

# TOTALLY GEODESIC SUBMANIFOLDS OF TEICHMÜLLER SPACE

ALEX WRIGHT

## 1. INTRODUCTION

**Main results.** Let  $\mathcal{T}_{g,n}$  and  $\mathcal{M}_{g,n}$  denote the Teichmüller and moduli space respectively of genus  $g$  Riemann surfaces with  $n$  marked points. The Teichmüller metric on these spaces is a natural Finsler metric that quantifies the failure of two different Riemann surfaces to be conformally equivalent. It is equal to the Kobayashi metric [Roy74], and hence reflects the intrinsic complex geometry of these spaces.

There is a unique holomorphic and isometric embedding from the hyperbolic plane to  $\mathcal{T}_{g,n}$  whose image passes through any two given points. The images of such maps, called Teichmüller disks or complex geodesics, are much studied in relation to the geometry and dynamics of Riemann surfaces and their moduli spaces.

A complex submanifold of  $\mathcal{T}_{g,n}$  is called totally geodesic if it contains a complex geodesic through any two of its points, and a subvariety of  $\mathcal{M}_g$  is called totally geodesic if a component of its preimage in  $\mathcal{T}_{g,n}$  is totally geodesic. Totally geodesic submanifolds of dimension 1 are exactly the complex geodesics.

Almost every complex geodesic in  $\mathcal{T}_{g,n}$  has dense image in  $\mathcal{M}_{g,n}$  [Mas82, Vee82]. We show that higher dimensional totally geodesic submanifolds are much more rigid.

**Theorem 1.1.** *The image in  $\mathcal{M}_{g,n}$  of a totally geodesic complex submanifold of  $\mathcal{T}_{g,n}$  of dimension greater than 1 is a closed totally geodesic subvariety of  $\mathcal{M}_{g,n}$ .*

One dimensional totally geodesic subvarieties of  $\mathcal{M}_{g,n}$  are called Teichmüller curves. There are infinitely many Teichmüller curves in each  $\mathcal{M}_{g,n}$ . We show that higher dimensional totally geodesic submanifolds are much more rare.

**Theorem 1.2.** *There are only finitely many totally geodesic submanifolds of  $\mathcal{M}_{g,n}$  of dimension greater than 1.*

**Context.** One source of totally geodesic submanifolds of  $\mathcal{M}_{g,n}$  is covering constructions, see [MMW17, Section 6] for a definition. The first example of a totally geodesic submanifold of dimension greater than 1 not coming from a covering construction was given in [MMW17], and two additional examples appear in [EMMW]. These three examples are totally geodesic surfaces in  $\mathcal{M}_{1,3}$ ,  $\mathcal{M}_{1,4}$  and  $\mathcal{M}_{2,1}$  respectively.

Work of Filip implies that any closed totally geodesic submanifold of  $\mathcal{M}_{g,n}$  is in fact a subvariety [Fil16]. Any real submanifold of  $\mathcal{T}_{g,n}$  that contains the Teichmüller disk between any pair of its points must in fact be a complex submanifold.

The inclusion of a totally geodesic complex submanifold into Teichmüller space must be an isometry for the Kobayashi metrics. Antonakoudis has shown that there is no holomorphic isometric immersion of a bounded symmetric domain of dimension greater than 1 into Teichmüller space [Ant17b], and that any isometry of a complex disk into Teichmüller space is either holomorphic or antiholomorphic [Ant17a].

**Elements of the proofs.** If  $N$  is a subset of moduli or Teichmüller space, define  $QN$  to be the locus of quadratic differentials which generate Teichmüller disks contained in  $N$ . Typically  $N$  will be a totally geodesic subvariety or submanifold, in which case we may view  $QN$  as the cotangent bundle to  $N$ . Note that  $QN$  is stratified according to the number of zeros and poles of the quadratic differential.

For every quadratic differential on a Riemann surface, either the quadratic differential is the square of an Abelian differential, or there is a unique double cover on which the lift of the quadratic differential is the square of an Abelian differential. The double cover is equipped with an involution. We call the Abelian differential together with this choice of involution the square root of the quadratic differential.

