CATEGORY THEORY

Definition 0.1. A **category** $\mathcal{C}$ consists of a class[1] of objects, written $\text{ob}(\mathcal{C})$, a class of morphisms between objects, written $\text{Hom}(\mathcal{C})$. Furthermore, for any three objects $a, b, c \in \text{ob}(\mathcal{C})$, there must be a binary operation $\text{Hom}(a, b) \times \text{Hom}(b, c) \to \text{Hom}(a, c)$, called **composition**, satisfying the following two properties:

i. (*associativity*) If $f \in \text{Hom}(a, b)$, $g \in \text{Hom}(b, c)$, and $h \in \text{Hom}(c, d)$, then $h \circ (g \circ f) = (h \circ g) \circ f$.
ii. (*identity*) For every $x \in \text{ob}(\mathcal{C})$, there exists $1_x \in \text{Hom}(x, x)$, called the identity morphism for x, such that for all $a, b \in \text{ob}(\mathcal{C})$, for all $f \in \text{Hom}(a, x)$ and $g \in \text{Hom}(x, b)$, $1_x \circ f = f$ and $g \circ 1_x = g$.

1. Let $\text{Mat}_{\mathbb{R}}$ be the category whose objects are positive integers and whose morphisms are matrices with entries in $\mathbb{R}$ (i.e. $\text{Hom}(m, n)$ is the set of all $m \times n$ matrices over $\mathbb{R}$). Composition is by matrix multiplication. Prove that $\text{Mat}_{\mathbb{R}}$ is a category.[2]
2. Prove that any partially ordered set (poset) $(P, \leq)$ is a category. (What are the morphisms?)
3. Let G be a group. Prove that we can think of G as a category by taking $\text{ob}(G) = \{*\}$ and $\text{Hom}_G(*, *) = G$ with composition given by the binary operation of G.

Definition 0.2. For any category $\mathcal{C}$, we may define the **dual (or opposite) category** $\mathcal{C}^{op}$ with $\text{ob}(\mathcal{C}^{op}) = \text{ob}(\mathcal{C})$ and whose morphisms are in bijection with the morphisms of $\mathcal{C}$. More precisely, for every $f \colon x \to y$ in $\text{Hom}(\mathcal{C})$, we define $f^{op} \colon y \to x$ in $\text{Hom}(\mathcal{C}^{op})$.

In other words, $\mathcal{C}^{op}$ has the same objects and morphisms of $\mathcal{C}$, except that all the morphisms "point in the opposite direction."

4. Give descriptions of the composition laws of $\text{Mat}_{\mathbb{R}}^{op}$ and P^{op}, where $(P, \leq)$ is a poset.

Let Fin be the category whose objects are finite sets, morphisms are functions, and composition is the usual composition of functions. For any function $f \colon X \to Y$, define a matrix $M(f)$ whose rows and columns are indexed by Y and X respectively, and where the entry in position $(y, x) \in Y \times X$ is given by the formula

$$M(f)_{y,x} = \begin{cases} 1 & y = f(x) \\ 0 & y \neq f(x). \end{cases}$$

5. Check that $M(1_X)$ is an identity matrix when $1_X \colon X \to X$ is the identity function.
6. For any pair of composable functions f, g, show that $M(f \circ g) = M(f) \cdot M(g)$.

These two properties of M make it look like a homomorphism—or perhaps a representation, since the outputs are matrices. Any such assignment of morphisms to morphisms that sends identities to identities and preserves composition is called a functor.

[1] If you don't know what a class is, you can think of it as a generalization of a set. For example, Russell's paradox tells us that there is no set of all sets, but we want to be able to define a category whose objects are sets, and we do that with classes. There is a class of all sets.

[2] Here we have explicityly chosen to work with $\mathbb{R}$, but you can replace $\mathbb{R}$ with any unital ring R to define Mat_R, the category of matrices over R.

