REPRESENTATION STABILITY REFERENCES

1. About this document

This document was assembled by the organizers of the 2016 AIM workshop on representation stability (Andrew Putman, Steven Sam, David Speyer, Andrew Snowden) to serve as a guide to the literature in the subject. We thought this could be useful to newcomers to the area since, although the subject is still young, there is already a sizeable literature. For each paper here, we have either written a brief description or included the abstract; we had hoped to summarize each paper, but there were simply too many of them. We have also marked a small number of papers with a star (★) to indicate that, in our opinion, they would be a good place for a newcomer to start reading.

2. FI-modules

Homology of **FI**-modules
Thomas Church, Jordan S. Ellenberg
arXiv:1506.01022

The authors study what they call **FI**-module homology. (From the viewpoint of twisted commutative algebras, this is just Tor with the residue field.) The main result bounds the Castelnuovo–Mumford regularity of an FI-module in terms of the degrees of generators and relations. The result holds over any base ring, and the proof is combinatorial. □

★ **FI**-modules and stability for representations of symmetric groups
Thomas Church, Jordan S. Ellenberg, Benson Farb
Duke Math. J. **164** (2015), no. 9, 1833–1910. arXiv:1204.4533

This paper introduced **FI**-modules, and developed a number of basic results about them (when working over a field of characteristic 0), including noetherianity and eventual polynomiality of dimension. It went on to give a great number of applications of this theory, e.g., to the cohomology of configuration spaces. □

★ **FI**-modules over Noetherian rings
Thomas Church, Jordan S. Ellenberg, Benson Farb, Rohit Nagpal
Geom. Top. **18** (2014), no. 5, 2951–2984, arXiv:1210.1854

This is the first paper to obtain interesting results about **FI**-modules over arbitrary rings. The authors prove noetherianity of **FI**-modules over arbitrary (commutative) noetherian coefficient rings, and prove polynomiality of dimension over arbitrary fields. □

A long exact sequence for homology of **FI**-modules
Wee Liang Gan
arXiv:1602.08873

From the abstract: We construct a long exact sequence involving the homology of an **FI**-module. Using the long exact sequence, we give two methods to bound the Castelnuovo–Mumford regularity

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of an **FI**-module which is generated and related in finite degree. We also prove that for an **FI**-module which is generated and related in finite degree, if it has a nonzero higher homology, then its homological degrees are strictly increasing (starting from the first homological degree). □

A remark on **FI**-module homology
Wee Liang Gan, Liping Li
arXiv:1505.01777

From the abstract: We show that the **FI**-homology of an **FI**-module can be computed via a Koszul complex. As an application, we prove that the Castelnuovo–Mumford regularity of a finitely generated torsion **FI**-module is equal to its degree. □

Upper bounds of homological invariants of $\mathbf{FI}_G$-modules
Liping Li
arXiv:1512.05879

From the abstract: In this paper we use a homological approach to obtain upper bounds for a few homological invariants of $\mathbf{FI}_G$-modules V. These upper bounds are expressed in terms of the generating degree and torsion degree, which measure the top and socle of V under actions of non-invertible morphisms in the category respectively. □

Depth and the Local Cohomology of $\mathbf{FI}_G$-modules
Liping Li, Eric Ramos
arXiv:1602.04405

From the abstract: In this paper we describe a machinery for homological calculations of representations of $\mathbf{FI}_G$, and use it to develop a local cohomology theory over any commutative Noetherian ring. As an application, we show that the depth introduced by the second author coincides with a more classical invariant from commutative algebra, and obtain upper bounds of a few important invariants of $\mathbf{FI}_G$-modules in terms of torsion degrees of their local cohomology groups. □

Filtrations and Homological degrees of **FI**-modules
Liping Li, Nina Yu
arXiv:1511.02977

From the abstract: Let k be a commutative Noetherian ring. In this paper we consider filtered modules of the category **FI** firstly introduced by Nagpal. We show that a finitely generated **FI**-module V is filtered if and only if its higher homologies all vanish, and if and only if a certain homology vanishes. Using this homological characterization, we characterize finitely generated **FI**-modules V whose projective dimension is finite, and describe an upper bound for it. Furthermore, we give a new proof for the fact that V induces a finite complex of filtered modules, and use it as well as a result of Church and Ellenberg to obtain another upper bound for homological degrees of tV. □