Let  $\Omega N$  be the locus of square roots of quadratic differentials in the largest dimensional stratum of  $QN$ . The following ingredient in our analysis may be of independent interest.

**Theorem 1.3.** *If  $N$  is a totally geodesic subvariety of moduli space, then  $\Omega N$  is transverse to the isoperiodic foliation.*

Theorem 1.3 is equivalent to saying that there is no nonconstant path in  $\Omega N$  along which absolute periods of the Abelian differentials are locally constant. See [McM14] for a definition of the isoperiodic foliation, which is also known as the kernel foliation, the absolute period foliation, and the rel foliation.

The proof of Theorem 1.3 uses results on cylinder deformations from [Wri15] and a classical result on Jenkins-Strebel differentials. Theorem 1.2 follows from Theorem 1.3 and recent finiteness results of Eskin-Filip-Wright [EFW].

The proof of Theorem 1.1 also uses Theorem 1.3 and results of [EFW]. A key tool is the computation of the algebraic hull of the Kontsevich-Zorich cocycle from [EFW].

**Acknowledgements.** This paper was inspired by comments of Curt McMullen, who in particular suggested the possibility that Theorem 1.1 might be true. The author thanks Ben Dozier, Alex Eskin, Simion Filip, Steve Kerckhoff, Vlad Markovic, Curt McMullen, Ronen Mukamel, and Mike Wolf for helpful conversations.

This research was carried out in part at the ICERM conference “Cycles on Moduli Spaces, Geometric Invariant Theory, and Dynamics”, and was conducted during the period while the author served as a Clay Research Fellow.

## 2. PROOF OF THEOREMS 1.2 AND 1.3

We use notation consistent with [MMW17]. We assume some familiarity with recent results on the  $GL(2, \mathbb{R})$  action on the Hodge bundle.

If  $N$  is a totally geodesic subvariety of moduli space,  $\Omega N$  is an example of an affine invariant submanifold; these are subvarieties of a stratum of  $\Omega\mathcal{M}_{g'}$  (for some  $g' > 0$ ) that are locally equal to a finite union of subspaces defined over  $\mathbb{R}$  in period coordinates [EMM15, Fil16]. The tangent space  $\Omega N(Y, \omega)$  to an affine invariant submanifold  $\Omega N$  at a point  $(Y, \omega)$  is a subspace of relative cohomology  $H^1(Y, \Sigma, \mathbb{C})$ , where  $\Sigma$  is the set of zeros of  $\omega$ . Let  $p$  denote the map from relative to absolute cohomology. The rank is defined to be half the dimension of  $p$  of the tangent space [Wri14]. This is an integer because  $p$  of the tangent space is symplectic [AEM17].

To prove Theorem 1.3 we will compare the dimension of  $\Omega N$  to that of  $N$ , using the following two results to get a lower bound on the dimension of  $N$ .

An affine subspace is any translation of a vector subspace. A Jenkins-Strebel differential is an Abelian or quadratic differential that is the union of horizontal cylinders and their boundaries; these are also known as horizontally periodic differentials. Unless specified other, all references to (co)dimensions will be over  $\mathbb{C}$ .

**Theorem 2.1.** *Any affine invariant submanifold  $\Omega N$  of rank  $r$  contains a set of Jenkins-Strebel differentials whose image in local period coordinates is a open subset of an affine subspace  $S$  of codimension  $r$ ,*

such that circumferences of horizontal cylinders are constant on this subset.

The affine subspace  $S$  is the translate of a subspace  $L$  such that  $p(L)$  is a Lagrangian in  $p(\Omega N(Y, \omega))$  and such that  $\ker(p) \cap \Omega N(Y, \omega) \subset L$ .

Theorem 2.1 can be viewed as a black box coming from [Wri15], however we provide specific references to [Wri15].

*Proof.* [Wri15, Theorem 1.10] asserts the existence of a horizontally periodic  $(Y, \omega) \in N$  such that the core curves of the horizontal cylinders span a subspace of the dual space of  $\Omega N(Y, \omega)$  of dimension  $r$ . The subspace  $L$  is the subspace of  $\Omega N(Y, \omega)$  that annihilates all these core curves. Deforming  $(Y, \omega)$  in any direction in  $L$ , the periods of the core curves of the horizontal cylinders remain constant. Hence all the horizontal cylinders of  $(Y, \omega)$  persist on any such sufficiently small deformation, and remain horizontal and of constant circumference.