Definition 0.3. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A **(covariant) functor** $\mathcal{F}$ is a mapping from $\mathcal{C}$ to $\mathcal{D}$ that associates every object $x \in \mathrm{ob}(\mathcal{C})$ to an object $\mathcal{F}(x) \in \mathrm{ob}(\mathcal{D})$ and associates to every morphism $f\colon x \to y$ in $\mathrm{Hom}(\mathcal{C})$ a morphism $\mathcal{F}(f)\colon \mathcal{F}(x) \to \mathcal{F}(y)$ in $\mathrm{Hom}(\mathcal{D})$ so that $\mathcal{F}(1_x) = 1_{\mathcal{F}(x)}$ for all $x \in \mathrm{ob}(\mathcal{C})$ and for all $f \in \mathrm{Hom}(x,y)$ and $g \in \mathrm{Hom}(y,z)$, $\mathcal{F}(g \circ f) = \mathcal{F}(g) \circ \mathcal{F}(f)$.

7. The source category for the functor M in Problems 5 & 6 is Fin. What is the target category?
8. Explain how a homomorphism of groups can be considered a functor.
9. If G is a group, explain why a functor $G \to \mathrm{Vec}_{\mathbb{C}}$ [3] is the same as a representation of G.

If A is a $p \times q$ matrix over $\mathbb{C}$, define a $p^2 \times q^2$ matrix $\otimes^2 A$ by the formula

$$(\otimes^2 A)_{(p_1,p_2),(q_1,q_2)} = A_{p_1,q_1} \cdot A_{p_2,q_2}.$$

10. Show that $\otimes^2$ is a functor. What is its source category? Target category?
11. Is it true that $\otimes^2(A+B) = \otimes^2(A) + \otimes^2(B)$?

Write FI for the category whose objects are finite sets and whose morphisms are injections.

12. Write the definition of what you think it should mean for a functor $\mathcal{F}\colon \mathcal{C} \to \mathcal{D}$ to be an **inclusion**.
13. Describe an inclusion functor $i\colon \mathrm{FI} \to \mathrm{Fin}$.
14. Write the definition of how to compose functors $\mathcal{F}\colon \mathcal{C} \to \mathcal{D}$ and $\mathcal{G}\colon \mathcal{D} \to \mathcal{E}$. For any $x \in \mathrm{ob}(\mathcal{C})$, what is $\mathcal{G} \circ \mathcal{F}(x)$? For any morphism $f\colon x \to y$, what is $\mathcal{G} \circ \mathcal{F}(f)$?
15. Does every functor $\mathrm{FI} \to \mathrm{Ab}$ [4] factor through i? In other words, for every functor $\mathcal{F}\colon \mathrm{FI} \to \mathrm{Ab}$, does there exist a functor $\mathcal{G}\colon \mathrm{Fin} \to \mathrm{Ab}$ so that $\mathcal{F} = \mathcal{G} \circ i$?

For any function $f\colon X \to Y$ between finite sets, define $T(f) = [1]$, the 1×1 matrix with entry 1. For every finite set X, let N_X be the $1 \times X$ matrix $[1 \quad 1 \quad \cdots \quad 1]$.

16. Show that T defines a functor $\mathrm{Fin} \to \mathrm{Ab}$. It is the **trivial representation** of Fin.
17. Show that $N_Y \cdot M(f) = T(f) \cdot N_X$.

Thinking of M and T as representations, N then looks like an intertwiner—a map of representations. Any time a family of morphisms intertwines the outputs of two functors in this way, it is called a **natural transformation**. In this case, the family $N_\bullet$ is a natural transformation from M to T. The matrix N_X is called the **component** at X.

Definition 0.4. Suppose that $\mathcal{F}$ and $\mathcal{G}$ are both functors from $\mathcal{C}$ to $\mathcal{D}$. A **natural transformation** η from $\mathcal{F}$ to $\mathcal{G}$ is family of morphisms that satisfies the following:

i. To every $x \in \mathrm{ob}(\mathcal{C})$, η associates a morphism $\eta_x\colon \mathcal{F}(x) \to \mathcal{G}(x)$ between objects of $\mathcal{D}$. (η_x is the **component** of η at x.)
ii. For every morphism $f\colon x \to y$ in $\mathrm{Hom}(\mathcal{C})$, we must have $\eta_y \circ \mathcal{F}(f) = \mathcal{G}(f) \circ \eta_x$.

Let $Z_n\colon \mathrm{Fin} \to \mathrm{Fin}$ be the "n-tuples" functor sending X to the set of ordered tuples $(x_1, \ldots, x_n)$ with $x_i \in X$, and where we let morphisms act by the rule $f(x_1, \ldots, x_n) = (f(x_1), \ldots, f(x_n))$.