FI-modules and the cohomology of modular representations of symmetric groups
Rohit Nagpal
arXiv:1505.04294

Let M be an **FI**-module over a field of characteristic p. One can then consider the group cohomology $\mathrm{H}^t(S_n, M_n)$. The author proves that, for t fixed, the dimensions of these spaces are eventually periodic with period a power of p. This generalizes a classical result of Nakaoka that states that the spaces stabilize when M is the trivial **FI**-module. As an application, new results are obtained about the mod p cohomology of unordered configuration spaces. Along the way, the author proves several useful structural results about FI-modules in positive characteristic. In forthcoming work, Nagpal–Snowden give a cleaner and more conceptual proof of the main theorem. □

Homological Invariants of **FI**-modules and $\mathbf{FI}_G$-modules
Eric Ramos
arXiv:1511.03964

From the abstract: We explore a theory of depth for $\mathbf{FI}_G$-modules which are presented in finite degrees. Using this theory, we prove results about the regularity, and provide novel bounds on stable ranges of **FI**-modules, making effective a theorem of Nagpal and thereby refining the stable range in results of Church, Ellenberg, and Farb. □

Generalized Representation Stability and $\mathbf{FI}_d$-modules
Eric Ramos
arXiv:1606.02673

From the abstract: In this note we consider the complex representation theory of $\mathbf{FI}_d$, a natural generalization of the category **FI** of finite sets and injections. We prove that finitely generated $\mathbf{FI}_d$-modules exhibit behaviors in the spirit of Church–Farb representation stability theory, generalizing a theorem of Church, Ellenberg, and Farb which connects finite generation of **FI**-modules to representation stability. □

On the degree-wise coherence of $\mathbf{FI}_G$-modules
Eric Ramos
arXiv:1606.04514

From the abstract: In this work we study a kind of coherence condition on $\mathbf{FI}_G$-modules, which generalizes the usual notion of finite generation. We prove that a module is coherent, in the appropriate sense, if and only if its generators, as well as its torsion, appears in only finitely many degrees. Using this technical result, we prove that the category of coherent $\mathbf{FI}_G$-modules is abelian, independent of any assumptions on the group G, or the coefficient ring k. Following this, we consider applications towards the local cohomology theory of $\mathbf{FI}_G$-modules, introduced by Li and the author in previous work. □

$\mathbf{FI}_{\mathcal{W}}$-modules and stability criteria for representations of classical Weyl groups
Jennifer C. H. Wilson
J. Algebra **420** (2014), 269–332. arXiv:1309.3817

From the abstract: In this paper we develop machinery for studying sequences of representations of any of the three families of classical Weyl groups, extending work of Church, Ellenberg, Farb, and Nagpal on the symmetric groups S_n to the signed permutation groups B_n and the even-signed permutation groups D_n. For each family W_n, we present an algebraic framework where a sequence V_n of W_n-representations is encoded into a single object we call an $\mathbf{FI}_W$-module. We prove that if an $\mathbf{FI}_W$-module V satisfies a simple finite generation condition then the structure of the sequence is highly constrained. One consequence is that the sequence is uniformly representation stable in the sense of Church–Farb, that is, the pattern of irreducible representations in the decomposition of each V_n eventually stabilizes in a precise sense. Using the theory developed here we obtain new results about the cohomology of generalized flag varieties associated to the classical Weyl groups, and more generally the r-diagonal coinvariant algebras.