The proof of [Wri15, Theorem 1.10] in [Wri15, Section 8] gives that for the  $(Y, \omega)$  that is specially chosen in the proof, any sufficiently small deformation of  $(Y, \omega)$  in the direction in  $L$  does not create any new cylinders. Indeed, [Wri15, Section 8] gives that any such deformation can be obtained by certain cylinder deformations of the horizontal cylinders of  $(Y, \omega)$ . Thus these deformations remain Jenkins-Strebel.

The proof of [Wri15, Theorem 1.10] gives that  $p(L)$  is Lagrangian. Since  $L$  has codimension  $r$  and  $p(L)$  has dimension  $r$ , it follows that  $\ker(p) \cap \Omega N(Y, \omega) \subset L$ .  $\square$

Problems on the existence and uniqueness of Jenkins-Strebel differentials have been extensively studied, see for example [Gar77, HM79, Jen57, Liu04, Str84, Wol95]. Here we require only the following uniqueness result. See for example Theorem 20.3 and the remarks after Lemma 20.3 in [Str84] for an expository account of the argument.

**Lemma 2.2.** *Let  $X \in \mathcal{M}_{g,n}$  be a Riemann surface. If two Jenkins-Strebel differentials  $q, q'$  on  $X$  have the same core curves of cylinders, and corresponding cylinders have the same circumference, then  $q = q'$ .*

A point of  $\Omega N$  consists of a translation surface  $(Y, \omega)$  and an involution  $J$  that negates  $\omega$ , such that  $Y/J \in N$ . There is a map from  $\Omega N$  to  $QN$ , because  $\omega^2$  defines a quadratic differential on  $Y/J$ . In turn there is a forgetful map from  $QN$  to  $\mathcal{M}_{g,n}$  obtained by forgetting the quadratic differential but remembering the location of the poles. We will refer frequently to the composite of these two maps, which gives a map  $\Omega N$  to  $\mathcal{M}_{g,n}$ . For notational simplicity we will omit  $J$  from our notation; there is no harm for our arguments in assuming it is the

only involution on  $Y$  negating  $\omega$ , as our arguments would not be any different were this not to be the case.

*Proof of Theorem 1.3.* Suppose  $N$  has dimension  $d$ . Since  $N$  is totally geodesic, there is a  $d - 1$  dimensional family of complex geodesics in  $N$  passing through each point of  $N$ , so we get that  $QN$  has dimension  $2d$ . Hence  $\Omega N$  also has dimension  $2d$ .

Let  $r$  be the rank of  $\Omega N$ . By definition rank is at most half the dimension of  $\Omega N$ , so  $r \leq d$ . By Theorem 2.1 there is a  $2d - r$  dimensional family of Jenkins-Strebel differentials in  $\Omega N$ , and hence also  $QN$ , with constant circumferences. By Lemma 2.2 we see that the dimension of  $N$  is at least  $2d - r$ . The inequalities  $2d - r \leq d$  and  $r \leq d$  give  $r = d$ . By definition of rank, it follows that the projection of the tangent space of  $\Omega N$  to absolute cohomology has the same dimension as  $\Omega N$ . Since leaves of the isoperiodic foliation are tangent to the kernel of this projection, we get that  $\Omega N$  is transverse to the isoperiodic foliation.  $\square$

*Proof of Theorem 1.2 given Theorem 1.3.* It is proved in [EFW] that each stratum of Abelian differentials contains at most finitely many affine invariant submanifolds of rank at least 2. By Theorem 1.3, if  $N$  is a totally geodesic submanifold of dimension at least 2, then  $\Omega N$  has rank at least 2.

Since  $\Omega N$  determines  $N$ , and there are a finite list of strata that may contain  $\Omega N$  for  $N$  a totally geodesic submanifold of  $\mathcal{M}_{g,n}$ , the result follows.  $\square$

### 3. PROOF OF THEOREM 1.1

This section requires the results and arguments from the previous section.