18. Show that the rule $(x_1, x_2, x_3) \mapsto (x_1, x_3)$ defines a natural transformation $Z_3 \to Z_2$
19. Show that the rule $(x_1, x_2) \mapsto (x_2, x_1, x_1, x_2)$ defines a natural transformation $Z_2 \to Z_4$.

[3] $\mathrm{Vec}_{\mathbb{C}}$ is the category whose objects are $\mathbb{C}$-vector spaces and whose morphisms are linear transformations.
[4] Ab is the category whose objects are abelian groups and whose morphisms are homomorphisms.

20. Find at least 16 distinct natural transformations $Z_2 \to Z_4$. Are there any others?

Definition 0.5. Let $\mathcal{C}$ be a category. Two objects $x, y \in \mathrm{ob}(\mathcal{C})$ are said to be **isomorphic** if there are morphisms $f \in \mathrm{Hom}(x, y)$ and $g \in \mathrm{Hom}(y, x)$ so that $g \circ f = 1_x$ and $f \circ g = 1_y$, in which case f is called an **isomorphism from** x **to** y, and g is called an **isomorphism from** y **to** x. An object $i \in \mathrm{ob}(\mathcal{C})$ is called **initial** if $|\mathrm{Hom}(i, c)| = 1$ for all $c \in \mathrm{ob}(\mathcal{C})$. An object $t \in \mathcal{C}$ is called **terminal** if $|\mathrm{Hom}(c, t)| = 1$ for all $c \in \mathrm{ob}(\mathcal{C})$.

The following two exercises show that initial and terminal objects are "unique up to unique isomorphism."

21. If $i_1, i_2 \in \mathrm{ob}(\mathcal{C})$ are initial objects, show that i_1 and i_2 are isomorphic in $\mathcal{C}$, and moreover, that the isomorphism $f : i_1 \to i_2$ is unique.
22. Prove that a terminal object of $\mathcal{C}$ is an initial object of $\mathcal{C}^{op}$. Deduce that if $t_1, t_2 \in \mathcal{C}$ are terminal, then there is a unique isomorphism $t_1 \xrightarrow{\sim} t_2$.

Definition 0.6. Let $\mathcal{D}$ be a category. The objects of the **functor category** $[\mathcal{D}, \mathcal{C}]$ are functors $\mathcal{F} : \mathcal{D} \to \mathcal{C}$. Morphisms in $[\mathcal{D}, \mathcal{C}]$ are natural transformations. Explicitly, if $\mathcal{F}, \mathcal{G} : \mathcal{D} \to \mathcal{C}$ are functors, then $\varphi \in \mathrm{Hom}(\mathcal{F}, \mathcal{G})$ is a family of morphisms $\varphi_d : \mathcal{F}(d) \to \mathcal{G}(d)$ so that $\varphi_b \circ \mathcal{F}(h) = \mathcal{G}(h) \circ \varphi_a$ for all $a, b \in \mathcal{D}$ and $h \in \mathrm{Hom}(a, b)$.

23. Define an associative composition law for natural transformations.
24. If $\mathcal{h} : \mathcal{E} \to \mathcal{D}$ is a functor, check that precomposition with $\mathcal{h}$ defines a functor

$$\mathcal{h}^* : [\mathcal{D}, \mathcal{C}] \to [\mathcal{E}, \mathcal{C}].$$

Definition 0.7. An object 0 in a category $\mathcal{C}$ is called a **zero object** if it is both an initial and a terminal object in $\mathcal{C}$.

25. Prove that R-Mod [5] has a zero object.

Definition 0.8. If x_1 and x_2 are objects in a category $\mathcal{C}$, the **product** of x_1 and x_2, should it exist, is an object $x_1 \times x_2 \in \mathrm{ob}(\mathcal{C})$ together with morphisms $\pi_1 : x_1 \times x_2 \to x_1$ and $\pi_2 : x_1 \times x_2 \to x_2$ satisfying the following property:

For any $y \in \mathrm{ob}(\mathcal{C})$ and pair of morphisms $f_1 \in \mathrm{Hom}(y, x_1)$ and $f_2 \in \mathrm{Hom}(y, x_2)$, there exists a unique morphism $f \in \mathrm{Hom}(y, x_1 \times x_2)$ so that $\pi_2 \circ f = f_2$ and $\pi_1 \circ f = f_1$.

