have an exact sequence

$$0 \to T \to M \to \Sigma^n(M) \to C \to 0$$

where T is the torsion submodule of M and C has degree $< d$. By induction, C has finite injective dimension. We have already explained that T has finite injective dimension. And $\Sigma^n(M)$ is projective, and thus injective (Theorem 4.5), and thus has injective dimension 0. Thus M has finite injective dimension. □

Corollary 4.7. *Let M and N be* **FI***-modules, with N finitely generated. Then* $\mathrm{Ext}^i(M, N) = 0$ *for* $i \gg 0$.

Proof of the shift theorem

We begin by proving some results on $\Sigma^k \mathbf{P}_d$; in fact, this is where all the real work is. Fix a set L of size k, and for an **FI**-module M identify $(\Sigma^k M)(S)$ with $M(S \amalg L)$. Thus $\mathbf{P}_d(S) = \mathbf{C}[\mathrm{Hom}_{\mathbf{FI}}([d], S)]$ and $(\Sigma^k \mathbf{P}_d)(S) = \mathbf{C}[\mathrm{Hom}_{\mathbf{FI}}([d], S \amalg L)]$. Let $\Pi \colon \Sigma^k \mathbf{P}_d \to \mathbf{P}_d$ be defined as follows: given $f \in \mathrm{Hom}([d], S \amalg L)$, define $\Pi(e_f) = e_f$ if the image of f is contained in S, and $P(e_f) = 0$ otherwise. This provides a section to the canonical map $\mathbf{P}_d \to \Sigma^k \mathbf{P}_d$, and realizes $\mathbf{P}_d$ as a summand of $\Sigma^k \mathbf{P}_d$.

If M is an **FI**-module and $g \colon S \to T$ is an injection then, by definition, there is an induced map $g_* \colon M(S) \to M(T)$. The **FI**-module $\Sigma^k \mathbf{P}_d$ has more structure though: given an injection $g \colon S \amalg L \to T \amalg L$, there is an induced map $g_* \colon (\Sigma^k \mathbf{P}_d)(S) \to (\Sigma^k \mathbf{P}_d)(T)$. If M is an **FI**-submodule of $\Sigma^k \mathbf{P}_d$ then it not need be closed under these additional operations. The following proposition, which is the key to our proof of the theorem, provides some compensation.

Proposition 4.8. *Let S be a set of size d, let T be an arbitrary finite set, and let $g \colon S \amalg L \to T \amalg L$ be an injection. Given $x \in (\Sigma^k \mathbf{P}_d)(S)$, there exist permutations $\tau_1, \ldots, \tau_r$ of $S \amalg L$ and injections $h_1, \ldots, h_r \colon S \to T$ such that*

$$\Pi(g_*(x)) = \sum_{i=1}^{r} (h_i)_*(\Pi(\tau_i(x))).$$

Proof. Recall that $(\Sigma^k \mathbf{P}_d)(S) = \mathbf{C}[\mathrm{Hom}([d], S \amalg L)]$. Write $x = \sum_A x_A$, where the sum is over subsets A of $S \amalg L$ of cardinality d and x_A is a linear combination of basis vectors e_f with image A. Note that there is a unique subset of $S \amalg L$ of cardinality d not meeting L, namely S, and so $\Pi(x) = x_S$.

Let $A_1, \ldots, A_r$ be the subsets of $S \amalg L$ of cardinality d not meeting $g^{-1}(L)$. Then $\Pi(g_*(x)) = \sum_{i=1}^r g_*(x_{A_i})$. Let τ_i be a permutation of $S \amalg L$ such that $\tau_i(A_i) = S$. Then $\tau_i \cdot x_{A_i} = \Pi(\tau_i \cdot x)$. Let $h_i \colon S \to T$ be the restriction of $g \circ \tau_i^{-1}$ to S. Note that $g(\tau_i^{-1}(S)) = g(A_i) \subset T$ since A_i does not meet $g^{-1}(L)$, and so h_i does map into T. Then $(h_i)_*(\tau_i x_{A_i}) = g_*(x_{A_i})$. The stated formula now follows. □

Proposition 4.9. *Let K be a submodule of $\mathbf{P}_d$ generated in degrees $\le d+k$. Then $\Pi(\Sigma^k K) \subset \mathbf{P}_d$ is generated in degree d.*

Proof. Since K is generated in degrees $\le d+k$, every element of $(\Sigma^k K)(T) = K(T \amalg L)$ is a linear combination of elements of the form $g_*(x)$, where $g\colon S \amalg L \to T \amalg L$ is an injection and $x \in (\Sigma^k K)(S)$. It suffices therefore to show that $\Pi(g_*(x))$ is generated by degree d elements of $\Pi(\Sigma^k K)$. From the previous proposition, we have

$$\Pi(g_*(x)) = \sum_{i=1}^{r} (h_i)_*(\Pi(\tau_i(x))).$$

As $\tau_i(x) \in (\Sigma^k K)(S)$, the result follows. □

Remark 4.10. The previous propositions apply equally well with $\mathbf{P}_d$ replaced by $\mathbf{P}_d^{\oplus s}$, for any $s \ge 0$. □

Lemma 4.11. *Let M be an* **FI***-module generated in degree d. Then for k sufficiently large, $\Sigma^k M \cong P \oplus N$ where P is a projective* **FI***-module generated in degree d and N is generated in degrees $< d$.*

Proof. Choose a presentation

$$0 \to K \to \mathbf{P}_d^{\oplus s} \to M \to 0.$$

Suppose that K is generated in degrees $\le k+d$. Applying Σ^k, we obtain an exact sequence

$$0 \to \Sigma^k(K) \to \Sigma^k(\mathbf{P}_d^{\oplus s}) \to \Sigma^k(M) \to 0.$$

Now, the middle term decomposes as $\mathbf{P}_d^{\oplus s} \oplus Q$, where Q is generated in degrees $< d$. Let $K' = \Pi(\Sigma^k(K))$. This is generated in degree d by the previous proposition. Let $\overline{Q}$ be the image of Q in $\Sigma^k(M)$. We then have an exact sequence

$$0 \to K' \to \mathbf{P}_d^{\oplus s} \to \Sigma^k(M)/\overline{Q} \to 0.$$

Since K' is a submodule of $\mathbf{P}_d^{\oplus s}$ generated in degree d, it is a summand (exercise!) and thus projective. Thus the inclusion $K' \to \mathbf{P}_d^{\oplus s}$ splits (Exercise 2.22), and so $P = \Sigma^k(M)/\overline{Q}$ is projective. We thus have $\Sigma^k(M) \cong P \oplus \overline{Q}$, and the result follows. □

Theorem 4.2 now follows easily by induction on the degree of generation.

Exercises

Sundry

Exercise 4.1 ($\star$)**.** The shift functor can obviously be defined on **FB**-modules. Suppose that M and N are **FB**-modules. Construct a natural isomorphism

$$\Sigma(M \circledast N) \cong [\Sigma(M) \circledast N] \oplus [M \circledast \Sigma(N)]\,.$$

Thus Σ is a "derivation," in a kind of categorical sense. □

Exercise 4.2 ($\star$)**.** Determine the degrees of the **FI**-modules $\mathbf{L}_\lambda$ and $\mathbf{P}_\lambda$. ☐

Exercise 4.3 ($\star\star$)**.** Construct an injective resolution of $\mathbf{L}_1$. ☐

Exercise 4.4 ($\star\star$)**.** Show that every injective **FI**-module is a direct sum of $\mathbf{P}_\lambda$'s and $\mathbf{I}_\lambda$'s (defined in Exercise 2.25). ☐

Exercise 4.5 ($\star\star\star$)**.** Let M be an **FI**-module. Define $t_i(M) = \sup\{n \mid \mathrm{Tor}_i(M)_n \neq 0\}$, where Tor was defined in the exercises of Lecture 3. Define the *regularity* of M, denoted $\rho(M)$, to be the supremum of $t_i(M) - i$ over $i \geq 0$. Show that every finitely generated **FI**-module has finite regularity. ☐

Exercise 4.6 ($\star\star\star$)**.** Show that the autoequivalence group of the category of **FI**-modules is trivial. (That is, show that any equivalence of categories $F\colon \mathbf{Mod}_{\mathbf{FI}} \to \mathbf{Mod}_{\mathbf{FI}}$ is isomorphic to the identity functor.) ☐

Local cohomology

For an **FI**-module M, let $\Gamma(M)$ be the torsion submodule of M. It is not difficult to see that Γ is a left-exact functor from the category $\mathbf{Mod}_{\mathbf{FI}}$ to itself. We can therefore consider its right-derived functors $\mathrm{R}^\bullet\Gamma$, which are collectively called *local cohomology*.

Exercise 4.7 ($\star$)**.** Verify that Γ is indeed a left-exact functor. ☐

Exercise 4.8 ($\star\star$)**.** Let M be an **FI**-module.

(a) Suppose that M is finitely generated. Show that $\mathrm{R}^i\Gamma(M)$ is finitely generated for all i, and vanishes for $i \gg 0$.

(b) Suppose that M is torsion. Show that $\Gamma(M) = M$ and $\mathrm{R}^i\Gamma(M) = 0$ for all $i > 0$.

(c) Show that $\mathrm{R}^i\Gamma(M) = 0$ for all i if and only if M is projective. ☐

Exercise 4.9 ($\star\star$)**.** Compute $(\mathrm{R}^i\Gamma)(\mathbf{L}_1)$ for all i. ☐

The Grothendieck group

In what follows, we put $\mathrm{K} = \mathrm{K}(\mathbf{Mod}_{\mathbf{FI}}^{\mathrm{fg}})$. Recall that for a partition λ we have a simple **FI**-module $\mathbf{M}_\lambda$ and a projective **FI**-module $\mathbf{P}_\lambda$.

Exercise 4.10 ($\star\star$)**.** Show that we have a well-defined map $\gamma\colon \mathrm{K} \to \mathrm{K}$ given by

$$\gamma([M]) = \sum_{i \geq 0} (-1)^i [\mathrm{R}^i\Gamma(M)].$$

Determine what γ does to $[\mathbf{M}_\lambda]$ and $[\mathbf{P}_\lambda]$. ☐

Exercise 4.11 ($\star\star\star$)**.** Show that the classes $[\mathbf{M}_\lambda]$ and $[\mathbf{P}_\lambda]$ form a $\mathbf{Z}$-basis for K. ☐

Exercise 4.12 ($\star\star$)**.** Show that the tensor product $\circledast$ of **FB**-modules gives $\Lambda = \mathrm{K}(\mathbf{Mod}_{\mathbf{FB}}^{\mathrm{fg}})$ the structure of a commutative ring, and K the structure of a Λ-module. Using Exercise 4.11, show that K is free of rank two as a Λ-module. ☐

Character polynomials

Let $\xi_{i,n} \colon \mathfrak{S}_n \to \mathbf{N}$ be the function that maps a permutation to the number of i-cycles in its cycle decomposition. Let $f_\bullet = \{f_n \colon \mathfrak{S}_n \to \mathbf{C}\}_{n \geq 0}$ be a sequence of functions. We say that $f_\bullet$ is *strictly polynomial* (resp. *weakly polynomial*) if there exists a polynomial $F(x_1, x_2, \ldots)$ such that $f_n = F(\xi_{1,n}, \xi_{2,n}, \ldots)$ holds for all (resp. all sufficiently large) n. We then call F the *character polynomial* of $f_\bullet$. For an **FB**-module M, we let $\mathrm{ch}(M_n) \colon \mathfrak{S}_n \to \mathbf{C}$ be the character of M_n, and let $\mathrm{ch}(M)$ be the sequence $\{\mathrm{ch}(M_n)\}_{n \geq 0}$.

Exercise 4.13 ($\star$)**.** Show that $\mathrm{ch}(\mathbf{L}_1)$ is weakly polynomial (and not strictly polynomial). □

Exercise 4.14 ($\star\star$)**.** Show that character polynomials are unique: that is, show that if F is a polynomial such that $F(\xi_{1,n}, \xi_{2,n}, \ldots) = 0$ for all $n \gg 0$ then $F = 0$. □

Exercise 4.15 ($\star\star\star$)**.** Let M be a finitely generated **FI**-module. Show that $\mathrm{ch}(M)$ is weakly polynomial, and strictly polynomial if M is projective. □

Exercise 4.16 ($\star\star$)**.** Let M be a finitely generated **FI**-module and F be its character polynomial (i.e., the character polynomial of $\mathrm{ch}(M)$). Show that

$$\mathrm{ch}(M) - F(\xi_1, \xi_2, \ldots) = \sum_{i \geq 0} (-1)^i \, \mathrm{ch}(\mathrm{R}^i \Gamma M).$$

In other words, the failure of the character polynomial to give the character at small values of n is governed by local cohomology. □

Lecture 5

Generic **FI**-modules

When working with a $\mathbf{C}[t]$-module M, it is often convenient to pass to the vector space $\mathbf{C}(t) \otimes_{\mathbf{C}[t]} M$. This kills torsion, and moves to a situation where one can use basic linear algebra.

We have seen that **FI**-modules are analogous to (graded) $\mathbf{C}[t]$-modules. It is therefore reasonable to ask if **FI** has some kind of "fraction field." Unfortunately, at least to my knowledge, the answer is no. However, this turns out to not be a serious obstacle: one can still define "modules" over this non-existent "field"! We call these "generic **FI**-modules." Passing from an **FI**-module to its generic counterpart is exactly analogous to replacing a $\mathbf{C}[t]$-module M with the vector space $\mathbf{C}(t) \otimes_{\mathbf{C}[t]} M$, and just as useful.

References: [Ga] is one of the original references for generalities on Serre quotients, and I still find it useful. For the specific case of **FI**-modules, see [SS1] (though be warned that this is not written in the language of **FI**-modules).

A warm-up problem

Let R be an integral domain and let K be its fraction field. We now investigate the following problem: how can we recover the category $\mathbf{Vec}_K$ of K-vector spaces from the category $\mathbf{Mod}_R$ of R-modules?