Let  $\tilde{N}$  be a totally geodesic submanifold of  $\mathcal{T}_{g,n}$  of dimension  $d > 1$ . Let  $N$  denote the projection of  $\tilde{N}$  to moduli space. Let  $Q\bar{N}$  be the closure of  $QN$ . Note that  $QN$  and  $Q\bar{N}$  are  $GL(2, \mathbb{R})$  invariant. The goal of this section is to show that  $Q\bar{N} = QN$ , which implies  $N$  is closed and hence establishes Theorem 1.1. In order to find a contradiction, we assume  $Q\bar{N} \neq QN$ . By [EMM15], each stratum of  $Q\bar{N}$  is an affine invariant submanifold. Since  $QN$  is properly contained in  $Q\bar{N}$ , we see that  $Q\bar{N}$  must have dimension strictly greater than  $2d$ .

The rough idea of the proof of Theorem 1.1 is to consider all tangent spaces of totally geodesic submanifolds of dimension  $d$  through each point of  $\bar{N}$ . Some version of this gives an equivariant subvariety of a Grassmanian bundle. Using [EFW] we wish to show this subvariety is

very large, so roughly speaking there are totally geodesic submanifolds of  $\overline{N}$  through every point and in so many directions that we are able to contradict Theorem 1.3. The first step is to show that there is at least one totally geodesic submanifold through each point of  $\overline{N}$ .

**Lemma 3.1.** *Suppose that  $L_k$  are totally geodesic submanifolds of  $\mathcal{T}_{g,n}$  of constant dimension, and that  $X_k \in L_k$  converge to  $X$ . Let  $P_k$  denote the cotangent space to  $L_k$  at  $X_k$ , and suppose that  $P_k$  converge to a subspace  $P$  of the cotangent space of  $\mathcal{T}_{g,n}$  at  $X$ . Then there is a totally geodesic submanifold of  $\mathcal{T}_{g,n}$  that passes through  $X$  and whose cotangent space at  $X$  is  $P$ .*

*Proof.* Let  $L$  be the set of all limit points of sequences  $Y_k$  with  $Y_k \in L_k$ . If  $\lim Y_k$  and  $\lim Y'_k$  are two such points of  $L$ , then since the complex geodesic from  $Y_k$  to  $Y'_k$  lies in  $L_k$ , we get that the complex geodesic from  $\lim Y_k$  to  $\lim Y'_k$  lies in  $L$ . (This can for example be proven as in the last paragraph of this proof.)

Let  $Q_1\mathcal{T}_{g,n}$  be the bundle of quadratic differentials over  $\mathcal{T}_{g,n}$  of norm less than 1. There is a well known continuous map  $E : Q_1\mathcal{T}_{g,n} \rightarrow \mathcal{T}_{g,n}$  that maps  $(Y, q)$  to the unique Riemann surface  $Y'$  such that there is a Teichmüller mapping  $Y \rightarrow Y'$  with initial quadratic differential  $q$  and stretch factor  $(1 + \|q\|)/(1 - \|q\|)$ . The restriction of  $E$  to the quadratic differentials of norm less than 1 on any fixed Riemann surface is a homeomorphism to  $\mathcal{T}_{g,n}$ . See for example [FM12, Chapter 11] for a review of this material.

Since  $L_k$  is totally geodesic, it contains the image of  $P_k$  under  $E$ . By invariance of domain, this image is a real manifold of real dimension equal to the real dimension of  $P_k$ , so we see that  $L_k$  is equal to the image of  $P_k$ .

The restriction of  $E$  to the preimage in  $Q_1\mathcal{T}_{g,n}$  of any compact subset of  $\mathcal{T}_{g,n}$  (under the standard projection  $Q_1\mathcal{T}_{g,n} \rightarrow \mathcal{T}_{g,n}$ ) is a proper map. Hence we get that  $L$  is the image of  $P$  under  $E$ . By invariance of domain,  $L$  is a real manifold of dimension equal to the real dimension of  $P$ . Since  $L$  is totally geodesic,  $L$  must in fact be a complex submanifold.  $\square$

**Corollary 3.2.** *For every  $(X, q) \in Q\overline{N}$  there is at least one  $d$  dimensional totally geodesic submanifold  $L$  such that the Teichmüller disk generated by  $q$  is contained in  $L$  and  $QL \subset Q\overline{N}$ .*

Note that  $L$  is not assumed to be closed (a priori it may be dense in  $\overline{N}$ ), and it is not assumed to be unique.