26. Prove that for any $x_1, x_2 \in \mathrm{ob}(R\text{-Mod})$, $x_1 \times x_2$ exists in R-Mod.

Definition 0.9. If x_2 and x_2 are objects in a category $\mathcal{C}$, the **coproduct** of x_1 and x_2, should it exist, is an object $x_1 \oplus x_2 \in \mathrm{ob}(\mathcal{C})$ together with morphisms $i_1 : x_1 \to x_1 \times x_2$ and $i_2 : x_2 \to x_1 \times x_2$ satisfying the following property:

For any $y \in \mathrm{ob}(\mathcal{C})$ and pair of morphisms $g_1 \in \mathrm{Hom}(x_1, y)$ and $g_2 \in \mathrm{Hom}(x_2, y)$ there exists a unique morphism $g \in \mathrm{Hom}(x_1 \oplus x_2, y)$ so that $g \circ i_1 = g_1$ and $g \circ i_2 = g_2$.

27. Prove that for any $x_1, x_2 \in \mathrm{ob}(R\text{-Mod})$, $x_1 \oplus x_2$ exists in R-Mod.

Definition 0.10. A **zero morphism** in a category $\mathcal{C}$ is a morphism $0_{xy} \in \mathrm{Hom}(x, y)$ satisfying the following:

i. For all $w \in \mathrm{ob}(\mathcal{C})$ and $f_1, f_2 \in \mathrm{Hom}(w, x)$, $0_{xy} \circ f_1 = 0_{xy} \circ f_2$.
ii. For all $z \in \mathrm{ob}(\mathcal{C})$ and $g_1, g_2 \in \mathrm{Hom}(y, z)$, $g_1 \circ 0_{xy} = g_2 \circ 0_{xy}$.

[5] For a unital ring R, R-Mod is the category whose objects are left R-modules and whose morphisms are homomorphisms of left R-modules.

28. Prove that for all $x, y \in \text{ob}(R\text{-Mod})$, there exists a zero morphism $0_{xy} \in \text{Hom}(x, y)$.

Definition 0.11. Given a morphism $f \in \text{Hom}(x, y)$ of a category $\mathcal{C}$, the **kernel** of f, should it exist, is an object $k \in \text{ob}(\mathcal{C})$ and a morphism $g_k \in \text{Hom}(k, x)$ satisfying the following:

i. $f \circ g_k = 0_{ky}$.
ii. For all $z \in \text{ob}(\mathcal{C})$ and $g \in \text{Hom}(z, x)$ satisfying $f \circ g = 0_{zy}$, there exists a unique $u \in \text{Hom}(z, k)$ such that $g_k \circ u = g$.

29. Prove that every morphism $f \in \text{Hom}(R\text{-Mod})$ has a kernel.

Definition 0.12. Given a morphism $f \in \text{Hom}(x, y)$ of a category $\mathcal{C}$ satisfying $g \circ f = 0_{xz}$, the **cokernel** of f, should it exist, is an object $q \in \text{ob}(\mathcal{C})$ and a morphism $g_q \in \text{Hom}(y, q)$ satisfying the following:

i. $g_q \circ f = 0_{xq}$.
ii. For all $z \in \text{ob}(\mathcal{C})$ and $g \in \text{Hom}(y, z)$ there exists a unique $u \in \text{Hom}(q, z)$ so that $g = u \circ g_q$.

30. Prove that every morphism $f \in \text{Hom}(R\text{-Mod})$ has a cokernel.

Definition 0.13. A **monomorphism** of a category $\mathcal{C}$ is a morphism $f \in \text{Hom}(x, y)$ satisfying the property that for all $z \in \text{ob}(\mathcal{C})$ and $g_1, g_2 \in \text{Hom}(z, x)$, if $f \circ g_1 = f \circ g_2$, then $g_1 = g_2$.

An **epimorphism** of a $\mathcal{C}$ is a morphism $h \in \text{Hom}(x, y)$ satisfying the property that for all $z \in \text{ob}(\mathcal{C})$ and $g_1, g_2 \in \text{Hom}(y, z)$, if $g_1 \circ h = g_2 \circ h$, then $g_1 = g_2$.