We analyze the algebraic structure of the category of $\mathbf{FI}_W$-modules, and introduce restriction and induction operations that enable us to study interactions between the three families of groups. We use this theory to prove analogues of Murnaghan's 1938 stability theorem for Kronecker coefficients for the families B_n and D_n. The theory of $\mathbf{FI}_W$-modules gives a conceptual framework for stability results such as these. □

3. Representations of categories

Noetherianity and rooted trees
Daniel Barter
arXiv:1509.04228

Let $\mathcal{C}$ be the category whose objects are finite trees (morphisms are given by certain embeddings of trees). The author studies representations of this category, and proves a noetherianity result. The key combinatorial result underlying the proof is Kruskal's tree theorem. □

Noetherian property of infinite EI categories
Wee Liang Gan, Liping Li
arXiv:1407.8235

From the abstract: It is known that finitely generated **FI**-modules over a field of characteristic 0 are Noetherian. We generalize this result to the abstract setting of an infinite EI category satisfying certain combinatorial conditions. □

Koszulity of directed categories in representation stability theory
Wee Liang Gan, Liping Li
arXiv:1411.5308

From the abstract: In the first part of this paper, we study Koszul property of directed graded categories. In the second part of this paper, we prove a general criterion for an infinite directed category to be Koszul. We show that infinite directed categories in the theory of representation stability are Koszul over a field of characteristic zero. □

Coinduction functor in representation stability theory
Wee Liang Gan, Liping Li
arXiv:1502.06989

From the abstract: We study the coinduction functor on the category of **FI**-modules and its variants. Using the coinduction functor, we give new and simpler proofs of (generalizations of) various results on homological properties of **FI**-modules. We also prove that any finitely generated projective **VI**-module over a field of characteristic 0 is injective. □

On central stability
Wee Liang Gan, Liping Li
arXiv:1504.07675

From the abstract: We give a simple proof, in a general setting, that finitely presented modules are centrally stable. □

Homological degrees of representations of categories with shift functors
Liping Li
arXiv:1507.08023

From the abstract: Let k be a commutative Noetherian ring and $\mathcal{C}$ be a locally finite k-linear category equipped with a self-embedding functor of degree 1. We show under a moderate condition that finitely generated torsion representations of $\mathcal{C}$ are super finitely presented (that is, they have projective resolutions each term of which is finitely generated). In the situation that these self-embedding functors are genetic functors, we give upper bounds for homological degrees of finitely generated torsion modules. These results apply to quite a few categories recently appearing in representation stability theory. In particular, when k is a field of characteristic 0, we obtain another upper bound for homological degrees of finitely generated **FI**-modules. □

Representation stability and finite linear groups
Andrew Putman, Steven V Sam
arXiv:1408.3694

This paper develops a framework for studying representation stability properties of congruence subgroups of things like $\mathbf{SL}_n(\mathbf{Z})$, mapping class groups of surfaces, etc. The main technical tool introduced are the matrix versions of **FI**, like **VI**, **VIC**, **SI** which depend on a ring R. Their representations are shown to be noetherian when R is a finite commutative ring. □

★ Gröbner methods for representations of combinatorial categories
Steven V Sam, Andrew Snowden
J. Amer. Math. Soc., to appear, arXiv:1409.1670

This paper develops general combinatorial methods for proving noetherianity and rationality of Hilbert series for representations of categories. The main tools are Gröbner bases and the theory of formal languages. A number of applications are given, such as: (1) a proof of the Lannes–Schwartz artinian conjecture; (2) an improvement of Snowden's work on syzygies of Segre embeddings; and (3) basic results about degree 1 twisted commutative algebras in positive characteristic (greatly generalizing the earlier results of Church–Ellenberg–Farb–Nagpal on **FI**-modules in positive characteristic). □

Representations of categories of G-maps
Steven V Sam, Andrew Snowden
arXiv:1410.6054

Let G be a group. Let $\mathbf{FA}_G$ be the category of free G-sets with finitely many orbits and arbitrary G-equivariant maps. Let $\mathbf{FI}_G$ and $\mathbf{FS}_G$ be the subcategories where the maps are injective and surjective. The authors study the representations of these categories. The two most interesting results are: (1) Noetherianity holds for $\mathbf{k}[\mathbf{FI}_G]$-modules if and only if the group algebra $\mathbf{k}[G^n]$ is left-noetherian for all n. This condition holds if G is polycyclic-by-finite, and these may be the only groups satisfying it. (2) A strong rationality result for Hilbert series of $\mathbf{FS}_G$-modules is proven. Note that $\mathbf{FS}_G$-modules generalize Δ-modules, and this result simultaneously strengthens and generalizes Snowden's earlier result about Hilbert series of Δ-modules. □