To begin, observe that if M is any R-module then there is an associated K-vector space $[M] = K \otimes_R M$. Every K-vector space V arises in this way (up to isomorphism): just take $M = V$, regarded as an R-module; or pick a basis for V and take M to be the free R-module with that basis.

The main problem, then, is to understand morphisms: specifically, how can we "see" the K-linear maps $[M] \to [N]$ in terms of M and N? Any map $M \to N$ of R-modules induces a K-linear map $[M] \to [N]$; in other words, there is a map

$$\mathrm{Hom}_R(M, N) \to \mathrm{Hom}_K([M], [N]).$$

This map is typically not injective or surjective. There are essentially two reasons for this:

Reason 1. Let N_{tors} denote the torsion submodule of N. Then $[N] \cong [N/N_{\mathrm{tors}}]$, and so $\mathrm{Hom}_K([M],[N]) = \mathrm{Hom}_K([M],[N/N_{\mathrm{tors}}])$. Thus a map $f\colon M \to N/N_{\mathrm{tors}}$ of R-modules induces a map $[f]\colon [M] \to [N]$, but f may not lift to a map $M \to N$. For example, take $M = N/N_{\mathrm{tors}}$ and f to be the identity; this fails to split provided N_{tors} is not a summand of N. Moreover, if the composition $M \xrightarrow{f} N \to N/N_{\mathrm{tors}}$ is zero then $[f] = 0$. For example, if $N = N_{\mathrm{tors}}$, this will be the case.

Reason 2. Let M' be a submodule of M such that M/M' is torsion. Then $[M] \cong [M']$, and so $\mathrm{Hom}_K([M],[N]) = \mathrm{Hom}_K([M'],[N])$. Thus a map $f\colon M' \to N$ of R-modules induces a map $[f]\colon [M] \to [N]$, but f may not extend to a map $M \to N$. For example, just take $M = R$ and $M' = N$ a proper ideal of R (that is not a summand) and f the identity map. Moreover, it is possible that $f \neq 0$ but $f|_{M'} = 0$. For example, just take $M = N$ a non-zero torsion module, $M' = 0$, and f the identity.

We now turn the above observations into a precise statement. Let M be an R-module. An *s-modification* of M is an inclusion $M' \to M$ such that M/M' is torsion. A *q-modification* of M is a surjection $M \to M''$ with torsion kernel. (The letters "s" and "q" stand for sub and quotient.) We now have:

Proposition 5.1. *Let M and N be R-modules.*

(a) Let $M' \to M$ be an s-modification and let $N \to N'$ be a q-modification. Then any R-linear map $M' \to N'$ naturally induces a K-linear map $[M] \to [N]$.

(b) Every K-linear map $[M] \to [N]$ is induced from an R-linear map $M' \to N'$ between modifications as above.

(c) Let $f, g\colon M' \to N'$ be R-linear maps between modifications. Then $[f] = [g]$ if and only if there is an s-modification $M'' \to M'$ and a q-modification $N' \to N''$ such that the compositions $M'' \to M \rightrightarrows N \to N''$ coincide.

More succinctly: we have a natural isomorphism

$$\varinjlim \mathrm{Hom}_R(M', N') \to \mathrm{Hom}_K([M],[N]),$$

where the direct limit is taken over all modifications as above.

Proof. Left to Exercise 5.2. ☐

We can summarize our discussion as follows. Define a category $\mathcal{C}$ as follows. The objects are formal symbols $[M]$, one for each R-module M. The Hom-sets are defined by

$$\mathrm{Hom}_{\mathcal{A}}([M],[N]) = \varinjlim \mathrm{Hom}_R(M', N'),$$

where the limit is taken over all s-modifications M' and all q-modifications N'. Then the above discussion shows that the functor $\mathcal{C} \to \mathbf{Vec}_K$ taking $[M]$ to $K \otimes_R M$ is an equivalence of categories. We have thus accomplished our goal of formally constructing the category $\mathbf{Vec}_K$ from the category $\mathbf{Mod}_R$.

Serre quotients

Serre subcategories

Let $\mathcal{A}$ be an abelian category. A *Serre subcategory* of $\mathcal{A}$ is a full abelian subcategory $\mathcal{B}$ with the following property: if

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

is an exact sequence in $\mathcal{A}$ and two of the objects belong to $\mathcal{B}$ then so does the third. In other words, any sub or quotient of an object in $\mathcal{B}$ must belong to $\mathcal{B}$, and any extension of objects in $\mathcal{B}$ must belong to $\mathcal{B}$. Intuitively, one thinks of $\mathcal{B}$ as a category of "torsion" objects.

Example 5.2. The category $\mathbf{Mod}_R^{\mathrm{tors}}$ of torsion R-modules is a Serre subcategory of $\mathbf{Mod}_R$ (for R an integral domain). □

Example 5.3. Let $T\colon \mathcal{A} \to \mathcal{C}$ be an exact functor between abelian categories. Define the *kernel* of T, denoted $\ker(T)$, to be the full subcategory of $\mathcal{A}$ spanned by objects M such that $T(M) = 0$. Then $\ker(T)$ is a Serre subcategory of $\mathcal{A}$ (Exercise 5.1). We note that the previous example is of this form: taking $\mathcal{A} = \mathbf{Mod}_R$, $\mathcal{C} = \mathbf{Vec}_K$, and $T(M) = K \otimes_R M$, we have $\ker(T) = \mathbf{Mod}_R^{\mathrm{tors}}$. □

The quotient construction

Fix a Serre subcategory $\mathcal{B}$ of $\mathcal{A}$. We can then carry over our notions of modifications. Precisely, given an object M of $\mathcal{A}$, an *s-modification* of M is an inclusion $M' \to M$ with cokernel in $\mathcal{B}$, and a *q-modification* is a surjection $M \to M''$ with kernel in $\mathcal{B}$.

We now define a new category, denoted $\mathcal{A}/\mathcal{B}$ and called the *Serre quotient* category, as follows. The objects of $\mathcal{A}/\mathcal{B}$ are simply the objects of $\mathcal{A}$. Given objects M and N of $\mathcal{A}$, we define

$$\mathrm{Hom}_{\mathcal{A}/\mathcal{B}}(M, N) = \varinjlim \mathrm{Hom}_{\mathcal{A}}(M', N'),$$

where the direct limit is taken over the s-modifications $M' \to M$ and q-modifications $N \to N'$ as usual. There is a natural functor $T\colon \mathcal{A} \to \mathcal{A}/\mathcal{B}$, called the *quotient functor*, that is the identity on objects and takes a morphism to the natural element it defines in the direct limit. The following proposition summarizes the most rudimentary properties of this construction:

Proposition 5.4. *We have the following:*

(a) The quotient category $\mathcal{A}/\mathcal{B}$ is abelian.

(b) The functor T is exact.

(c) Let $f\colon M \to N$ be a morphism in $\mathcal{A}$. Then $T(f)$ is an isomorphism if and only if $\ker(f)$ and $\mathrm{coker}(f)$ belong to $\mathcal{B}$.

The quotient category has the following universal property:

Proposition 5.5. *Let $\mathcal{C}$ be an abelian category and let $\Phi\colon \mathcal{A} \to \mathcal{C}$ be an exact functor $\mathcal{B} \subset \ker(\Phi)$. Then there exists a unique (up to isomorphism) functor $\Psi\colon \mathcal{A}/\mathcal{B} \to \mathcal{C}$ such that $\Phi \cong \Psi \circ T$.*

Example 5.6. The discussion in the previous section shows that $\mathbf{Mod}_R / \mathbf{Mod}_R^{\mathrm{tors}}$ is equivalent to $\mathbf{Vec}_K$. This is one of the basic examples of Serre quotients. □

The section functor

Let $\mathcal{A}$ be an abelian category, $\mathcal{B}$ a Serre subcategory, and $T\colon \mathcal{A} \to \mathcal{A}/\mathcal{B}$ the quotient functor. We say that $\mathcal{B}$ is a *localizing subcategory* if T admits a right adjoint. In this case, the right adjoint is called the *section functor*, and denoted S. The following proposition summarizes the key properties of the section functor:

Proposition 5.7. *Assume $\mathcal{B}$ is a localizing subcategory. Then:*

(a) The section functor S is left-exact, and, in fact, continuous.

(b) For $N \in \mathcal{A}/\mathcal{B}$, the natural map $T(S(N)) \to N$ is an isomorphism.

(c) For $M \in \mathcal{A}$, the kernel and cokernel of the natural map $M \to S(T(M))$ belong to $\mathcal{B}$.

The functor $\mathbf{S}\colon \mathcal{A} \to \mathcal{A}$ given by $\mathbf{S} = S \circ T$ is called the *saturation* or *localization* functor. It is also left-exact. There is a canonical morphism $M \to \mathbf{S}(M)$, which is the subject of part (c) of the above proposition. Intuitively, $\mathbf{S}(M)$ is obtained by killing the torsion submodule of M and then forming the "biggest possible" torsion extension of the result. We say that M is *($\mathcal{B}$-)saturated* if this map is an isomorphism.

Example 5.8. The Serre subcategory $\mathbf{Mod}_R^{\mathrm{tors}}$ of $\mathbf{Mod}_R$ is localizing. Identifying the quotient category with $\mathbf{Vec}_K$, the section functor is the canonical functor $\mathbf{Vec}_K \to \mathbf{Mod}_R$ (which simply regards a K-vector space as an R-module). □

A recognition result

In many situations, a Serre quotient category is equivalent to some more concrete category, e.g., $\mathbf{Mod}_R / \mathbf{Mod}_R^{\mathrm{tors}}$ is equivalent to $\mathbf{Vec}_K$. The following result can often be used to establish such equivalences:

Proposition 5.9. *Let $\mathcal{A}$ and $\mathcal{C}$ be an abelian categories and let $T\colon \mathcal{A} \to \mathcal{C}$ be a functor. Suppose that:*

(a) T is exact.

(b) T has a right adjoint $S\colon \mathcal{C} \to \mathcal{A}$.

(c) The co-unit $T \circ S \to \mathrm{id}_{\mathcal{C}}$ is an isomorphism.

Then $\ker(T)$ is a localizing subcategory of $\mathcal{A}$ and the functor $\mathcal{A}/\ker(T) \to \mathcal{C}$ induced by T is an equivalence.

Grothendieck categories

Grothendeick abelian categories are an especially nice class of abelian categories; see Appendix A. They behave well with respect to Serre quotients, as we now explain.

Let $\mathcal{A}$ be a Grothendieck abelian category and let $\mathcal{B}$ be a Serre subcategory. Then $\mathcal{B}$ is a localizing subcategory if and only if it is closed under arbitrary direct sums. This is a very nice result: it is very easy to check if $\mathcal{B}$ is closed under direct sums, and then one gets "for free" the existence of the section functor. Assume $\mathcal{B}$ is a localizing subcategory. Then $\mathcal{B}$ is itself a Grothendieck category, as in the quotient $\mathcal{A}/\mathcal{B}$.

Essentially all abelian categories appearing in these notes are Grothendieck, and all Serre subcategories are closed under direct sums. Thus the scenario described above is very common.

Application to FI-modules

We now apply the above theory to the case of **FI**-modules. The category $\mathbf{Mod}_{\mathbf{FI}}^{\text{tors}}$ of torsion **FI**-modules is a Serre subcategory of $\mathbf{Mod}_{\mathbf{FI}}$. We define the category of *generic* **FI***-modules*, denoted $\mathbf{Mod}_{\mathbf{FI}}^{\text{tors}}$, to be the Serre quotient $\mathbf{Mod}_{\mathbf{FI}} / \mathbf{Mod}_{\mathbf{FI}}^{\text{tors}}$. Since $\mathbf{Mod}_{\mathbf{FI}}$ is a Grothendieck category and $\mathbf{Mod}_{\mathbf{FI}}^{\text{tors}}$ is closed under arbitrary direct sums, it follows that $\mathbf{Mod}_{\mathbf{FI}}^{\text{tors}}$ is a localizing subcategory and that $\mathbf{Mod}_{\mathbf{FI}}^{\text{gen}}$ is a Grothendieck abelian category. We write T and S for the quotient and section functors, as usual. Define an object of $\mathbf{Mod}_{\mathbf{FI}}^{\text{gen}}$ to be *finitely generated* if it is isomorphic to an object of the form $T(M)$ with M a finitely generated **FI**-module (though see Exercise 5.24 for more about this).

The following proposition gives the most basic structural facts about generic **FI**-modules.

Proposition 5.10. *We have the following:*

(a) The object $T(\mathbf{L}_\lambda)$ is simple in $\mathbf{Mod}_{\mathbf{FI}}^{\text{gen}}$.

(b) Every finitely generated object of $\mathbf{Mod}_{\mathbf{FI}}^{\text{gen}}$ has finite length, and admits a composition series where the successive quotients have the form $T(\mathbf{L}_\lambda)$.

(c) Every simple object of $\mathbf{Mod}_{\mathbf{FI}}^{\text{gen}}$ is isomorphic to $T(\mathbf{L}_\lambda)$ for some λ.

Proof. (a) Let $\overline{N}$ be a non-zero subobject of $T(\mathbf{L}_\lambda)$. By Exercise 5.4, we have $\overline{N} = T(N)$ for some subobject N of $\mathbf{L}_\lambda$. Since $\overline{N}$ is non-zero, so is N. Suppose n is such that $N_n \neq 0$. Then N_m is non-zero for all $m \geq n$ since $\mathbf{L}_\lambda$ is torsion-free. But $\mathbf{L}_{\lambda,m}$ is irreducible, and so $N_m = \mathbf{L}_{\lambda,m}$. It follows that $(\mathbf{L}_\lambda/N)_m = 0$ for all $m \geq n$, and so $\mathbf{L}_\lambda/N$ is torsion. Thus $T(\mathbf{L}_\lambda/N) = 0$, and so $T(\mathbf{L}_\lambda) = T(N)$ since T is exact. Thus $T(\mathbf{L}_\lambda)$ has no non-zero proper subobjects, and is therefore simple.