31. Prove that every monomorphism f of R-Mod can be realized as the kernel of a morphism g.
32. Prove that every epimorphism h of R-Mod can be realized as the cokernel of a morphism g.

Definition 0.14. An **abelian category** is a category $\mathcal{A}$ that satisfies the following properties:

i. $\mathcal{A}$ has a zero object.
ii. For all $x, y \in \text{ob}(\mathcal{A})$, $x \times y \in \text{ob}(\mathcal{A})$ and $x \oplus y \in \text{ob}(\mathcal{A})$.
iii. Every morphism $f \in \text{Hom}(\mathcal{A})$ has a kernel and a cokernel.
iv. All monomorphisms and epimorphisms of $\mathcal{A}$ can be realized as kernels or cokernels, respectively.

Combining Problems 25-32, you have just proven that R-Mod is an abelian category. Abelian categories can be thought of as a generalization of module categories, and much of the algebra that holds for modules can be extended to abelian categories. More precisely, the properties of an abelian category are what one needs to define exact sequences.

An FI-**module** is a functor $\mathcal{F}$ from FI to $\text{Vec}_{\mathbb{C}}$ [6].

33. Prove that the category of FI-modules, the functor category $[\text{FI}, \text{Vec}_{\mathbb{C}}]$, is an abelian category.

Definition 0.15. The **Grothendieck group** of an abelian category $\mathcal{A}$ is the abelian group generated by all isomorphism classes $[x]$ of objects of $\mathcal{A}$ and with the relation $[a] - [b] + [c] = 0$ if $0 \to a \to b \to c \to 0$ is an exact sequence in $\mathcal{A}$.

[6] The more general definition of an FI-module replaces $\text{Vec}_{\mathbb{C}}$ with R-Mod.

34. Compute the Grothendieck groups of the category of finitely generated $\mathbb{C}$-vector spaces and the category of finitely generated abelian groups.

Definition 0.16. Given an object $c \in \mathcal{C}$, the **representable functor** $\mathrm{Hom}(c, -): \mathcal{C} \to \mathrm{Set}$[7] is the functor that sends b to $\mathrm{Hom}(c, b)$, and for any morphism $f: a \to b$, $\mathrm{Hom}(c, f): \mathrm{Hom}(c, a) \to \mathrm{Hom}(c, b)$ sends $g \in \mathrm{Hom}(c, a)$ to $g \circ f \in \mathrm{Hom}(c, b)$.

In the next two problems, we prove Yoneda's lemma. Let $\mathcal{H}, \mathcal{F}: \mathcal{C} \to \mathrm{Set}$ be functors, and suppose $\mathcal{H} = \mathrm{Hom}(c, -)$ is represented by some $c \in \mathrm{ob}(\mathcal{C})$. Note that $\mathcal{H}(c) = \mathrm{Hom}(c, c)$ has a special element $1_c: c \to c$, the identity morphism on c.

35. For every $x \in \mathcal{F}(c)$, show that there is a unique natural transformation $\varphi: \mathcal{H} \implies \mathcal{F}$ with $\varphi_c(1_c) = x$.
36. Explain why we might say that $\mathcal{H}$ is a "free $\mathcal{C}$-module on a single generator at $c \in \mathcal{C}$."

Let $\mathcal{S}$ be the poset category with objects $\{2\}, \{2, 3\}, \{1, 2, 3\}, \{0, 1, 2, 3\}, \{1, 2, 3, 4\}$ and where morphisms are inclusions. Define a functor $\mathcal{F}: \mathcal{S} \to \mathrm{Ab}$ by the formula

$$S \in \mathcal{S} \qquad \mathcal{F}(S) = \ker\left(\mathbb{Z}S \xrightarrow{\epsilon} \mathbb{Z}\right)$$

where $\mathbb{Z}S$ denotes the $\mathbb{Z}$-module of formal $\mathbb{Z}$-linear combinations of elements of S, and the map $\epsilon: \mathbb{Z}S \to \mathbb{Z}$ is defined on basis vectors $s \in S$ by $\epsilon(s) = 1$.