Uniformly presented vector spaces
John D. Wiltshire-Gordon
arXiv:1406.0786

Let **FA** be the category whose objects are finite sets and whose morphisms are arbitrary functions. This author studies representations of **FA** (and calls them "uniformly presented vector spaces") and proves a number of results. For instance, over a field, finitely generated **FA**-modules have finite length. □

Categories of dimension zero
John D. Wiltshire-Gordon
arXiv:1508.04107

Let $\mathcal{C}$ be a category. The category $\mathrm{Rep}_{\mathbf{k}}(\mathcal{C})$ is said to have Krull dimension 0 if (roughly) finitely generated objects have finite length. This paper gives a combinatorial criterion on $\mathcal{C}$ for $\mathrm{Rep}_{\mathbf{k}}(\mathcal{C})$ to have Krull dimension 0. □

4. Twisted commutative algebras

Noetherianity of some degree two twisted commutative algebras
Rohit Nagpal, Steven V Sam, Andrew Snowden
Selecta Math. (N.S.) **22** (2016), no. 2, 913–937, arXiv:1501.06925

Let A be the twisted commutative algebra $\mathrm{Sym}(\mathrm{Sym}^2(\mathbf{C}^\infty))$. This paper proves that A is noetherian. Note that A is unbounded, and none of the general methods developed so far can prove the noetherianity of A. The method here is not at all combinatorial, but more algebraic, and specific to characteristic 0. □

★ GL-equivariant modules over polynomial rings in infinitely many variables
Steven V Sam, Andrew Snowden
Trans. Amer. Math. Soc. **368** (2016), 1097–1158, arXiv:1206.2233

Let $A = \mathbf{C}[x_1, x_2, \ldots]$ be the polynomial ring in infinitely many variables over the complex numbers. The group GL_∞ acts on A by linear substitutions. This paper studies the category of A-modules equipped with a compatible polynomial action of GL_∞, and gives a very complete picture of the category. Note that this category is equivalent to the category of **FI**-modules over $\mathbf{C}$. □

★ Introduction to twisted commutative algebras
Steven V Sam, Andrew Snowden
arXiv:1209.5122

This paper is an introduction to the theory of twisted commutative algebras. □

A remark on a conjecture of Derksen
Andrew Snowden
J. Commut. Algebra **6** (2014), no. 1, 109–112. arXiv:1211.3483

H. Derksen formulated a conjecture on the degrees of syzygies of invariant rings for finite groups. This very short paper points out that a weaker statement can be obtained with very little work using ideas from tca's. (Later, Chardin–Symonds arXiv:1410.0150 found a counterexample to Derksen's conjecture, but proved a very slightly weaker statement.) □

5. Algebraic topology

Homological stability for configuration spaces of manifolds
Thomas Church
Invent. Math. **188** (2012) 2, 465–504. arXiv:1103.2441

From the abstract: Let $C_n(M)$ be the configuration space of n distinct ordered points in M. We prove that if M is any connected orientable manifold (closed or open), the homology groups $\mathrm{H}_i(C_n(M); \mathbf{Q})$ are representation stable in the sense of Church–Farb. Applying this to the trivial representation, we obtain as a corollary that the unordered configuration space $B_n(M)$ satisfies classical homological stability: for each i, $\mathrm{H}_i(B_n(M); \mathbf{Q})$ is isomorphic to $\mathrm{H}_i(B_{n+1}(M); \mathbf{Q})$ for $n > i$. This improves on results of McDuff, Segal, and others for open manifolds. Applied to closed manifolds, this provides natural examples where rational homological stability holds even though integral homological stability fails.