(b) Let M be a finitely generated **FI**-module. By Exercise 3.2, there is a filtration $0 = F^0 \subset \cdots \subset F^n = M$ such that each F^i/F^{i-1} is either simple or a (non-zero) submodule of some Spechtral **FI**-module $\mathbf{L}_{\lambda(i)}$. It follows that $T(F^i)/T(F^{i-1})$ is either 0 or isomorphic to $T(\mathbf{L}_{\lambda(i)})$, by (a). The result follows.

(c) Let $T(M)$ be a simple object of $\mathbf{Mod}_{\mathbf{FI}}^{\mathrm{gen}}$. By definition, $T(M)$ is non-zero, and so M is not torsion. Let $N \subset M$ be a finitely generated submodule that is not torsion (e.g., take the submodule generated by a non-torsion element). Then $T(N)$ is a non-zero subobject of $T(M)$, and thus equal to $T(M)$ by simplicity. By part (b), $T(N)$ has the form $T(\mathbf{L}_\lambda)$ for some λ, which completes the proof. □

We thus see that if M is a finitely generated **FI**-module then $T(M)$ is simple, and the simple constituents exactly record the asymptotic behavior of M_n for n large.

Exercises

Many of these exercises are stated for a general abelian category $\mathcal{A}$ and Serre subcategory $\mathcal{B}$. If you aren't comfortable in this generality, just take $\mathcal{A} = \mathbf{Mod}_{\mathbf{FI}}$ and $\mathcal{B} = \mathbf{Mod}_{\mathbf{FI}}^{\mathrm{tors}}$.

General properties of Serre quotients

Exercise 5.1 ($\star$)**.** Prove the claim in Example 5.3: if $T \colon \mathcal{A} \to \mathcal{C}$ is an exact functor of abelian categories then $\ker(T)$ is a Serre subcategory of $\mathcal{A}$. □

Exercise 5.2 ($\star$)**.** Prove Proposition 5.1. □

Exercise 5.3 ($\star$)**.** Let $\mathcal{B}$ be a localizing subcategory of $\mathcal{A}$ and let $N \in \mathcal{A}/\mathcal{B}$. Show that $S(N)$ is saturated. □

Exercise 5.4 ($\star\star$)**.** Let $\mathcal{B} \subset \mathcal{A}$ be a Serre subcategory, let M be an object of $\mathcal{A}$ and let $\overline{N}$ be a subobject of $T(M)$. Show that there is a subobject N of M such that $\overline{N} = T(N)$. □

Exercise 5.5 ($\star\star$)**.** Let $\mathcal{B} \subset \mathcal{A}$ be a localizing subcategory. Show that $M \in \mathcal{A}$ is saturated if and only if $\mathrm{Ext}^i_{\mathcal{A}}(T, M) = 0$ for all $T \in \mathcal{B}$ and all $i \in \{0, 1\}$. □

Exercise 5.6 ($\star\star$)**.** Let $\mathcal{B} \subset \mathcal{A}$ be a Serre subcategory. Show that there is a canonical exact sequence

$$\mathrm{K}(\mathcal{B}) \to \mathrm{K}(\mathcal{A}) \to \mathrm{K}(\mathcal{A}/\mathcal{B}) \to 0,$$

where $\mathrm{K}(-)$ denotes the Grothendieck group. □

Examples of Serre quotients

Exercise 5.7 ($\star\star$)**.** Let R be a commutative ring and let S be a multiplicative subset of R. Let $\mathcal{A} = \mathbf{Mod}_R$ be the category of R-modules and let $\mathcal{B}$ be the full subcategory spanned by modules M satisfying the following property: for every $x \in M$ there exists $s \in S$ such that $sx = 0$. Show that $\mathcal{B}$ is a localizing subcategory of $\mathcal{A}$ and that $\mathcal{A}/\mathcal{B}$ is equivalent to $\mathbf{Mod}_{S^{-1}R}$. □

Exercise 5.8 ($\star\star$)**.** Let R be a finitely generated graded $\mathbf{C}$-algebra with $R_0 = \mathbf{C}$ and $R_i = 0$ for $i < 0$. Let $\underline{\mathbf{Mod}}_R$ denote the category of graded R-modules, and let $\underline{\mathbf{Mod}}_{R,0}$ be the full subcategory spanned by modules M such that every element is annihilated by a power of the irrelevant ideal R_+. Show that $\underline{\mathbf{Mod}}_{R,0}$ is a localizing subcategory of $\underline{\mathbf{Mod}}_R$ and that the quotient category is canonically equivalent to $\mathrm{QCoh}(\mathrm{Proj}(R))$. Here $\mathrm{QCoh}(-)$ denotes the category of quasi-coherent sheaves on a scheme. □

Exercise 5.9 (⋆⋆)**.** Let X be a topological space. Let $\mathrm{PSh}(X)$ (resp. $\mathrm{Sh}(X)$) denote the category of presheaves (resp. sheaves) of abelian groups on X. Let $\mathrm{PSh}(X)_0$ be the full subcategory of $\mathrm{PSh}(X)$ spanned by presheaves for which all stalks vanish. Show that $\mathrm{PSh}(X)_0$ is a localizing subcategory of $\mathrm{PSh}(X)$ and that the quotient category is canonically equivalent to $\mathrm{Sh}(X)$. □

Exercise 5.10 (⋆⋆)**.** Let X be a topological space, let $Z \subset X$ be a closed subset, and let $U = X \setminus Z$ be the complement. Let $\mathrm{Sh}_Z(X)$ denote the full subcategory of $\mathrm{Sh}(X)$ spanned by sheaves with support contained in Z. Show that $\mathrm{Sh}_Z(X)$ is a localizing subcategory of $\mathrm{Sh}(X)$ and that the quotient category is canonically equivalent to $\mathrm{Sh}(U)$. □

Exercise 5.11 (⋆⋆)**.** Let X be an algebraic variety (or, more generally, a quasi-compact quasi-separated scheme), let Z be a closed subset, and let $U = X \setminus Z$. Let $\mathrm{QCoh}_Z(X)$ be the full subcategory of $\mathrm{QCoh}(X)$ spanned by sheaves whose support is (set-theoretically) contained in Z. Show that $\mathrm{QCoh}_Z(X)$ is a localizing subcategory of $\mathrm{QCoh}(X)$ and that the quotient category is canonically equivalent to $\mathrm{QCoh}(U)$. □

Exercise 5.12 (⋆⋆⋆)**.** Let $\mathcal{A}$ be an abelian category having enough injectives and let $\mathcal{C}$ be another abelian category. Consider the category $\mathrm{Fun}(\mathcal{A}, \mathcal{C})$ of all functors $\mathcal{A} \to \mathcal{C}$, which is itself an abelian category. Let $\mathrm{Lex}(\mathcal{A}, \mathcal{C})$ be the full subcategory spanned by the left-exact functors.

(a) Let $F\colon \mathcal{A} \to \mathcal{C}$ be a functor. Show that there exists a left-exact functor $T(F)$ and a natural transformation $F \to T(F)$ that is universal among maps from F to left-exact functors. Hint: let M be an object of $\mathcal{A}$, and choose an exact sequence

$$0 \to M \to I \to J$$

where I and J are injective objects. Define $T(F)(M)$ to be the kernel of the map $F(I) \to F(J)$.

(b) Show that $\mathrm{Lex}(\mathcal{A}, \mathcal{C})$ is an abelian category and that T is exact. Warning: $\mathrm{Lex}(\mathcal{A}, \mathcal{C})$ is abelian and a subcategory of $\mathrm{Fun}(\mathcal{A}, \mathcal{C})$, but is not (in general) an *abelian subcategory* of $\mathrm{Fun}(\mathcal{A}, \mathcal{C})$!

(c) Show that T induces an equivalence $\mathrm{Fun}(\mathcal{A}, \mathcal{C})/\ker(T) \to \mathrm{Lex}(\mathcal{A}, \mathcal{C})$. □

Injective objects and localization

Let $\mathcal{B}$ be a localizing subcategory of $\mathcal{A}$. We say that $\mathcal{B} \subset \mathcal{A}$ satisfies *property (Inj)* if every injective in $\mathcal{B}$ remains injective in $\mathcal{A}$.

Exercise 5.13 (⋆)**.** Let $I \in \mathcal{A}/\mathcal{B}$ be injective. Show that $S(I) \in \mathcal{A}$ is injective. □

Exercise 5.14 (⋆)**.** Show that $\mathcal{B}$ satisfies (Inj) if and only if for every $M \in \mathcal{B}$ there is an injection $M \to I$ where I belongs to $\mathcal{B}$ and is injective in $\mathcal{A}$. □

Exercise 5.15 (⋆⋆)**.** Give a direct proof (without using any hard results) that $\mathbf{Mod}_{\mathbf{FI}}^{\mathrm{tors}} \subset \mathbf{Mod}_{\mathbf{FI}}$ satisfies property (Inj). □

Exercise 5.16 (⋆⋆)**.** Suppose that $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{A}/\mathcal{B}$ have enough injectives and that $\mathcal{B} \subset \mathcal{A}$ satisfies (Inj). Let $I \in \mathcal{A}$ be injective. Show that $T(I)$ is injective in $\mathcal{A}/\mathcal{B}$ and that I decomposes as $I_1 \oplus I_2$ where I_1 belongs to $\mathcal{B}$ and I_2 is saturated. □

Exercise 5.17 (⋆⋆⋆)**.** Let R be a noetherian commutative ring and let S be a multiplicative. Let $\mathcal{B}$ be the subcategory of $\mathcal{A} = \mathbf{Mod}_R$ spanned by R-modules M such that $S^{-1}M = 0$. Show that $\mathcal{B} \subset \mathcal{A}$ satisfies (Inj). Conclude that if I is an injective R-module then $S^{-1}I$ is an injective $S^{-1}R$-module. □

Local cohomology and saturation

For the following exercises, we fix a Grothendieck abelian category $\mathcal{A}$ and a localizing subcategory $\mathcal{B}$ satisfying (Inj). The quotient category $\mathcal{A}/\mathcal{B}$ is then Grothendieck, and the quotient functor T is cocontinuous.

Exercise 5.18 (⋆⋆)**.** We say that $M \in \mathcal{A}$ is *derived saturated* if it is saturated and $\mathrm{R}^i\mathbf{S}(M) = 0$ for $i > 0$. Show that M is derived saturated if and only if $\mathrm{Ext}^i_{\mathcal{A}}(T, M) = 0$ for all $T \in \mathcal{B}$ and all $i \geq 0$. □

Exercise 5.19 (⋆⋆)**.** For $M \in \mathcal{A}$, let $\Gamma(M)$ be the maximal submodule of M that belongs to $\mathcal{B}$. This is a left-exact functor of M, and so we can consider its right-derived functors $\mathrm{R}^\bullet\Gamma$, which, as before, we refer to as *local cohomology*.

(a) Show that there is a canonical 4-term short exact sequence

$$0 \to \Gamma(M) \to M \to \mathbf{S}(M) \to \mathrm{R}^1\Gamma(M) \to 0.$$

Conclude that M is saturated if and only if $\mathrm{R}^i\Gamma(M) = 0$ for $i \in \{0, 1\}$.

(b) Show that there is a canonical isomorphism $\mathrm{R}^i\mathbf{S}(M) \cong \mathrm{R}^{i+1}\Gamma(M)$.

(c) More generally, if M is a bounded below complex in $\mathcal{A}$, show that there is a canonical distinguished triangle

$$\mathrm{R}\Gamma(M) \to M \to \mathrm{R}\mathbf{S}(M) \to$$

in the derived category of $\mathcal{A}$, and explain how to recover (a) and (b) from this. □

Generic FI-modules

Exercise 5.20 (⋆)**.** Show that projective **FI**-modules are saturated. □

Exercise 5.21 (⋆⋆)**.** Give an example of a **FI**-modules $N \subset M$ such that N and M are saturated but M/N is not saturated. □

Exercise 5.22 (⋆⋆)**.** Classify the injective objects in $\mathbf{Mod}_{\mathbf{FI}}^{\mathrm{gen}}$. □

Exercise 5.23 (⋆⋆)**.** Show that $\mathbf{Mod}_{\mathbf{FI}}^{\mathrm{gen}}$ has no non-zero projective objects. □

Exercise 5.24 (⋆⋆). Show that the definition of finite generation in $\mathbf{Mod}_{\mathbf{FI}}^{\mathrm{gen}}$ given in the lecture coincides with the intrinsic notion of finite generation in $\mathbf{Mod}_{\mathbf{FI}}^{\mathrm{gen}}$ as defined in Appendix A. □

Exercise 5.25 (⋆⋆⋆). Show that Spechtral **FI**-modules $\mathbf{L}_\lambda$ are saturated. (Hint: Exercise B.4 may be helpful.) □

Exercise 5.26 (⋆⋆⋆). Let M be a finitely generated torsion-free **FI**-module. Show that M admits a finite length filtration such that each graded piece is a submodule of a Spechtral module. □

The infinite symmetric group

Let $\mathfrak{S}_\infty = \bigcup_{n\geq 1} \mathfrak{S}_n$. Define $\mathfrak{S}_{\infty-n}$ to be the subgroup of $\mathfrak{S}$ fixing each of the numbers $1, \ldots, n$. We say that a representation of $\mathfrak{S}_\infty$ is *smooth* if every element x is fixed by $\mathfrak{S}_{\infty-n}$ for some n (depending on x), and we write $\mathbf{Rep}^{\mathrm{sm}}(\mathfrak{S}_\infty)$ for the category of smooth representations. The basic example of a smooth representation is $\mathbf{C}^\infty = \bigcup_{n\geq 1} \mathbf{C}^n$ with $\mathfrak{S}_\infty$ acting by permuting the basis vectors.

Exercise 5.27 (⋆). Show that the tensor product of two smooth representations is again smooth. □

Exercise 5.28 (⋆⋆). Show that every smooth representation is a quotient of a direct sum of tensor powers of $\mathbf{C}^\infty$. □

Exercise 5.29 (⋆⋆⋆). Let M be an **FI**-module. Define

$$T(M) = \varinjlim_{n\to\infty} M_n,$$

where the direct limit is formed with respect to the standard transition maps (i.e., those coming from the standard inclusion $i_n \colon [n] \to [n+1]$).