37. What does $\mathcal{F}$ do to morphisms of $\mathcal{S}$?
38. For each $S \in \mathcal{S}$, show that $\mathcal{F}(S)$ is a free $\mathbb{Z}$-module by finding a basis.
39. For every inclusion $i: S \subseteq T$, write $\mathcal{F}(i)$ as a matrix.

Definition 0.17. An **isomorphism of categories** is given by a pair of functors $\mathcal{F}: \mathcal{C} \to \mathcal{D}$ and $\mathcal{G}: \mathcal{D} \to \mathcal{C}$ so that $\mathcal{G} \circ \mathcal{F}$ is the identity functor on $\mathcal{C}$ and $\mathcal{F} \circ \mathcal{G}$ is the identity functor on $\mathcal{D}$.

The terminal category $*$ is the category with a single object and a single morphism (the identity morphism of the object).

40. Find an isomorphism of categories $\psi: \mathcal{C} \to [*, \mathcal{C}]$.
41. Show that every category $\mathcal{C}$ has a unique functor $t: \mathcal{C} \to *$.

Let $t: \mathcal{E} \to *$ be the unique functor from $\mathcal{E}$ to the terminal category. Let $\mathcal{F}: \mathcal{E} \to \mathcal{C}$ be a functor. If the functor

$$\begin{array}{ccc} \mathcal{C} & \to & \mathrm{Set} \\ c & \mapsto & \mathrm{Hom}(\mathcal{F}, \psi(c) \circ t) \end{array}$$

is representable, then the representing object, which is an object of $\mathcal{C}$, is written $\mathrm{colim}_{\mathcal{E}}\mathcal{F}$; it is called the **colimit of** $\mathcal{F}$. The colimit may also be written $t_!\mathcal{F}$. Eliding the isomorphism ψ, so that the symbol "c" may stand for an object $c \in \mathcal{C}$ or its corresponding functor $c: * \to \mathcal{C}$, the colimit satisfies

$$\mathrm{Hom}(t_!\mathcal{F}, c) \cong \mathrm{Hom}(\mathcal{F}, t^*c)$$

for all $c \in \mathcal{C}$.

42. Show that any two colimits of $\mathcal{F}$ are isomorphic.
43. If G is a group considered as a one-object category, and if $X: G \to \mathrm{Set}$ is a right G-set considered as a functor, show that $\mathrm{colim}_G X \cong X/G$.

The next questions investigate adjoint functors.

[7] Set is the category whose objects are sets whose morphisms are functions.

Definition 0.18. An **adjunction** of categories $\mathcal{C}$ and $\mathcal{D}$ is a pair of functors $\mathcal{F}\colon \mathcal{D} \to \mathcal{C}$ and $\mathcal{G}\colon \mathcal{C} \to \mathcal{D}$ so that for all $x \in \mathrm{ob}(\mathcal{C})$ and $y \in \mathrm{ob}(\mathcal{D})$ there is a natural isomorphism

$$\mathrm{Hom}_{\mathcal{C}}(\mathcal{F}(y), x) \cong \mathrm{Hom}_{\mathcal{D}}(x, \mathcal{G}(y)).$$

$\mathcal{F}$ is the **left adjoint functor** and $\mathcal{G}$ is the **right adjoint functor**.

Let $D\colon \mathrm{Fin} \to \mathrm{Fin}$ be the "doubling functor" that sends a set X to the product $X \times \{1, 2\}$.

44. Show that the functor $X \mapsto \mathrm{Hom}(DX, Y)$ is representable by finding a set Z so that $\mathrm{Hom}(DX, Y) \cong \mathrm{Hom}(X, Z)$ are isomorphic functors of X. The set Z is allowed to depend on Y, but not on X!
45. Show that any map $f\colon Y \to Y'$ induces a natural transformation $\mathrm{Hom}(D-, Y) \implies \mathrm{Hom}(D-, Y')$.
46. Let ΩY denote the set called Z earlier. Given any map $Y \to Y'$, use Yoneda's lemma to provide a map $\Omega Y \to \Omega Y'$. Show that this rule makes Ω into a functor.
47. Find a natural isomorphism

$$\mathrm{Hom}(DX, Y) \cong \mathrm{Hom}(X, \Omega Y)$$

where both sides are considered functors of X and Y.