**Lemma 3.3.** *Let  $L$  be a totally geodesic submanifold of  $\mathcal{M}_{g,n}$ . We assume  $L$  is complete but not closed. Then  $\Omega L$  is locally a countable union of subsets that are linear in period coordinates.*

Thus, we see that although the immersed manifold  $\Omega L$  may a priori be dense and may intersect itself, each “leaf” of  $\Omega L$  is defined by homogeneous linear equations with real coefficients in period coordinates.

*Proof.* Consider an open subset of  $QL$ , which can be considered as a manifold immersed into the bundle of quadratic differentials over  $\mathcal{M}_{g,n}$ . For each open subset of  $QL$ , its image in  $\Omega L$  is a locally  $GL(2, \mathbb{R})$  invariant submanifold. An argument attributed to Kontsevich [Mö108, Proposition 1.2] gives that this local piece of submanifold must be linear.

Note that [Mö108, Proposition 1.2] requires that the piece of submanifold be analytic. This is automatically true of any totally geodesic submanifold. Indeed, the proof of Lemma 3.1 shows that every totally geodesic submanifold is the image of a particular map  $E$ , which is analytic by the analytic dependence on parameters in the Measurable Riemann Mapping Theorem.  $\square$

Consider the bundle over  $\Omega\bar{N}$  whose fiber over a point  $(Y, \omega)$  consists of all  $2d$ -dimensional subspaces defined over  $\mathbb{R}$  of  $\Omega\bar{N}(Y, \omega)$  that contain  $\omega$ . Let  $R$  be the total space of this bundle, and denote fibers by  $R(Y, \omega)$ .

Consider the subset  $R'$  of  $R$  consisting of those subspaces  $V$  such that the restriction of the map from  $\Omega\bar{N} \rightarrow \mathcal{M}_{g,n}$  to a neighbourhood of  $\omega$  in the subspace  $V$  has derivative of rank at most  $d$  at every point.

**Lemma 3.4.**  *$R'$  is closed.*

*Proof.* This follows directly from the definition, since having rank at most  $d$  is a closed condition.  $\square$

**Lemma 3.5.** *Suppose  $(Y, \omega, V) \in R$  and the image in  $\mathcal{M}_{g,n}$  of a neighborhood of  $\omega$  in  $V$  is an open subset of a totally geodesic submanifold of dimension  $d$ . Then  $(Y, \omega, V) \in R'$ .*

*Proof.* This is immediate from the definition, because a map to a manifold of dimension  $d$  can have rank at most  $d$ .  $\square$

**Lemma 3.6.** *Every fiber of  $R'$  is nonempty.*

*Proof.* This follows from Corollary 3.2 and Lemmas 3.3 and 3.5.  $\square$

Suppose  $(Y, \omega, V) \in R$ , and  $(Y', \omega') = g(Y, \omega)$  for some  $g \in GL(2, \mathbb{R})$ . Since  $V$  is real linear and contains  $\omega$ , and since the relative cohomology class of  $\omega'$  is a linear combination of  $\text{Re}(\omega)$  and  $\text{Im}(\omega)$ , we get

$\omega' \in V$ . Hence  $GL(2, \mathbb{R})$  acts on  $R$  by the usual action on Abelian differentials and parallel transport (the Gauss-Manin connection) on the  $2d$ -dimensional subspaces, and  $R'$  is  $GL(2, \mathbb{R})$  invariant.

**Lemma 3.7.** *Let  $U$  be a connected neighborhood of a point  $p$  in  $\mathbb{C}^a$ , let  $\mathcal{M}$  be a complex manifold, and let  $f : U \rightarrow \mathcal{M}$  be analytic. For any  $k$  and  $d$ , let  $\mathcal{V}_{k,d}(p)$  be the set of  $k$  planes  $V$  through  $p$  such that  $f$  restricted to  $V \cap U$  has derivative of rank at most  $d$  at every point of  $V \cap U$ . Then*

- (1)  $\mathcal{V}_{k,d}(p)$  is a subvariety of the variety of  $k$  planes through  $p$ , and
- (2) in coordinates provided by the Plücker embedding,  $\mathcal{V}_{k,d}(p)$  is defined by a (possibly infinite) set of homogeneous polynomials that vary analytically with  $p \in U$ .