(a) Show that $T(M)$ naturally carries a smooth representation of $\mathfrak{S}_\infty$.

(b) Show that T defines an exact functor $T \colon \mathbf{Mod}_{\mathbf{FI}} \to \mathbf{Rep}^{\mathrm{sm}}(\mathfrak{S}_\infty)$.

(c) Describe the right-adjoint to T.

(d) Show that T induces an equivalence $\mathbf{Mod}_{\mathbf{FI}}^{\mathrm{gen}} \to \mathbf{Rep}^{\mathrm{sm}}(\mathfrak{S}_\infty)$. □

We let $\mathfrak{M}_{\lambda[\infty]} = T(\mathbf{L}_\lambda)$. By the previous exercise, and our work on $\mathbf{Mod}_{\mathbf{FI}}^{\mathrm{gen}}$, these are exactly the simple smooth $\mathfrak{S}_\infty$-representations.

Define the *specialization functor*

$$\Gamma_n \colon \mathbf{Rep}^{\mathrm{sm}}(\mathfrak{S}_\infty) \to \mathbf{Rep}(\mathfrak{S}_n), \qquad V \mapsto V^{\mathfrak{S}_{\infty-n}}.$$

This is a left-exact functor, and so we can consider its right-derived functors $\mathrm{R}^i\Gamma_n$.

Exercise 5.30 (⋆⋆). Let λ and n be given.

(a) Show that $\Gamma_n(\mathfrak{M}_{\lambda[\infty]}) = \mathfrak{M}_{\lambda[n]}$, recalling our convention that $\mathfrak{M}_{\lambda[n]} = 0$ if $\lambda[n]$ is not a partition. (Exercise 5.25 may be helpful!)

(b) Show that there exists N such that $\mathrm{R}^i\Gamma_n(\mathfrak{M}_{\lambda[\infty]}) = 0$ for $i > 0$ and $n > N$. □

Exercise 5.31 (⋆⋆⋆)**.** Show that Γ_n is a tensor functor, that is, if V and W are smooth $\mathfrak{S}_\infty$-representations then the natural map $\Gamma_n(V) \otimes \Gamma_n(W) \to \Gamma_n(V \otimes W)$ is an isomorphism. (This should be surprising—it is very unusual for the invariants in a tensor product to be the tensor product of invariants!) □

Lecture 6

Schur–Weyl duality

Schur–Weyl duality provides a link between the representation theory of symmetric groups and general linear groups. The theory shows that **FI**-modules can be dramatically reinterpreted as $\mathbf{GL}_\infty$-equivariant modules over the infinite variable polynomial ring $\mathbf{C}[x_1, x_2, \ldots]$. This is a very useful perspective, as it allows us to bring in tools from algebraic geometry. This lecture gives an introduction to this circle of ideas.

References: for generalities of Schur–Weyl duality and tca's, see [SS2]; for the specific applications to **FI**-modules, see [SS1].

Polynomial representations

Schur functors

For a vector space V, one has the well-known decomposition

$$V^{\otimes 2} = \mathrm{Sym}^2(V) \oplus \textstyle\bigwedge^2(V).$$

Here $\mathrm{Sym}^2(V)$ is spanned by the symmetric tensors in $V^{\otimes 2}$, i.e., those of the form $v \otimes w + w \otimes v$, while $\bigwedge^2(V)$ is spanned by the skew-symmetric tensors, i.e., those of the form $v \otimes w - w \otimes v$. It is clear that $\mathrm{Sym}^2(V)$ and $\bigwedge^2(V)$ are $\mathbf{GL}(V)$-subrepresentations of $V^{\otimes 2}$, and so this is a decomposition of $\mathbf{GL}(V)$-representations.

This decomposition naturally generalizes to higher tensor powers, as follows. The symmetric group $\mathfrak{S}_n$ acts on $V^{\otimes n}$ by permuting the tensor factors, and this action commutes with that of $\mathbf{GL}(V)$. In other words, $V^{\otimes n}$ is naturally a representation of $\mathfrak{S}_n \times \mathbf{GL}(V)$. By basic representation theory, we therefore get a decomposition

$$V^{\otimes n} = \bigoplus_{|\lambda|=n} \mathfrak{M}_\lambda \otimes \mathbf{S}_\lambda(V)$$

where $\mathbf{S}_\lambda(V)$ is the multiplicity space of $\mathfrak{M}_\lambda$ in $V^{\otimes n}$, and a representation of $\mathbf{GL}(V)$. Explicitly, we have

$$\mathbf{S}_\lambda(V) = \mathrm{Hom}_{\mathfrak{S}_n}(\mathfrak{M}_\lambda, V^{\otimes n}).$$

Note that we also have

$$\mathbf{S}_\lambda(V) = (\mathfrak{M}_\lambda \otimes_{\mathbf{C}} V^{\otimes n})^{\mathfrak{S}_n},$$

since representations of $\mathfrak{S}_n$ are self-dual. The representation $\mathbf{S}_\lambda(V)$ is completely understood:

Proposition 6.1. *Let $n \geq 0$ be an integer and λ be a partition with $r = \ell(\lambda)$ non-zero parts. If $n \geq r$ then $\mathbf{S}_\lambda(\mathbf{C}^n)$ is the irreducible representation of $\mathbf{GL}_n$ with highest weight $(\lambda_1, \ldots, \lambda_n)$. If $n < r$ then $\mathbf{S}_\lambda(V) = 0$.*

The construction $\mathbf{S}_\lambda(V)$ is functorial in V: that is, if $V \to W$ is a linear map, there is a canonical induced map $\mathbf{S}_\lambda(V) \to \mathbf{S}_\lambda(W)$. Thus $\mathbf{S}_\lambda$ defines a functor $\mathbf{Vec} \to \mathbf{Vec}$. These functors are called *Schur functors*, and are extremely important in representation theory.

Polynomial representations of $\mathbf{GL}_\infty$

Let $\mathbf{GL}_\infty = \bigcup_{n \geq 1} \mathbf{GL}_n$. This group naturally acts on $\mathbf{C}^\infty = \bigcup_{n \geq 1} \mathbf{C}^n$, which we refer to as the *standard representation*. We say that a representation of $\mathbf{GL}_\infty$ is *polynomial* if it can be realized as a subquotient of a (possibly infinite) direct sum of tensor powers of the standard representation. We let $\mathbf{Rep}^{\mathrm{pol}}(\mathbf{GL}_\infty)$ denote the category of polynomial representations. It is easy to see that it is an abelian category and closed under arbitrary direct sums and finite tensor products.

As an abelian category, the structure of $\mathbf{Rep}^{\mathrm{pol}}(\mathbf{GL}_\infty)$ is fairly simple:

Proposition 6.2. *The category $\mathbf{Rep}^{\mathrm{pol}}(\mathbf{GL}_\infty)$ is semi-simple. The simple objects are the representations $\mathbf{S}_\lambda(\mathbf{C}^\infty)$, as λ varies over all partitions.*

Proof. It's easy to see from Proposition 6.1 that $\mathbf{S}_\lambda(\mathbf{C}^\infty)$ is irreducible. The decomposition

$$(\mathbf{C}^\infty)^{\otimes n} = \bigoplus_{|\lambda|=n} \mathfrak{M}_\lambda \otimes \mathbf{S}_\lambda(\mathbf{C}^\infty)$$

thus shows that $(\mathbf{C}^\infty)^{\otimes n}$ is semi-simple, and that its irreducible constituents are the $\mathbf{S}_\lambda(\mathbf{C}^\infty)$ with $|\lambda| = n$. Since any direct sum or subquotient of semi-simple objects is semi-simple, it follows that $\mathbf{Rep}^{\mathrm{pol}}(\mathbf{GL}_\infty)$ is semi-simple. Moreover, any simple object must occur as a constituent of some $(\mathbf{C}^\infty)^{\otimes n}$, and is therefore one of the $\mathbf{S}_\lambda(\mathbf{C}^\infty)$. □

Polynomial functors

A *polynomial functor* $\mathbf{Vec} \to \mathbf{Vec}$ is a functor that is (isomorphic to) a (perhaps infinite) direct sum of $\mathbf{S}_\lambda$'s. We write **Pol** for the category of polynomial functors. It is not difficult to see that this is a semi-simple abelian category and the $\mathbf{S}_\lambda$'s are the simple objects. It follows that the functor

$$\mathbf{Pol} \to \mathbf{Rep}^{\mathrm{pol}}(\mathbf{GL}_\infty), \qquad F \mapsto F(\mathbf{C}^\infty)$$

is an equivalence of categories. Thus polynomial functors are just a different way of viewing the category $\mathbf{Rep}^{\mathrm{pol}}(\mathbf{GL}_\infty)$. The advantage of this perspective is that one can evaluate a polynomial functor on $\mathbf{C}^n$ for any n, which affords a great deal of flexibility. The disadvantage is that functors are a bit more abstract and complicated than representations.

Schur–Weyl duality

Schur–Weyl duality is a relationship between the representation theory of symmetric groups and general linear groups. In our formulation, it takes the form of an equivalence of categories between $\mathbf{Rep}(\mathfrak{S}_*)$ and $\mathbf{Rep}^{\mathrm{pol}}(\mathbf{GL}_\infty)$. There are many interesting aspects of this equivalence to understand, though we only discuss a few.

The functor Φ

Consider the functor

$$\Phi\colon \mathbf{Rep}(\mathfrak{S}_*) \to \mathbf{Rep}^{\mathrm{pol}}(\mathbf{GL}_\infty), \qquad V_* \mapsto \bigoplus_{n \ge 0} (V_n \otimes_{\mathbf{C}} (\mathbf{C}^\infty)^{\otimes n})^{\mathfrak{S}_n}.$$

Essentially by definition, Φ takes the simple object $\mathbf{M}_\lambda$ of $\mathbf{Rep}(\mathfrak{S}_*)$ to the simple object $\mathbf{S}_\lambda(\mathbf{C}^\infty)$ of $\mathbf{Rep}^{\mathrm{pol}}(\mathbf{GL}_\infty)$. We thus see that Φ is a bijection on (isomorphism classes of) simple objects, and is therefore an equivalence since each side is semi-simple.

The quasi-inverse Ψ

We now recall some terminology. Let $T \subset \mathbf{GL}_\infty$ be the set of diagonal matrices, the standard maximal torus. For a sequence $\lambda = (\lambda_1, \lambda_2, \ldots)$ of integers which is eventually 0 (a *weight*), we let $\chi_\lambda\colon T \to \mathbf{C}^\times$ be the map that takes the diagonal matrix with entries $(\alpha_1, \alpha_2, \ldots)$ to $\prod_{i \ge 1} \alpha_i^{\lambda_i}$. If V is a representation of $\mathbf{GL}_\infty$ and $v \in V$, we see that v is a *weight vector* of weight λ is $tv = \chi_\lambda(t)v$ for all $t \in T$. We let V_λ be the set of all weight vectors of weight λ, which is called the *λ weight space*. If V is a polynomial representation then V is the direct sum of its weight spaces, over all weights, and no weight using a negative number appears.

Let 1^n be the weight $(1, \ldots, 1, 0, 0, \ldots)$, where there are n 1's. The homomorphism χ_{1^n} is normalized by $\mathfrak{S}_n \subset \mathbf{GL}_\infty$, that is, if $\sigma \in \mathfrak{S}_n$ and $t \in T$ then $\chi_{1^n}(\sigma t \sigma^{-1}) = \chi_{1^n}(t)$. It follows that if v is a 1^n-weight vector then so is σv. In other words, V_{1^n} is a representation of $\mathfrak{S}_n$.

We now define a functor

$$\Psi\colon \mathbf{Rep}^{\mathrm{pol}}(\mathbf{GL}_\infty) \to \mathbf{Rep}(\mathfrak{S}_*), \qquad V \mapsto (V_{1^n})_{n \ge 0}.$$

This is a quasi-inverse to Φ (Exercise 6.2).

Applications to tca's

Recall that a *twisted commutative algebra* (tca) is a commutative algebra object in the tensor category $(\mathbf{Rep}(\mathfrak{S}_*), \circledast)$. It turns out that the equivalences Φ and Ψ are compatible with tensor products (Exercise 6.3). It follows that tca's can equivalently be thought of as commutative algebra objects in $\mathbf{Rep}^{\mathrm{pol}}(\mathbf{GL}_\infty)$. An algebra A in this category is simply a commutative $\mathbf{C}$-algebra in the usual sense equipped with an action of $\mathbf{GL}_\infty$ (by algebra homomorphisms) such that A forms a polynomial representation of $\mathbf{GL}_\infty$.

FI-modules from the GL point of view

The equivalence

Let $\mathbf{A}$ be the tca $\mathbf{C}[t]$, where t has degree 1 and all symmetric group actions are trivial. Recall that $\mathbf{A}$-modules are equivalent to $\mathbf{FI}$-modules (Exercise 2.8). Let $\mathbf{B} = \Phi(\mathbf{A})$. Since the trivial representation of $\mathfrak{S}_n$ corresponds to $\mathrm{Sym}^n(\mathbf{C}^\infty)$ under Schur–Weyl duality, we see that $\mathbf{B} = \bigoplus_{n \geq 0} \mathrm{Sym}^n(\mathbf{C}^\infty)$. This suggests that $\mathbf{B} = \mathrm{Sym}(\mathbf{C}^\infty)$, the symmetric algebra on $\mathbf{C}^\infty$. In fact, this is the case: $\mathbf{A}$ is the symmetric algebra on $\mathbf{M}_1$ in $\mathbf{Rep}(\mathfrak{S}_*)$, and so (since Φ is compatible with tensor products), $\Phi(\mathbf{A})$ is the symmetric algebra on $\Phi(\mathbf{M}_1) = \mathbf{C}^\infty$. We thus see that $\mathbf{A}$-modules (and thus $\mathbf{FI}$-modules) are equivalent to $\mathbf{B}$-modules.