To prove the main theorem, we introduce the notion of monotonicity for a sequence of S_n-representations, which is of independent interest. Monotonicity provides a new mechanism for proving representation stability using spectral sequences. The key technical point in the main theorem is that certain sequences of induced representations are monotone. □

★ Representation theory and homological stability
Thomas Church, Benson Farb
Adv. in Math. (2013) 250–314. arXiv:1008.1368

From the abstract: We introduce the idea of *representation stability* (and several variations) for a sequence of representations V_n of groups G_n. A central application of the new viewpoint we introduce here is the importation of representation theory into the study of homological stability. This makes it possible to extend classical theorems of homological stability to a much broader variety of examples. Representation stability also provides a framework in which to find and to predict patterns, from classical representation theory (Littlewood–Richardson and Murnaghan rules, stability of Schur functors), to cohomology of groups (pure braid, Torelli and congruence groups), to Lie algebras and their homology, to the (equivariant) cohomology of flag and Schubert varieties, to combinatorics (the $(n+1)^{(n-1)}$ conjecture). The majority of this paper is devoted to exposing this phenomenon through examples. In doing this we obtain applications, theorems and conjectures.

Beyond the discovery of new phenomena, the viewpoint of representation stability can be useful in solving problems outside the theory. In addition to the applications given in this paper, it is applied in [CEF] to counting problems in number theory and finite group theory. Representation stability is also used in [C] to give broad generalizations and new proofs of classical homological stability theorems for configuration spaces on oriented manifolds. □

Generating the Johnson filtration
Thomas Church, Andrew Putman
Geom. Topol. **19** (2015) 2217–2255. arXiv:1311.7150

From the abstract: For $k \geq 1$, let $\mathrm{Torelli}_g^1(k)$ be the kth term in the Johnson filtration of the mapping class group of a genus g surface with one boundary component. We prove that for all k, there exists some $G_k \geq 0$ such that $\mathrm{Torelli}_g^1(k)$ is generated by elements which are supported on subsurfaces whose genus is at most G_k. We also prove similar theorems for the Johnson filtration of $\mathrm{Aut}(F_n)$ and for certain mod-p analogues of the Johnson filtrations of both the mapping class group and of $\mathrm{Aut}(F_n)$. The main tools used in the proofs are the related theories of **FI**-modules (due to the first author together with Ellenberg and Farb) and central stability (due to the second author), both of which concern the representation theory of the symmetric groups over **Z**. □

Algebraic structures on cohomology of configuration spaces of manifolds with flows
Jordan S. Ellenberg, John D. Wiltshire-Gordon
arXiv:1508.02430

From the abstract: Let $\mathrm{PConf}^n M$ be the configuration space of ordered n-tuples of distinct points on a smooth manifold M admitting a nowhere-vanishing vector field. We show that the ith cohomology group with coefficients in a field $\mathrm{H}^i(\mathrm{PConf}^n M, k)$ is an N-module, where N is the category of noncommutative finite sets introduced by Pirashvili and Richter. Studying the representation theory of N, we obtain new polynomiality results for the cohomology groups $\mathrm{H}^i(\mathrm{PConf}^n M, k)$. In the case of unordered configuration space $\mathrm{Conf}^n M = (\mathrm{PConf}^n M)/S_n$ and rational coefficients, we show that cohomology dimension in fixed degree is nondecreasing. □

Representation stability for families of linear subspace arrangements
Nir Gadish
arXiv:1603.08547

From the abstract: Church–Ellenberg–Farb used the language of **FI**-modules to prove that the cohomology of certain sequences of hyperplane arrangements with S_n-actions satisfies representation stability. Here we lift their results to the level of the arrangements themselves, and define when a collection of arrangements is "finitely generated". Using this notion we greatly widen the stability results to:

1) General linear subspace arrangements, not necessarily of hyperplanes.

2) A wide class of group actions, replacing **FI** by a general category C. We show that the cohomology of such collections of arrangements satisfies a strong form of representation stability, with many concrete applications. For this purpose we develop a theory of representation stability and generalized character polynomials for wide classes of groups. We apply this theory to get classical cohomological stability of quotients of linear subspace arrangements with coefficients in certain constructible sheaves. □

Representation stability for cohomology of configuration spaces in $\mathbf{R}^d$
Patricia Hersh, Victor Reiner
Int. Math. Res. Not. IMRN, to appear, arXiv:1505.04196

From the abstract: This paper studies representation stability in the sense of Church and Farb for representations of the symmetric group S_n on the cohomology of the configuration space of n ordered points in $\mathbf{R}^d$. This cohomology is known to vanish outside of dimensions divisible by $d-1$; it is shown here that the S_n-representation on the $i(d-1)$st cohomology stabilizes sharply at $n = 3i$ (resp. $n = 3i + 1$) when d is odd (resp. even).