We now unpack the above discussion. The algebra $\mathbf{B}$ introduced above is simply the infinite variable polynomial ring $\mathbf{C}[x_1, x_2, \ldots]$, with $\mathbf{GL}_\infty$ acting by linear substitutions. By a $\mathbf{B}$-module, we always mean a $\mathbf{B}$-module in the category $\mathbf{Rep}^{\mathrm{pol}}(\mathbf{GL}_\infty)$. Thus, a $\mathbf{B}$-module is a $\mathbf{GL}_\infty$-equivariant module M over the ring $\mathbf{C}[x_1, x_2, \ldots]$ such that M is a polynomial representation of $\mathbf{GL}_\infty$.

The equivalence $\mathbf{Mod}_{\mathbf{B}} \to \mathbf{Mod}_{\mathbf{FI}}$ can be realized directly, as follows. Let M be a $\mathbf{B}$-module, and let N be the corresponding $\mathbf{FI}$-module. Then $N_n = M_{1^n}$, the 1^n weight space of M. The transition map $N_n \to N_{n+1}$ corresponds is induced by mutliplication by x_{n+1} on M. Note that x_{n+1} has weight $(0, \ldots, 0, 1, 0, 0, \ldots)$, where the 1 is in the $n+1$ position, and so if $m \in M$ has weight 1^n then $x_{n+1}m$ has weight 1^{n+1}.

Simple, torsion, and projective modules

It is easy to see what the simple $\mathbf{FI}$-modules correspond to on the $\mathbf{B}$-module side. The object $\mathbf{M}_\lambda$ of $\mathbf{Rep}(\mathfrak{S}_*)$ corresponds to $\mathbf{S}_\lambda(\mathbf{C}^\infty)$ of $\mathbf{Rep}^{\mathrm{pol}}(\mathbf{GL}_\infty)$. When we regard $\mathbf{M}_\lambda$ as an $\mathbf{FI}$-module, we take all transition maps to be zero. Thus, when we regard $\mathbf{S}_\lambda(\mathbf{C}^\infty)$ as a $\mathbf{B}$-module, all variables act by 0. In other words, we simply regard $\mathbf{S}_\lambda(\mathbf{C}^\infty)$ as a module over $\mathbf{B}/\mathbf{B}_+ = \mathbf{C}$, where $\mathbf{B}_+$ is the ideal $(x_1, x_2, \ldots)$.

It follows from the above discussion that a finite length $\mathbf{B}$-module is annihilated by $(\mathbf{B}_+)^n$ for some n. We thus see that torsion $\mathbf{FI}$-modules exactly correspond to $\mathbf{B}$-modules in which every element is annihilated by some power of $\mathbf{B}_+$; we refer to these as torsion $\mathbf{B}$-modules.

It is also easy to see what the projective $\mathbf{FI}$-modules correspond to. Recall (from Exercise 2.20) that the $\mathbf{FI}$-module $\mathbf{P}_\lambda$ corresponds to the $\mathbf{A}$-module $\mathbf{A} \circledast \mathbf{M}_\lambda$. It follows that this corresponds to the $\mathbf{B}$-module $\Phi(\mathbf{A} \circledast \mathbf{M}_\lambda) = \mathbf{B} \otimes \mathbf{S}_\lambda(\mathbf{C}^\infty)$.

The generic category revisited

The $\mathbf{B}$-module perspective on $\mathbf{FI}$-modules is extremely powerful since it allows us to bring in tools from algebraic geometry. We now illustrate this by describing the generic category.

Phase one

We can regard a $\mathbf{B}$-module as a $\mathbf{GL}_\infty$-equivariant quasi-coherent sheaf on the "variety" $\mathrm{Spec}(\mathbf{C}[x_1, x_2, \dots]) = \mathbf{A}^\infty$ (with a polynomiality condition). Torsion $\mathbf{B}$-modules correspond to those supported (set-theoretically) at the point 0. Thus the generic category $\mathbf{Mod}_\mathbf{B}^{\mathrm{gen}} = \mathbf{Mod}_\mathbf{B} / \mathbf{Mod}_\mathbf{B}^{\mathrm{tors}}$ can be interpreted as equivariant sheaves on the complement $\mathbf{A}^\infty \setminus \{0\}$ (Exercise 5.11). (That exercise doesn't actually apply here since $\mathbf{A}^\infty \setminus \{0\}$ is too big, but it serves as a sense of intuition, which can actually be made to work.)

To understand $\mathbf{GL}_\infty$-equivariant sheaves on $\mathbf{A}^\infty \setminus \{0\}$, we should first undestand the finite variable case, namely $\mathbf{GL}_n$-equivariant sheaves on $\mathbf{A}^n \setminus \{0\}$. For this we appeal to the following general theorem. Suppose that an algebraic group G acts transitively on a variety X. Then the category of G-equivariant quasi-coherent sheaves on X is equivalent to the category of representations of the stabilizer G_x, for any $x \in X$. The equivalence takes a sheaf to its fiber at x. We thus see that $\mathbf{GL}_n$-equivariant quasi-coherent sheaves on $\mathbf{A}^n \setminus \{0\}$ correspond to representations of the general affine group $\mathbf{GA}_n$, which we define as the stabilizer of the vector $e_1^* = (1, 0, \dots, 0) \in \mathbf{A}^n$. Note that $\mathbf{A}^n$ here is really $\mathrm{Spec}(\mathrm{Sym}(\mathbf{C}^n)) = (\mathbf{C}^n)^*$, and so $\mathbf{GL}_n$ is acting through the dual representation; thus $\mathbf{GA}_n$ is the subgroup of $\mathbf{GL}_n$ consisting of matrices with first row $(1, 0, 0, \dots)$.

Going back to the infinite variable case, this suggests that $\mathbf{Mod}_\mathbf{B}^{\mathrm{gen}}$ should be equivalent to the category of representations of $\mathbf{GA}_\infty$. This is almost correct: we just have to impose the polynomiality constraint. We define a *polynomial representation* of $\mathbf{GA}_\infty$ to be one that occurs as a subquotient of a direct sum of tensor powers of the standard representation $\mathbf{C}^\infty$. The correct statement is then:

Theorem 6.3. *We have an equivalence of categories* $\mathbf{Mod}_\mathbf{B}^{\mathrm{gen}} \cong \mathbf{Rep}^{\mathrm{pol}}(\mathbf{GA}_\infty)$.

Phase two

[finish describing equivalence between $\mathbf{Rep}^{\mathrm{pol}}(\mathbf{GA}_\infty)$ and $\mathbf{Mod}_\mathbf{B}^{\mathrm{tors}}$]

Exercises

Polynomial functors and Schur–Weyl duality

Exercise 6.1 (⋆)**.** Show that the space $\mathbf{S}_{(2,1)}(V)$ is identified with the kernel of the multiplication map $\mathrm{Sym}^2(V) \otimes V \to \mathrm{Sym}^3(V)$. □

Exercise 6.2 (⋆⋆)**.** Show that Φ and Ψ are quasi-inverse to each other (i.e., the composition in either direction is isomorphic to the identity functor). □

Exercise 6.3 (⋆⋆)**.** Show that Φ are compatible with tensor products. That is, for $\mathfrak{S}_*$-representations V and W, construct a canonical isomorphism

$$\Phi(V \circledast W) \cong \Phi(V) \otimes \Phi(W)$$ □

Exercise 6.4 (⋆⋆)**.** In Lecture 4, we defined the shift operator on **FI**-modules. One can also shift **FB**-modules or $\mathfrak{S}_*$-representations in the same manner. Determine what the shift operator corresponds to on $\mathbf{Rep}^{\mathrm{pol}}(\mathbf{GL}_\infty)$ under Schur–Weyl duality. □

Exercise 6.5 (⋆⋆)**.** Let V be a finite dimensional vector space. Show that under Schur–Weyl duality, the $\mathfrak{S}_n$-representation $V^{\otimes n}$ corresponds to the $\mathbf{GL}_\infty$-representation $\mathrm{Sym}^n(V \otimes \mathbf{C}^\infty)$. □

Exercise 6.6 (⋆⋆)**.** Let $\mathcal{M}$ be the set of perfect matchings on the vertex set $[2n]$. (Recall that a perfect matching is an undirected graph such that each vertex belongs to exactly one edge.) Show that under Schur–Weyl duality, the $\mathfrak{S}_{2n}$-representation $\mathbf{C}[\mathcal{M}]$ corresponds to the $\mathbf{GL}_\infty$-representation $\mathrm{Sym}^n(\mathrm{Sym}^2(\mathbf{C}^\infty))$. □

Exercise 6.7 (⋆⋆)**.** Decompose $\mathbf{S}_\lambda(\mathbf{C}^\infty) \otimes \mathrm{Sym}^k(\mathbf{C}^\infty)$ into irreducible $\mathbf{GL}_\infty$-representations, for arbitrary λ and k. □

Exercise 6.8 (⋆⋆⋆)**.** Show that the composition of two polynomial functors is again a polynomial functor. Via Schur–Weyl duality, this operation can be transferred to one on $\mathbf{Rep}(\mathfrak{S}_*)$. Describe this operation directly. □

Exercise 6.9 (⋆⋆⋆)**.** Let V and W be polynomial representations. Show that $\ell(V \otimes W) \le \ell(V) + \ell(W)$. In particular, if V and W are bounded then so is $V \otimes W$. (This exercise is easy if one "cheats" and uses the Littlewood–Richardson rule. However, it can be done using only what we have covered!) □

B-modules

Exercise 6.10 (⋆⋆)**.** Describe the **B**-module corresponding to $\mathbf{L}_1$ as best you can. □

Exercise 6.11 (⋆⋆)**.** Show that the equivalence $\mathbf{Mod}_{\mathbf{FI}}^{\mathrm{gen}} \to \mathbf{Mod}_{\mathbf{FI}}^{\mathrm{tors}}$ takes $T(\mathbf{P}_\lambda)$ to $\mathbf{I}_\lambda$ and $T(\mathbf{L}_\lambda)$ to $\mathbf{M}_\lambda$. □

Exercise 6.12 (⋆⋆)**.** Let M be a bounded polynomial functor, and let $n > \ell(M)$. Show that the map

$$\{\text{subobjects of } M\} \to \{\text{subspaces of } M(\mathbf{C}^n)\}, \qquad N \mapsto N(\mathbf{C}^n)$$

is injective. Using this observation (and Exercise 6.9), show that if A is a finitely generated and bounded tca then A is noetherian. In particular, this gives a direct proof of the noetherian property for **B**, which translates to a new proof of the noetherian property for **FI**. (This argument comes from [Sn].) □

Exercise 6.13 (⋆⋆)**.** Show that the tensor product $\otimes_{\mathbf{B}}$ on $\mathbf{Mod}_{\mathbf{B}}$ induces a well-defined exact tensor product on $\mathbf{Mod}_{\mathbf{B}}^{\mathrm{gen}}$. (This is a kind of "generic flatness" for **B**-modules.) □

Exercise 6.14 (⋆⋆⋆)**.** Using the geometric point of view, relate $(\mathrm{R}^i\Gamma)(\mathbf{L}_\lambda)$ to the cohomology of a $\mathbf{GL}_n$-equivariant sheaf on $\mathbf{P}^{n-1}$. Then compute this cohomology group explicitly using the Borel–Weil–Bott Theorem.

This exercise takes a lot of work to do carefully, so at least initially you should not worry too much about being rigorous. The details are worked out completely (in the $\mathbf{FI}_d$ case) in [SS5, §5.5], which culminates in [SS5, Corollary 5.20]. □

Another model for the generic category

Let $\mathbf{K} = \mathrm{Frac}(\mathbf{B}) = \mathbf{C}(x_1, x_2, \dots)$ be the fraction field of $\mathbf{B}$. A *semi-linear representation* of $\mathbf{GL}_\infty$ over $\mathbf{K}$ is a $\mathbf{K}$-vector space V equipped with a $\mathbf{C}$-action of $\mathbf{GL}_\infty$ such that $g(av) = (ga)(gv)$ for $g \in \mathbf{GL}_\infty$, $a \in \mathbf{K}$, and $v \in V$. We say that an element in a semi-linear representation V if *polynomial* if the $\mathbf{C}$-linear subrepresentation it generates is a polynomial representation, and we say that V is *polynomial* if it is generated (as a semi-linear representation) by a family of polynomial elements. Let $\mathbf{Rep}^{\mathrm{pol}}_{\mathbf{K}}(\mathbf{GL}_\infty)$ be the category of polynomial semi-linear representations.

Exercise 6.15 (⋆⋆⋆)**.** Do the following:

(a) Show that $M \mapsto \mathbf{K} \otimes_{\mathbf{B}} M$ defines a functor $T\colon \mathbf{Mod}_{\mathbf{B}} \to \mathbf{Rep}^{\mathrm{pol}}_{\mathbf{K}}(\mathbf{GL}_\infty)$.

(b) Describe the right adjoint to the functor T.

(c) Show that T induces an equivalence $\mathbf{Mod}^{\mathrm{gen}}_{\mathbf{B}} \cong \mathbf{Rep}^{\mathrm{pol}}_{\mathbf{K}}(\mathbf{GL}_\infty)$. □

Exercise 6.16 (⋆⋆)**.** Using the description of the generic category and section functor from the previous exercise, give a direct proof that projective modules are saturated. □

Exercise 6.17 (⋆⋆⋆)**.** Consider the following tensor categories:

(a) $\mathbf{Rep}^{\mathrm{sm}}(\mathfrak{S}_\infty)$, with the usual tensor product.

(b) $\mathbf{Rep}^{\mathrm{pol}}(\mathbf{GA}_\infty)$, with the usual tensor product.

(c) $\mathbf{Rep}^{\mathrm{pol}}_{\mathbf{K}}(\mathbf{GL}_\infty)$ with the usual tensor product of semi-linear representations (define this!).

(d) $\mathbf{Mod}^{\mathrm{gen}}_{\mathbf{B}}$, with the tensor product induced by $\otimes_{\mathbf{B}}$ (Exercise 6.13).

(e) $\mathbf{Mod}^{\mathrm{tors}}_{\mathbf{B}}$ with the tensor product $\otimes_{\mathbf{B}}$.

(f) $\mathbf{Mod}^{\mathrm{tors}}_{\mathbf{B}}$, where the tensor product of M and N is defined as follows: first, form $M \otimes_{\mathbf{C}} N$ which is a module over $\mathbf{B} \otimes_{\mathbf{C}} \mathbf{B}$; then, restrict along the comultiplication map $\mathbf{B} \to \mathbf{B} \otimes_{\mathbf{C}} \mathbf{B}$ (this is the unique ring homomorphism that takes a basis vector x of $\mathbf{B}_1 = \mathbf{C}^\infty$ to $x \otimes 1 + 1 \otimes x$).

(g) $\mathbf{Mod}^{\mathrm{tors}}_{\mathbf{FI}}$ with the pointwise tensor product $\boxtimes$.

(h) $\mathbf{Mod}^{\mathrm{gen}}_{\mathbf{FI}}$ with the tensor product induced by $\boxtimes$.

We have shown that all the abelian categories occurring above are equivalent. Determine how the various tensor products correspond under the equivalences. □

Schur functors in tensor categories

Exercise 6.18 (⋆)**.** Let $\mathcal{A}$ be a $\mathbf{C}$-linear abelian category (meaning all Hom sets as $\mathbf{C}$-vector spaces) equipped with a symmetric tensor product $\otimes$. Explain how to define $\mathbf{S}_\lambda(M)$ for $M \in \mathcal{A}$. □

Exercise 6.19 (⋆⋆)**.** Let $\mathbf{Rep}(\mathfrak{S}_*)^{\mathrm{f}}$ be the category of finite length objects in $\mathbf{Rep}(\mathfrak{S}_*)$. Let $\mathcal{A}$ be a $\mathbf{C}$-linear symmetric tensor category, and let $T(\mathcal{A})$ be the category of additive symmetric tensor functors $\mathbf{Rep}(\mathfrak{S}_*)^{\mathrm{f}} \to \mathcal{A}$. Show that the functor

$$T(\mathcal{A}) \to \mathcal{A}, \qquad F \mapsto F(\mathbf{M}_1)$$

is an equivalence of categories. □

Remark 6.4. The above exercise says that $\mathbf{Rep}(\mathfrak{S}_*)^{\mathrm{f}}$ is the universal $\mathbf{C}$-linear symmetric tensor category, in the sense that giving an additive symmetric tensor functor $\mathbf{Rep}(\mathfrak{S}_*)^{\mathrm{f}} \to \mathcal{A}$ is the same as giving an object of $\mathcal{A}$. This is analogous to the fact that $\mathbf{C}[x]$ is the universal $\mathbf{C}$-algebra, in the sense that giving a $\mathbf{C}$-algebra homomorphism $\mathbf{C}[x] \to A$ is the same as giving an element of A. □

Positive characteristic

Much of our discussion of Schur–Weyl duality breaks down in positive characteristic. The following exercises elucidate the situation, to some degree.

Let $\mathbf{k} = \overline{\mathbf{F}}_p$, or any algebraically closed field of characteristic p. We consider $\mathbf{GL}_\infty$ over $\mathbf{k}$. Of course, $\mathbf{k}^\infty$ is still a representation, which we call the *standard representation*. We say that a representation is *polynomial* if it appears as a subquotient of a (possibly infinite) direct sum of tensor powers of the standard representation. Write $\mathbf{Rep}^{\mathrm{pol}}_{\mathbf{k}}(\mathbf{GL}_\infty)$ for this category. Also, write $\mathbf{Rep}_{\mathbf{k}}(\mathfrak{S}_*)$ for the representation categoy of $\mathfrak{S}_*$ over $\mathbf{k}$.

Exercise 6.20 (⋆)**.** Show that $\mathbf{Rep}^{\mathrm{pol}}_{\mathbf{k}}(\mathbf{GL}_\infty)$ is an abelian category, and that it is closed under arbitrary direct sums and finite tensor products. □

Exercise 6.21 (⋆)**.** Show that $\mathbf{Rep}^{\mathrm{pol}}_{\mathbf{k}}(\mathbf{GL}_\infty)$ is not semi-simple. (Hint: pth powers are always a good thing to look at in characteristic p!) This already shows that the situation is very different from characteristic 0. □

Exercise 6.22 (⋆⋆⋆)**.** Define

$$T\colon \mathbf{Rep}^{\mathrm{pol}}_{\mathbf{k}}(\mathbf{GL}_\infty) \to \mathbf{Rep}_{\mathbf{k}}(\mathfrak{S}_*) \qquad T(V)_n = \text{the } 1^n \text{ weight space in } V.$$

Show that T induces an equivalence

$$\mathbf{Rep}^{\mathrm{pol}}_{\mathbf{k}}(\mathbf{GL}_\infty)/\ker(T) \to \mathbf{Rep}_{\mathbf{k}}(\mathfrak{S}_*).$$

Give an explicit example of a non-zero object in $\ker(T)$. Thus T itself is not an equivalence, contrary to characteristic 0. □

Exercise 6.23 (⋆⋆)**.** Show that $\mathbf{Rep}^{\mathrm{pol}}_{\mathbf{k}}(\mathbf{GL}_\infty)$ and $\mathbf{Rep}_{\mathbf{k}}(\mathfrak{S}_*)$ are not abstractly equivalent. □

Exercise 6.24 (⋆⋆)**.** Let $B = \mathrm{Sym}(\mathbf{k}^\infty)$, regarded as an algebra in $\mathbf{Rep}^{\mathrm{pol}}_{\mathbf{k}}(\mathbf{GL}_\infty)$. Show that the category of B-modules in this category is not equivalent to the category of **FI**-modules over $\mathbf{k}$. □

Lecture 7

The spectrum of a tca

Motivation

Let R be a commutative ring. Recall that the *spectrum* of R, denoted $\mathrm{Spec}(R)$, is the set of prime ideals of R. For an ideal I of R, let $V(I) \subset \mathrm{Spec}(R)$ be the set of prime ideals $\mathfrak{p}$ that contain I. There is a natural topology on $\mathrm{Spec}(R)$, called the *Zariski topology*, in which the closed sets are exactly the $V(I)$'s.

Let M be an R-module. We define the *support* of M, denoted $\mathrm{Supp}(M)$, to be the subset of $\mathrm{Spec}(R)$ consisting of those primes ideals $\mathfrak{p}$ such that $M_{\mathfrak{p}} \neq 0$. If M is finitely generated, then $\mathrm{Supp}(M) = V(\mathrm{Ann}\, M)$, where $\mathrm{Ann}(M)$ denotes the annihilator of M; in particular, in this case, $\mathrm{Supp}(M)$ is a closed set. The support of a module is a very helpful invariant in understanding what a module looks like. For example, the Krull dimension of a module, which is defined to be the Krull dimension of its support, is a useful measure of how large a module is.

This suggests that to understand the module theory of more general tca's, we should try to define a notion of spectrum and support for them. This is the purpose of this lecture.

The main definitions

Primes in tca's: part 1

Recall that we have an equivalence of categories $\mathbf{Rep}(\mathfrak{S}_*) \cong \mathbf{Rep}^{\mathrm{pol}}(\mathbf{GL}_\infty)$ by Schur–Weyl duality. Commutative algebras in $\mathbf{Rep}(\mathfrak{S}_*)$ are called twisted commutative algebras (tca's), while commutative algebras in $\mathbf{Rep}^{\mathrm{pol}}(\mathbf{GL}_\infty)$ are called *GL-algebras*. We would like to define a notion of prime ideal in tca's and **GL**-algebras. For this, it suffices to define a notion of domain, since an ideal should be prime if and only if the quotient is a domain.

Let A be a tca and let B be the corresponding **GL**-algebra. There are two reasonable candidate definitions for domain:

(a) Given $x \in A_n$ and $y \in A_m$ both non-zero, their product $xy \in A_{n+m}$ is also non-zero.

(b) B is a domain in the usual sense (ignoring the **GL**-action).

It turns out that these conditions are not equivalent to each other (Exercise 7.8). We therefore need some way to determine which one should be considered "correct."

The categorical perspective

Our current dilemma is a situation where a more general categorical perspective can point the way. Suppose that $\mathcal{A}$ is a tensor category and A is a commutative algebra object in $\mathcal{A}$. We can then define an *c-ideal* of A to be an A-submodule of A. Given two ideals $\mathfrak{a}$ and $\mathfrak{b}$, we define their product $\mathfrak{ab}$ to be the image of the natural map $\mathfrak{a} \otimes_A \mathfrak{b} \to A \otimes_A A = A$. We say that A is a *c-domain* if $\mathfrak{ab} = 0$ implies $\mathfrak{a} = 0$ or $\mathfrak{b} = 0$ for ideals $\mathfrak{a}$ and $\mathfrak{b}$. Similarly, we say that an ideal $\mathfrak{p}$ is *c-prime* if $\mathfrak{ab} \subset \mathfrak{p}$ implies $\mathfrak{a} \subset \mathfrak{p}$ or $\mathfrak{b} \subset \mathfrak{p}$. We use the c- prefix for clarity, so that we can distinguish between these categorical definitions and the usual ones in situations where they both apply.

Let us now examine these definitions when $\mathcal{A}$ is the category of representations of a group G. A commutative algebra in $\mathcal{A}$ is simply a commutative algebra A equipped with an action of G by group homomorphisms. A c-ideal in A is simply an ideal that is stable by G. Given $x \in A$, we let $(x)_G$ denote the c-ideal it generates; this is just the ideal generated by the gx with $g \in G$. One easily verifies that A is a c-domain if and only if $(x)_G(y)_G = 0$ implies $x = 0$ or $y = 0$ for $x, y \in A$. We thus see that A is a c-domain if and only if $x(gy) = 0$ for all g implies $x = 0$ or $y = 0$. Similarly, we see that a c-ideal $\mathfrak{p}$ of A is c-prime if and only if $x(gy) \in \mathfrak{p}$ for all $g \in G$ implies $x \in \mathfrak{p}$ or $y \in \mathfrak{p}$.

Primes in tca's: part 2

The categorical perspective is helpful since it is invariant under equivalence of categories. In other words, it provides a definition of prime for tca's and **GL**-algebras that coincides under Schur–Weyl duality. It turns out that it yields the first candidate definition we discussed earlier. We now adopt that as our official definition, but still use the term c-prime for clarity.

We can now define our version spectrum for a tca A: we define the *c-spectrum* of A to be the set of c-primes, equipped with the obvious analog of the Zariski topology (closed sets are the $V(I)$ with I a c-ideal).

We can also define a notion of support for an A-module M. Let $\mathfrak{p}$ be a c-prime. We say define $M_{\mathfrak{p}} = 0$ to mean the following: for every finitely generated A-submodule N of M, there is a c-ideal $\mathfrak{a}$ of A not contained in $\mathfrak{p}$ such that $\mathfrak{a}N = 0$. We then define the *c-support* of M to be the set of c-primes $\mathfrak{p}$ such that $M_{\mathfrak{p}} \neq 0$. In fact, this definition makes sense in the abstract setting of an algebra in a tensor category.

Example 1: FI

Let B be the **GL**-algebra $\mathrm{Sym}(\mathbf{C}^\infty) = \mathbf{C}[x_i]$. As we have seen, B-modules correspond to **FI**-modules. The c-ideals of B are exactly the ideals $\mathfrak{a}_r = \bigoplus_{n \geq r} B_n$ for $r \in \mathbf{N}$, together with the zero ideal; note that $\mathfrak{a}_0$ is the unit ideal. One has $\mathfrak{a}_r\mathfrak{a}_s = \mathfrak{a}_{r+s}$. From this, it follows easily that the c-primes are exactly $\mathfrak{a}_1 = B_+$ and (0). Thus the c-spectrum of B has two points, namely (0) and B_+, the first of which is open and the second of which is closed. These two points correspond to the two pieces of the category $\mathbf{Mod}_{\mathbf{FI}}$, namely $\mathbf{Mod}_{\mathbf{FI}}^{\mathrm{gen}}$ and $\mathbf{Mod}_{\mathbf{FI}}^{\mathrm{tors}}$.

Example 2: $\mathbf{FI}_2$

Now let B be the $\mathbf{GL}$-algebra $\mathrm{Sym}(\mathbf{C}^\infty \oplus \mathbf{C}^\infty) = \mathbf{C}[x_i, y_i]_{i\ge 1}$. We can now write down a number of different interesting c-ideals that are prime:

- The zero ideal (0).
- The irrelevant ideal B_+.
- For $\alpha, \beta \in \mathbf{C}$, not both zero, the ideal generated by $\alpha x_i + \beta y_i$.
- The ideal generated by $x_i y_j - x_j y_i$.

Since these ideals are $\mathbf{GL}$-stable and prime, it follows that they are c-prime. In fact, it turns out that these are the only c-primes in this case.

We thus see that the c-spectrum of B consists of the above points. It can be organized as follows: the c-primes $\alpha x_i + \beta y_i$ are indexed by the point $[\alpha : \beta] \in \mathbf{P}^1$, while the final c-prime corresponds to the generic point of $\mathbf{P}^1$. Note that $\mathbf{P}^1$ can be thought of as the Grassmannian $\mathbf{Gr}_1(\mathbf{C}^2)$. The remaining two points can be thought of as $\mathbf{Gr}_0(\mathbf{C}^2)$ and $\mathbf{Gr}_2(\mathbf{C}^2)$. We thus see that the c-spectrum of B is, as a set, the disjoint union of $\mathbf{Gr}_k(\mathbf{C}^2)$ for $0 \le k \le 2$.

The form of the c-spectrum suggests that $\mathbf{Mod}_B$ might now split nicely into three pieces: one corresponding to each $\mathbf{Gr}_k(\mathbf{C}^2)$. This is indeed the case.

Here is one somewhat concrete consequence of the above picture. Recall (from Exercise 4.12) that $\Lambda = \mathrm{K}(\mathbf{Mod}_{\mathbf{FB}}^{\mathrm{fg}})$ is a ring, and that $\mathrm{K}(\mathbf{Mod}_{\mathbf{FI}}^{\mathrm{fg}})$ is a rank two module over Λ. Using the above picture, one can show that $\mathrm{K}(\mathbf{Mod}_B^{\mathrm{fg}})$ is naturally isomorphic to $\bigoplus_{k=0}^2 \Lambda \otimes \mathrm{K}(\mathbf{Gr}_k(\mathbf{C}^2))$, which turns out to be a free Λ-module of rank 4.