The result comes from analyzing S_n-representations known to control the cohomology: the Whitney homology of set partition lattices for d even, and the higher Lie representations for d odd. A similar analysis shows that the homology of any rank-selected subposet in the partition lattice stabilizes by $n \geq 4i$, where i is the maximum rank selected.

Further properties of the Whitney homology and more refined stability statements for S_n-isotypic components are also proven, including conjectures of J. Wiltshire-Gordon. □

Representation stability for homotopy groups of configuration spaces
Alexander Kupers, Jeremy Miller
J. Reine Angew. Math., to appear, arXiv:1410.2328

From the abstract: We prove that the dual rational homotopy groups of the configuration spaces of a 1-connected manifold of dimension at least 3 are uniformly representation stable in the sense of Church, and that their derived dual integral homotopy groups are finitely-generated as **FI**-modules in the sense of Church-Ellenberg-Farb. This is a consequence of a more general theorem relating properties of the cohomology groups of a 1-connected co-**FI**-space to properties of its dual homotopy groups. We also discuss several other applications. □

Twisted homological stability for configuration spaces
Martin Palmer
arXiv:1308.4397

From the abstract: Let M be an open, connected manifold. A classical theorem of McDuff and Segal states that the sequence of configuration spaces of n unordered, distinct points in M is homologically stable with coefficients in $\mathbf{Z}$: in each degree, the integral homology is eventually independent of n. The purpose of this note is to prove that this phenomenon also holds for homology with twisted coefficients. We first define an appropriate notion of finite-degree twisted coefficient system for configuration spaces and then use a spectral sequence argument to deduce the result from the untwisted homological stability result of McDuff and Segal. The result and the methods are generalisations of those of Betley for the symmetric groups. □

★ Stability in the homology of congruence subgroups
Andrew Putman
Invent. Math. **202** (2015), no. 3, 987–1027. arXiv:1201.4876

From the abstract: The homology groups of many natural sequences of groups $\{G_n\}_{n=1}^{\infty}$ (e.g., general linear groups, mapping class groups, etc.) stabilize as $n \to \infty$. Indeed, there is a well-known machine for proving such results that goes back to early work of Quillen. Church and Farb discovered that many sequences of groups whose homology groups do not stabilize in the classical sense actually stabilize in some sense as representations. They called this phenomena representation stability. We prove that the homology groups of congruence subgroups of $\mathrm{GL}_n(R)$ (for almost any reasonable ring R) satisfy a strong version of representation stability that we call central stability. The definition of central stability is very different from Church–Farb's definition of representation stability (it is defined via a universal property), but we prove that it implies representation stability. Our main tool is a new machine for proving central stability that is analogous to the classical homological stability machine. □

Homological stability for automorphism groups
Nathalie Wahl, Oscar Randal-Williams
arXiv:1409.3541

From the abstract: We prove a general homological stability theorem for families of automorphism groups in certain categories. We show that this theorem can be applied to all the classical examples of stable families of groups, such as the symmetric groups, general linear groups and mapping class groups, and obtain new stability theorems with twisted coefficients for the braid groups, automorphisms of free groups, unitary groups, mapping class groups of non-orientable surfaces and mapping class groups of 3-manifolds. We allow both polynomial and abelian twisted coefficients. □

Representation stability for the cohomology of the pure string motion groups
Jennifer C. H. Wilson
Algebr. Geom. Topol. **12** (2012), 909–931, arXiv:1108.1255

From the abstract: The cohomology of the pure string motion group PSigma_n admits a natural action by the hyperoctahedral group W_n. Church and Farb conjectured that for each $k > 0$, the sequence of degree k rational cohomology groups of PSigma_n is uniformly representation stable with respect to the induced action by W_n, that is, the description of the groups' decompositions into irreducible W_n representations stabilizes for $n \gg k$. We use a characterization of the cohomology groups given by Jensen, McCammond, and Meier to prove this conjecture. Using a transfer argument, we further deduce that the rational cohomology groups of the string motion group vanish in positive degree. We also prove that the subgroup of orientation-preserving string motions, also known as the braid-permutation group, is rationally cohomologically stable in the classical sense. □