The discussion of this section generalizes to $\mathbf{FI}_d$ in the obvious manner. In particular, $\mathrm{K}(\mathbf{Mod}_{\mathbf{FI}_d}^{\mathrm{fg}})$ is a free Λ-module of rank 2^d.

Exercises

c-primes

Exercise 7.1 ($\star$)**.** Consider $R = \mathbf{C}[x_1, x_2, \ldots]$ with $\mathfrak{S}_\infty$ acting in the obvious manner. Show that the ideal $(x_1^2, x_2^2, \ldots)$ is c-prime. □

Exercise 7.2 ($\star$)**.** Consider $R = \mathbf{F}_p[x_1, x_2, \ldots]$ with $\mathbf{GL}_\infty$ acting in the usual manner. Give an example of a non-radial c-prime. □

Exercise 7.3 ($\star$)**.** Let B be the $\mathbf{GL}$-algebra $\mathrm{Sym}(\mathbf{C}^d \otimes \mathbf{C}^\infty)$, where $\mathbf{GL}_\infty$ acts trivially on $\mathbf{C}^d$. Show that $\mathbf{Mod}_B$ is equivalent to $\mathbf{Mod}_{\mathbf{FI}_d}$. □

Exercise 7.4 (⋆⋆)**.** Suppose that a finite group G acts on a ring R. Show that any c-prime is radical. □

Exercise 7.5 (⋆⋆)**.** Explicitly determine the topology on the c-spectrum of the **GL**-algebra $\mathrm{Sym}(\mathbf{C}^\infty \oplus \mathbf{C}^\infty)$. □

Exercise 7.6 (⋆⋆⋆)**.** Suppose that a group G acts on a ring R. Define the *c-radical* of a c-ideal $\mathfrak{a}$ to be the sum of all c-ideals $\mathfrak{b}$ such that $\mathfrak{b}^n \subset \mathfrak{a}$ for some n. Show that the c-radical of $\mathfrak{a}$ is equal to the intersection of all c-primes containing $\mathfrak{a}$. □

Exercise 7.7 (⋆⋆⋆)**.** Consider the **GL**-algebra $\mathrm{Sym}(\mathbf{C}^d \otimes \mathbf{C}^\infty)$. Show that any c-prime is in fact prime. □

Exercise 7.8 (⋆⋆⋆)**.** Show that the $\mathbf{GL}_\infty$-ideal $\mathfrak{p}$ of $B = \mathrm{Sym}(\mathrm{Sym}^2(\mathbf{C}^\infty)) = \mathbf{C}[x_{i,j}]$ generated by $x_{1,1}^2$ is c-prime. In particular, $B/\mathfrak{p}$ is a **GL**-algebra that is not a domain but is a c-domain. □

Hilbert series

The following exercise gives the main theorem on Hilbert series of $\mathbf{FI}_d$-modules. It's not related to spectra, but I wanted to include it somewhere, so I'm sticking it here! This argument is from [Sn, §3], and will require some background.

Let T be the standard (diagonal) torus in $\mathbf{GL}_n$. We let $\mathbf{C}[\alpha_1^{\pm 1}, \ldots, \alpha_n^{\pm 1}]$ be the ring of Laurent polynomials, which we identify with the coordinate ring $\mathbf{C}[T]$. For an element f of this ring, we write $\int_T f d\alpha$ for its constant term; this is literally the integral of f over the maximal compact subgroup of T with respect to the normalized Haar measure. We let $f \mapsto \overline{f}$ be the automorphism of $\mathbf{C}[T]$ mapping α_i to α_i^{-1}, and put $|f|^2 = f \cdot \overline{f}$. We let $\Delta = \prod_{1 \le i < j \le n} (\alpha_j - \alpha_i)$ be the dicriminant.

Suppose that V is an algebraic representation of T. Then V breaks up into one-dimensional representations, each of which corresponds to a monomial in the α's. We define the *character* of V, denoted $\mathrm{ch}(V)$, to be the sum of these monomial. We define the character of a representation of $\mathbf{GL}_n$ to be the character of its restriction to T. For example, the character of the standard representation $\mathbf{C}^n$ is $\alpha_1 + \cdots + \alpha_n$ and the character of $\bigwedge^2(\mathbf{C}^n)$ is $\sum_{1 \le i < j \le n} \alpha_i \alpha_j$.

Let V and W be irreducible representations of $\mathbf{GL}_n$. *Weyl's integration formula* states that

$$\frac{1}{n!} \int_T \mathrm{ch}(V)\, \mathrm{ch}(W) |\Delta|^2 d\alpha = \begin{cases} 1 & \text{if } V \cong W \\ 0 & \text{otherwise} \end{cases}$$

In other words, the irreducible characters of $\mathbf{GL}_n$ are orthonormal on T with respect to the measure $\frac{1}{n!} |\Delta|^2 d\alpha$.

We need one more definition: for a graded representation $M = \bigoplus_{n \ge 0} M_n$ of T, we define its *equivariant Hilbert series* by

$$\mathrm{H}_{M,T}(t, \alpha) = \sum_{n \ge 0} \mathrm{ch}(M_n) t^n.$$

We regard it as a power series in t with coefficients in $\mathbf{C}[T]$.

Exercise 7.9 (⋆⋆⋆)**.** Let M be a finitely generated $\mathbf{FI}_d$-module. Let M' be the Schur–Weyl dual of M (thought of as a polynomial functor), and let $n \geq \ell(M')$.

(a) Establish the formula

$$\mathrm{H}_M(t) = \frac{1}{n!}\int_T \mathrm{H}_{M'(\mathbf{C}^n),T}(t,\alpha)\exp\left(\sum_{i=1}^n \overline{\alpha}_i\right)|\Delta|^2 d\alpha$$

(b) Show that

$$\mathrm{H}_{M'(\mathbf{C}^n),T} = \frac{q(t,\alpha)}{\prod_{i=1}^n (1-\alpha_i t)^d}$$

for some $q(t,\alpha) \in \mathbf{Q}[t, \alpha_1^{\pm 1}, \ldots, \alpha_n^{\pm 1}]$.

(c) Show that $\mathrm{H}_M(t) = \sum_{k=0}^d p_k(t)e^{kt}$ for polynomials $p_0, \ldots, p_d \in \mathbf{Q}[t]$.

(d) Show that there are polynomials $q_1, \ldots, q_d \in \mathbf{Q}[t]$ such that $\dim(M_n) = \sum_{k=1}^d q_k(n)k^n$ for $n \gg 0$. □

Appendix A

Abelian categories

Grothendieck abelian categories

Definitions

Let $\mathcal{A}$ be an abelian category. We say that $\mathcal{A}$ is a *Grothendieck abelian category* if it satisfies the following conditions:

(a) $\mathcal{A}$ has all direct sums: that is, the direct sum of any (small) family of objects in $\mathcal{A}$ exists. Equivalently, $\mathcal{A}$ is cocomplete (i.e., all small colimits exist).

(b) Filtered colimits (i.e., direct limits) in $\mathcal{A}$ are exact.

(c) $\mathcal{A}$ admits a cogenerator. This is a mild finiteness condition.

Grothendieck categories enjoy a number of pleasant properties, some of which we describe below. Many of the categories one naturally encounters (such as the category $\mathbf{Mod}_{\mathcal{C}}$ of $\mathcal{C}$-modules) are Grothendieck categories. It is therefore very useful to learn some of the general properties of these categories.

Examples

Here are some important examples of Grothendieck abelian categories:

- The category of left R-modules, for any ring R.
- The category of sheaves of abelian groups on X, for any topological space X.
- Let X be a variety, or, more generally, a quasi-compact quasi-separated scheme. Then the category $\mathrm{QCoh}(X)$ of quasi-coherent sheaves on X is a Grothendieck abelian category.
- Let $\mathcal{C}$ be a small category and $\mathcal{A}$ a Grothendieck abelian category. Then the functor category $\mathrm{Fun}(\mathcal{C}, \mathcal{A})$ is a Grothendieck abelian category. In particular, taking $\mathcal{A} = \mathbf{Vec}$, we see that the category $\mathbf{Mod}_{\mathcal{C}}$ of $\mathcal{C}$-modules is a Grothendieck abelian category.

- Let $\mathcal{A}$ be a Grothendieck abelian category and let $\mathcal{B}$ be a Serre subcategory closed under arbitrary direct sums. Then both $\mathcal{B}$ and the quotient category $\mathcal{A}/\mathcal{B}$ are Grothendieck abelian categories. Moreover, the quotient functor $T\colon \mathcal{A} \to \mathcal{A}/\mathcal{B}$ is cocontinuous, and $\mathcal{B}$ is localizing, that is, T has a right adjoint $S\colon \mathcal{A}/\mathcal{B} \to \mathcal{A}$.

The most common examples of abelian categories that are not Grothendieck are ones where a finiteness constraint has been imposed. For example, the following categories are *not* Grothendieck (except in some highly degenerate cases):

- The category of finitely generated left R-modules, for R a left noetherian ring.
- The category of coherent sheaves on a noetherian scheme.

Injectives and projectives

Let $\mathcal{A}$ be a Grothendieck category. Then $\mathcal{A}$ has enough injectives. In fact, more is true.

One corollary of the above property is that for any left-exact functor out of a Grothendieck abelian category one can form its right-derived functors.

It is not necessarily true that Grothendieck categories have enough projectives. For example, the category $\mathrm{QCoh}(\mathbf{P}^1)$ of quasi-coherent sheaves on $\mathbf{P}^1$ does not (it has no non-zero projectives).

Adjoint functors

Every functor between Grothendieck categories that should have an adjoint does. By this we mean the following: if $\Phi\colon \mathcal{A} \to \mathcal{B}$ is a functor between categories that has a left adjoint the Φ is continuous; if $\mathcal{A}$ and $\mathcal{B}$ are Grothendieck categories the converse is true, that is, if Φ is continuous then it automatically has a left adjoint. Similarly, any cocontinuous functor between Grothendieck abelian categories has a right adjoint.

Finiteness conditions

Let $\mathcal{A}$ be a Grothendieck category. We say that an object M of $\mathcal{A}$ is *finitely generated* if the following condition holds: if $\{N_i\}_{i \in I}$ is a collection of subobjects of M such that $M = \sum_{i \in I} N_i$ then there exists a finite subset $J \subset I$ such that $M = \sum_{j \in J} N_j$. The intuition behind this definition is that each of the finitely many generators of M should belong to a finite sum of N's, and therefore every element should. For $\mathcal{A} = \mathbf{Mod}_R$, this agrees with the usual notion of finite generation (Exercise A.4).

The fact that finite generation is an intrinsic property of objects can be quite convenient to know. For example, if we are ignorant of this definition, it may not be immediately clear how to define finite generation in an abstractly defined category like $\mathbf{Mod}_{\mathbf{FI}}^{\mathrm{gen}}$.

One can also define related finiteness conditions, like finitely presented or coherent, in any Grothendieck abelian category. These definitions can actually be made in arbitrary abelian categories, but may not behave as expected in general. (For example, in a general abelian category if one defines finite generation as we did above, it is not necessarily true that a quotient of a finitely generated object is finitely generated.)

Grothendieck groups

Let $\mathcal{A}$ be an abelian category. The *Grothendieck group* of $\mathcal{A}$, denoted $\mathrm{K}(\mathcal{A})$, is defined as follows. Let F be the free abelian group having for a basis the set $\mathrm{Ob}(\mathcal{A})$ of objects of $\mathcal{A}$ (we'll ignore the set-theoretic issues in this discussion); for $M \in \mathcal{A}$, write $\{M\}$ for the corresponding basis of F. Let $R \subset F$ be the subgroup generated by all elements of the form $\{M_2\} - \{M_1\} - \{M_3\}$ where

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

is an exact sequence in $\mathcal{A}$. Then $\mathrm{K}(\mathcal{A})$ is defined to be F/R. For an object M of $\mathcal{A}$, we let $[M]$ denote its class in $\mathrm{K}(\mathcal{A})$ (i.e., the image of $\{M\}$).

The Grothendieck group satisfies the following mapping property. Let B be an arbitrary abelian group. Then to give a group homomorphism $\Phi\colon \mathrm{K}(\mathcal{A}) \to B$ is equivalent to giving a function $\varphi\colon \mathrm{Ob}(\mathcal{A}) \to B$ that is additive in short exact sequences; that is, given a short exact sequence as above, we have $\varphi(M_2) = \varphi(M_1) + \varphi(M_3)$. The maps Φ and φ are related by $\varphi(M) = \Phi([M])$.

If $\mathcal{A}$ has all direct sums then $\mathrm{K}(\mathcal{A})$ is automatically zero (Exercise A.8). For this reason, we typically only consider the Grothendieck group of categories where we have imposed a finiteness constraint (like finite generation).