6. Algebraic geometry

★ Noetherianity up to symmetry
Jan Draisma
arXiv:1310.1705

This is a survey of various uses of equivariant noetherianity in algebraic geometry. It is very clearly written and a wonderful introduction to the aspects of the subject that it touches. □

Plücker varieties and higher secants of Sato's Grassmannian
Jan Draisma, Rob H. Eggermont
J. Reine Angew. Math., to appear. arXiv:1402.1667

This is similar to the Draisma–Kuttler paper but with Segre embeddings replaced by Plücker embeddings. □

Noetherianity for infinite-dimensional toric varieties
Jan Draisma, Rob H. Eggermont, Robert Krone, Anton Leykin
arXiv:1306.0828

This paper shows that certain infinite limits of toric ideals are finitely generated up to the symmetry group which acts on them. □

★ Bounded-rank tensors are defined in bounded degree
Jan Draisma, Jochen Kuttler
Duke Math. J. **163** (1), 35-63, 2014. arXiv:1103.5336

A special case of the main result: fix a field k, and a number r. Then there is a universal bound $C(r)$ so that the rth secant variety of a Segre embedding of a product of projective spaces is set-theoretically cut out by equations of degree $\leq C(r)$. The content is that this bound is independent of both the number of projective spaces and also their dimensions. The idea is to construct an infinite limit of the space of the Segre embedding together with a large group action and then to show that the equivariant closed subsets of interest satisfy the descending chain condition (equivariant noetherianity). □

Finiteness properties of congruence classes of infinite matrices
Robert Eggermont
Linear Algebra Appl. **484** (2015), 290–303, arXiv:1411.0526

This shows that any space which is a direct sum of a fixed number of copies of symmetric matrices, skew-symmetric matrices, and vector representations of $\mathbf{GL}(\infty)$ is equivariantly noetherian, i.e., any descending chain of equivariant Zariski closed subsets stabilizes. □

Ideals of bounded rank symmetric tensors are generated in bounded degree
Steven V Sam
Invent. Math., to appear, arXiv:1510.04904

This is similar in spirit to the Draisma–Kuttler and Draisma–Eggermont papers above, but deals with Veronese re-embeddings (actually of arbitrary projective schemes). The main difference is the existence of a bound for the generators of the prime ideal of the secant varieties, and not just generators up to radical. The idea is to put all of the coordinate rings together into a single algebraic structure (Hopf ring) and show that its ideals are finitely generated. □

Syzygies of Segre embeddings and Δ-modules
Andrew Snowden
Duke Math. J. **162** (2013), no. 2, 225–277, arXiv:1006.5248.

This paper introduces the notion of Δ-modules, an algebraic structure designed to act on vector spaces related to Segre embeddings of projective spaces, as the number of projective spaces and their dimensions vary. The main result is that each Tor group of the Segre embedding is finitely generated in characteristic 0. Intuitively, this says that there is a finite "master list" of p-syzygies (for each p) from which one can build up all other p-syzygies. □

7. Representation theory

A representation stability theorem for VI-modules

Wee Liang Gan, John Watterlond

arXiv:1602.00654

From the abstract: A **VI**-module gives rise to a sequence of representations of the finite general linear groups. We prove that the sequence obtained from any finitely generated **VI**-module over an algebraically closed field of characteristic zero is representation stable. □

Stability and periodicity in the modular representation theory of symmetric groups

Nate Harman

arXiv:1509.06414

From the abstract: We study asymptotic properties of the modular representation theory of symmetric groups and investigate modular analogs of stabilization phenomena in characteristic zero. The main results are equivalences of categories between certain abelian subcategories of representations of S_n and S_m for different n and m. We apply these results to obtain a structural result for **FI**-modules, and to prove a result conjectured by Deligne in a recent letter to Ostrik. □

★ Stability patterns in representation theory

Steven V Sam, Andrew Snowden

Forum Math. Sigma **3** (2015), e11, 108 pp., arXiv:1302.5859

This paper studies the stable representation theory of classical groups and the symmetric groups. Each case is similar, but handled separately. Here we just talk about the GL case. A category $\mathrm{Rep}^{\mathrm{alg}}(\mathrm{GL}_\infty)$ of "algebraic" representations is introduced as the primary object of study. This category is shown to be equivalent to the category of representations of the "upwards Brauer algebra" as well as the category of modules over a certain (bivariate) tca. Additionally, it is shown to be the universal tensor category equipped with a pair of objects admitting a pairing (the two objects being the standard representation and its restricted dual). Each of these descriptions is a useful point of view on the category. Finally, a specialization functor Γ_n is constructed to $\mathrm{Rep}(\mathrm{GL}_n)$, and its derived functors are studied. A Borel–Weil–Bott type theorem is established: if L is a simple object then $\mathrm{R}^i\Gamma_n(L)$ is non-zero for at most one value of i, and that non-zero value is necessarily irreducible; moreover, there are combinatorial rules to determine the exact behavior. □

8. Group theory

Stability results for Houghton groups

Peter Patzt, Xiaolei Wu

arXiv:1509.07639

From the abstract: We prove homological stability for a twisted version of the Houghton groups and their multidimensional analogues. Based on this, we can describe the homology of the Houghton groups and that of their multidimensional analogues over constant noetherian coefficients as an essentially finitely generated **FI**-module. □

9. S_∞-noetherianity

The papers in this section all prove more or less the same theorem: for any field $\mathbf{k}$ *and fixed* d, *the ring* $\mathbf{k}[x_{i,j}]_{i\in\mathbf{N},1\leq j\leq d}$ *is* S_∞*-noetherian. It seems that D. E. Cohen was the first to prove such a result (in 1967), and more recent authors were unaware of his work (at least initially). It should be noted that the various papers give different applications of this theorem.*

Finite generation of symmetric ideals
Matthias Aschenbrenner, Christopher Hillar
Trans. Amer. Math. Soc. **359** (2007), 5171–5192. arXiv:math/0411514

This paper proves the theorem for $d = 1$. As an application, the authors prove a stabilization theorem about a certain monomial ideal. This theorem was apparently motivated by a question from chemistry. □

On the laws of a metabelian variety
D. E. Cohen
J. Algebra **5** (1967), 267–273.

This is the first paper to prove the theorem, but only does the $d = 1$ case. An application is given to the "variety of groups" in the sense of universal algebra. A subvariety is a class of groups characterized by a collection of identities. For example, abelian groups form a subvariety since they are characterized by the identity $ab = ba$. For a long time, it was unknown if the variety of groups is noetherian (i.e., any descending chain of subvarieties stabilizes). It was eventually shown not to be, by explicit counterexamples. Cohen's result is that the variety of metabelian groups is noetherian. (Recall that a group is *metabelian* if its commutator subgroup is abelian.) More concretely, this means that any infinite list of group-theoretic identities that includes the metabelian identity $[[a, b], [c, d]] = 1$ is redundant. □

Closure relations, Buchberger's algorithm, and polynomials in infinitely many variables
D. E. Cohen
Computation theory and logic, (1987), 78–87.

This extends the results of the previous paper to $d > 1$. □

★ Finite Gröbner bases in infinite dimensional polynomial rings and applications
Christopher Hillar, Seth Sullivant
Adv. in Math. **221** (2012), 1–25. arXiv:0908.1777

This paper proves the theorem for arbitrary d. Various applications to algebraic statistics are given. □

Analog of Hilbert basis theorem for infinitely generated commutative algebras
A. Kemer
Asian–European J. Math. **1** (2008), 555–564.

From the abstract: Let $F[X]$ be the free commutative and associative algebra (with unity) over a field F generated by the infinite set X. We give a new proof of the theorem of Aschenbrenner–Hillar: The set of all ideals of $F[X]$ closed under the permutations of the generators from X satisfies the ascending chains condition. □