Exercises

Exercise A.1 ($\star$)**.** Let X be a topological space and let $\mathrm{PSh}(X)$ (resp. $\mathrm{Sh}(X)$) denote the category of presheaves (resp. sheaves) of abelian groups on X. Show that $\mathrm{PSh}(X)$ and $\mathrm{Sh}(X)$ are abelian categories, that $\mathrm{Sh}(X)$ is a full subcategory of $\mathrm{PSh}(X)$, but that $\mathrm{Sh}(X)$ is *not* an abelian subcategory of $\mathrm{PSh}(X)$. □

Projective and injective objects

Exercise A.2 ($\star$)**.** Show that any direct summand of a projective object is projective. Similarly for injectives. □

Exercise A.3 ($\star\star$)**.** Suppose that

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

is a short exact sequence in $\mathcal{A}$. Show that

$$\mathrm{pd}(M_2) \le \max(\mathrm{pd}(M_1), \mathrm{pd}(M_3)).$$

Formulate and proves similar inequalities that bound $\mathrm{pd}(M_1)$ and $\mathrm{pd}(M_3)$. □

Finiteness properties

Exercise A.4 (⋆)**.** Let R be a ring and let $\mathcal{A} = \mathbf{Mod}_R$ be the category of left R-modules. Show that the categorical notion of finite generation defined above agrees with the usual notion of finite generation. □

Exercise A.5 (⋆)**.** Let $\mathcal{A}$ be an abelian category. Show that the full subcategory of $\mathcal{A}$ spanned by the noetherian objects is an abelian subcategory of $\mathcal{A}$. □

Exercise A.6 (⋆)**.** Let $\mathcal{A}$ be a Grothendieck abelian category. Show that any quotient of a finitely generated object is finitely generated. □

Grothendieck groups

Exercise A.7 (⋆)**.** Let $\mathcal{A}$ be an abelian category such that every object has finite projective dimension. Show that the classes $[P]$, with P a projective object of $\mathcal{A}$, span $\mathrm{K}(\mathcal{A})$. □

Exercise A.8 (⋆)**.** Let $\mathcal{A}$ be an abelian category having all (infinite) direct sums. Show that $\mathrm{K}(\mathcal{A}) = 0$. □

Exercise A.9 (⋆⋆)**.** For each of the following abelian categories $\mathcal{A}$, describe $\mathrm{K}(\mathcal{A}^{\mathrm{fg}})$.

(a) The category of vector spaces over a field.

(b) The category of abelian groups.

(c) The category of R-modules, where $R = \mathbf{k}[x_1, \ldots, x_n]$ and $\mathbf{k}$ is a field.

(d) The category of complex representations of a finite group G. □

Exercise A.10 (⋆⋆)**.** Let $\mathcal{A}$ be a $\mathbf{k}$-linear abelian category, where $\mathbf{k}$ is a field (this means that all Hom sets are $\mathbf{k}$-vector spaces). Suppose that for any two objects M and N of $\mathcal{A}$ the group $\mathrm{Ext}^i(M, N)$ is finite dimensional over $\mathbf{k}$ and vanishes for $i \gg 0$. Show that there is a bilinear pairing

$$\langle , \rangle \colon \mathrm{K}(\mathcal{A}) \times \mathrm{K}(\mathcal{A}) \to \mathbf{Z}$$

defined by

$$\langle [M], [N] \rangle = \sum_{i \geq 0} (-1)^i \dim_{\mathbf{k}} \mathrm{Ext}^i(M, N).$$

This is called the *Ext pairing*. It is typically not symmetric. □

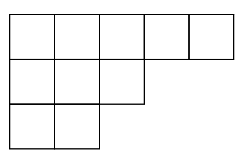
Appendix B

Representation theory of the symmetric group

Partitions

A *partition* is a sequence $\lambda = (\lambda_1, \lambda_2, \ldots)$ of non-negative integers that is weakly decreasing, i.e., $\lambda_1 \geq \lambda_2 \geq \cdots$, and eventually zero, i.e., $\lambda_i = 0$ for all $i \gg 0$. We often omit the zero entries of λ, and just write $\lambda = (\lambda_1, \ldots, \lambda_n)$ if $\lambda_{n+1} = 0$. We write $|\lambda|$ for the sum $\lambda_1 + \lambda_2 + \cdots$, which we call the *size* of λ. If $|\lambda| = n$, we also say that λ is a partition of n.

Let λ be a partition. The *Young diagram* of λ is a grid of boxes: the first row has λ_1 boxes, the second row λ_2 boxes, and so on. For example, the Young of the partition $\lambda = (5, 3, 2)$ is



Young diagrams provided a very useful way of visualizing partitions. We will freely pass between the two points of view. For instance, we often speak of a "box" in a partition, when we really mean the box in the associated Young diagram.

The Young diagram point of view

The standard basis

A tableau t is called *standard* if the rows and columns are increasing. We have the following important result about these tableaux:

Theorem B.1. *Let λ be a partition. Then the e_t with t a standard tableau of shape λ form a basis of $\mathfrak{M}_\lambda$.*

Corollary B.2. *The dimension of $\mathfrak{M}_\lambda$ is the number of standard tableaux of shape λ.*

For example, suppose $\lambda = (3, 2)$. The standard tableaux are

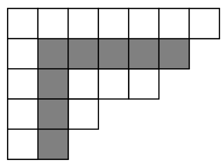
$$\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5\\\cline{1-2}\end{array}\qquad\begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&5\\\cline{1-2}\end{array}\qquad\begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&4\\\cline{1-2}\end{array}\qquad\begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&5\\\cline{1-2}\end{array}\qquad\begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4\\\cline{1-2}\end{array}$$

Thus $\mathfrak{M}_\lambda$ is five dimensional.

The hook length formula

Let λ be a partition. Then *hook* at a box in the Young diagram consists of the box itself, those boxes in the same row and to the right, and those boxes in the same column and below. The *hook length* of a box is the number of boxes in its hook.

For example, the following figure shows the hook of the box $(2, 2)$ in the partition $\lambda = (7, 6, 5, 5, 3, 2)$.



The hook length of this box is 8.

Theorem B.3 (Hook length formula)**.** *Let λ be a partition of n. Then*

$$\dim \mathfrak{M}_\lambda = \frac{n!}{\prod \textit{hook lengths}},$$

where the denominator is the product of the hook lengths of all boxes.

For example, consider the partition $\lambda = (5, 4, 2, 1)$ of $n = 12$. The hook lengths are as follows:

$$\begin{array}{|c|c|c|c|c|}\hline 8&6&4&3&1\\\hline 6&4&2&1\\\cline{1-4} 3&1\\\cline{1-2} 1\\\cline{1-1}\end{array}$$

We thus find

$$\dim \mathfrak{M}_\lambda = \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}{8 \cdot 6 \cdot 6 \cdot 4 \cdot 4 \cdot 3 \cdot 3 \cdot 2} = 11 \cdot 7 \cdot 5 \cdot 5 \cdot 3 = 5775$$

Exercises

Exercise B.1 ($\star$)**.** List the standard tableaux of shape $\lambda = (3, 1, 1)$. $\square$

Exercise B.2 ($\star$)**.** Compute $\dim \mathfrak{M}_{(6,6,3,2,1)}$ using the hook length formula. $\square$

Exercise B.3 (⋆⋆)**.** Recall that the *Catalan number* C_n is defined by

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

It is an integer. Show that $\dim \mathfrak{M}_{(n,n)} = C_n$. □

Exercise B.4 (⋆⋆)**.** Let λ be a partition of n and let μ be obtained from λ by removing a single box. By the Pieri rule, there is a unique $\mathfrak{S}_{n-1}$-equivariant map $\mathfrak{M}_\mu \to \mathfrak{M}_\lambda$. Describe this map on standard tableaux. □

Exercise B.5 (⋆⋆)**.** Let $n \geq 0$ be an integer and let $\mathcal{M}$ be the set of perfect matchings on the set $[2n]$; recall that a perfect matching is an undirected graph in which every vertex appears in exactly one edge. For $\Gamma \in \mathcal{M}$ we let e_Γ denote the corresponding basis vector of $\mathbf{C}\langle\mathcal{M}\rangle$. Suppose that $\{a, b\}$ and $\{c, d\}$ are two distinct edges in Γ. Let Γ' (resp. Γ'') be the graph obtained by replacing these two edges wtih $\{a, c\}$ and $\{b, d\}$ (resp. $\{a, d\}$, $\{b, c\}$). The *Plücker relation* is the equation

$$e_\Gamma + e_{\Gamma'} + e_{\Gamma''} = 0.$$

Let V be the quotient of $\mathbf{C}\langle\mathcal{M}\rangle$ obtained by imposing all of the Plücker relations. Show that $\mathrm{sgn} \otimes V$ is isomorphic to $\mathcal{M}_{n,n}$ as an $\mathfrak{S}_{2n}$-representation. (Hint: use the Garnir presentation of this Specht module.) □

Exercise B.6 (⋆⋆⋆)**.** Let G be a finite group and let H be a subgroup of G. Let $H\backslash G/H$ denote the set of double cosets.

(a) Let $H \times H$ act on $\mathbf{C}[G]$ by $(h, k) \cdot e_g = e_{hgk^{-1}}$. Show that $\mathbf{C}[H\backslash G/H]$ is naturally isomorphic to $\mathbf{C}[G]^{H\times H}$.

(b) Show that $\mathbf{C}[G]^{H\times H}$ is a subalgebra of $\mathbf{C}[G]$. Conclude that $\mathbf{C}[H\backslash G/H]$ is naturally an algebra, and describe the multiplication on it directly.

(c) Show that $\mathrm{End}_G(\mathbf{C}[G/H])$ is naturally identified with $\mathbf{C}[H\backslash G/H]$ as an algebra.

(d) Show that the linear map $i \colon \mathbf{C}[H\backslash G/H] \to \mathbf{C}[H\backslash G/H]$ taking HgH to $Hg^{-1}H$ is an anti-involution (that is, $i(xy) = i(y)i(x)$ and $i(i(x)) = x$). We call this the *Gelfand map*.

(e) Suppose that the Gelfand map is the identity. Conclude that $\mathbf{C}[H\backslash G/H]$ is commutative, and thus $\mathbf{C}[G/H]$ is multiplicity-free. This is called the *Gelfand trick*. □

Exercise B.7 (⋆⋆⋆)**.** Let $\mathcal{M}$ be the set of perfect matchings on $[2n]$. The symmetric group $\mathfrak{S}_{2n}$ naturally acts on $\mathcal{M}$. Let $\Gamma_0 \in \mathcal{M}$ be the matching with edges $\{i, i+n\}$ for $1 \leq i \leq n$, and let $H \subset \mathfrak{S}_{2n}$ be the stabilizer of Γ_0.

(a) Show that $H\backslash\mathcal{M}$ is canonically in bijection with the set of partitions of n.

(b) Show that every H-double coset in $\mathfrak{S}_{2n}$ is represented by an element of $\mathfrak{S}_n$ (regarded as the subgroup of $\mathfrak{S}_{2n}$ fixing all $i \geq n+1$). Furthermore, show that for $g, g' \in \mathfrak{S}_n$ we have $HgH = Hg'H$ if and only if g and g' are conjugate in $\mathfrak{S}_n$.

(c) Show that the Gelfand map on $\mathbf{C}[H\backslash\mathfrak{S}_{2n}/H]$ is the identity. Conclude that $\mathbf{C}[\mathcal{M}]$ is multiplicity-free as a $\mathfrak{S}_{2n}$-representation and contains exactly $p(n)$ irreducibles, where $p(n)$ is the number of partitions of n.

(d) Let $\mathcal{P}$ be the set of partitions of $2n$ into even parts. For $\lambda \in \mathcal{P}$, show that $\mathfrak{M}_\lambda$ occurs as a subrepresentation of $\mathbf{C}[\mathcal{M}]$. (This is rather difficult!) Conclude that $\mathbf{C}[\mathcal{M}] = \bigoplus_{\lambda\in\mathcal{P}} \mathfrak{M}_\lambda$.

(e) Applying Schur–Weyl duality, conclude that $\operatorname{Sym}^n \circ \operatorname{Sym}^2 = \bigoplus_{\lambda\in\mathcal{P}} \mathbf{S}_\lambda$. □

Appendix C

Notation

- **FB**: the category with objects finite sets and morphisms bijections
- **FI**: the category with objects finite sets and morphisms injections
- $\mathbf{FI}_d$: the category with objects finite sets and morphisms injections together with a d-coloring on the complement of the image
- **FIM**: the category with objects finite sets and morphisms injections together with a perfect matching on the complement of the image
- **FS**: the category with objects finite sets and morphisms surjections
- **OI**: the category with objects totally ordered finite sets and morphisms order-preserving injections
- **VI**: the category of finite dimensional vector s
- **Vec**: the category of vector spaces (almost always over the complex numbers)
- $\mathbf{Mod}_{\mathcal{C}}$: the category of $\mathcal{C}$-modules, that is, functors $\mathcal{C} \to \mathbf{Vec}$
- $\mathfrak{S}_n$: the symmetric group on n letters
- $\mathfrak{S}_\infty$: the infinite symmetric group, define as $\bigcup_{n\geq 1} \mathfrak{S}_n$
- $\mathbf{GL}_n$: the general linear group of rank n, i.e., the group of invertible $n \times n$ matrices
- $\mathbf{GL}_\infty$: the infinite general linear group, define as $\bigcup_{n\geq 1} \mathbf{GL}_n$
- $\mathcal{A}^{\mathrm{fg}}$: the category of finitely generated objects in an abelian category $\mathcal{A}$
- $\mathrm{K}(\mathcal{A})$: the Grothendieck group of an abelian category $\mathcal{A}$
- $\mathrm{Fun}(\mathcal{A}, \mathcal{B})$: the category of functors $\mathcal{A} \to \mathcal{B}$
- $\mathbf{P}_d$: the dth principal projective **FI**-module
- $\mathbf{P}_\lambda$: the indecomposable projective **FI**-module corresponding to λ
- $\mathbf{I}_d$: the dth principal injective **FI**-module
- $\mathbf{I}_\lambda$: the indecomposable injective **FI**-module corresponding to λ

- $\mathbf{L}_\lambda$: the Spechtral **FI**-module corresponding to λ
- $\mathbf{M}_\lambda$: the **FI**-module that is $\mathfrak{M}_\lambda$ in degree $|\lambda|$ and 0 in other degrees
- $\mathfrak{M}_\lambda$: the Specht module corresponding to the partition λ
- $|\lambda|$: the size of the partition λ, equal to $\lambda_1 + \lambda_2 + \cdots$, or the number of boxes in the Young diagram
- $\ell(\lambda)$: the number of non-zero parts in the partition λ, equal to the number of rows in the Young diagram
- $\mathrm{PSh}(X)$: the category of presheaves of abelian groups on a topological space X
- $\mathrm{Sh}(X)$: the category of sheaves of abelian groups on a topological space X.
- $\mathrm{QCoh}(X)$: the category of quasi-coherent sheaves on a scheme X

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