**Corollary 3.8.** *The fibers of  $R'$  are varieties.*

*Proof of Lemma 3.7.* Let  $S$  be the set of  $k$  planes through  $p$  equipped with a choice of basis for the tangent space to the  $k$  plane at  $p$ . It is equivalent to show that the set of  $(V, v_1, \dots, v_k) \in S$  for which  $f$  restricted to  $V \cap U$  has derivative of rank at least  $d+1$  at at least one point of  $V \cap U$  is a Zariski open subset of  $S$ .

We may assume  $\mathcal{M} = \mathbb{C}^b$ . Using the basis  $v_1, \dots, v_k$ , we may consider  $Df$  restricted to  $V \cap U$  as a matrix whose entries are analytic functions on  $U$ . If the derivative of  $f$  restricted to  $V \cap U$  has rank at least  $d+1$  at some point, then there is some  $d+1$  by  $d+1$  minor of this matrix whose determinant  $R$  is not identically zero.

Since  $R$  is nonzero, there is some  $\ell$  so that the  $\ell$ -th multivariate Taylor polynomial  $R_\ell$  of  $R$  centered at  $p$  is also nonzero. Each coefficient of  $R_\ell$  can be viewed as a polynomial function on  $S$ . (This polynomial depends on all partial derivatives of order at most  $\ell$  of  $f$  at  $p$ . Since  $f$  and  $p$  are fixed, all these numbers may be viewed as constants.) Let  $c : S \rightarrow \mathbb{C}$  be one of the nonzero coefficients of  $R_\ell$ .

$(V, v_1, \dots, v_k)$  is contained in the Zariski open set defined by  $c \neq 0$ . On this set,  $R_\ell \neq 0$  and hence  $R \neq 0$ , and hence the rank of the derivative of  $f$  restricted to the  $k$ -plane is at least  $d+1$  at some point. This proves the first statement.

For the second statement, note that  $R_\ell$  is an analytic function of  $p$ . □

**Lemma 3.9.** *For almost every  $(Y, \omega) \in \Omega\bar{N}$ ,  $R'(Y, \omega) \neq R(Y, \omega)$ . Furthermore,  $R'(Y, \omega)$  does not contain the set of all subspaces  $S'$  in  $R(Y, \omega)$  for which  $\ker(p) \cap S'$  has maximal dimension and such that the symplectic form restricted to  $p(S')$  is as degenerate as possible.*



“Maximal dimension” means maximal among all  $S' \in R(Y, \omega)$ . Note that the symplectic form restricted to  $p(S')$  can never be zero, since  $S'$  contains  $\omega$ . That the symplectic form is as close to degenerate as possible is equivalent to the dimension of the largest symplectic subspace of  $p(S')$  being as small as possible among all  $S' \in R(Y, \omega)$ .

*Proof.* First we prove that  $R'(Y, \omega) \neq R(Y, \omega)$  for almost every  $(Y, \omega)$ . Suppose in order to find a contradiction that  $R'(Y, \omega) = R(Y, \omega)$  for a positive measure set of  $(Y, \omega)$ . Since the  $GL(2, \mathbb{R})$  action is ergodic, and since  $R'$  is closed, in fact  $R'(Y, \omega) = R(Y, \omega)$  for every  $(Y, \omega)$  in  $\Omega\bar{N}$ .

Suppose  $\Omega\bar{N}$  has dimension  $2r + b$ , where  $r$  is the rank. By the assumption that  $Q\bar{N}$  has dimension strictly greater than  $2d$ , we get that  $2r + b > 2d$ .

By Theorem 2.1 and Lemma 2.2 there exists  $(Y, \omega) \in \Omega\bar{N}$  and an affine subspace  $S$  of  $\Omega\bar{N}(Y, \omega)$  that contains  $\omega$ , has dimension  $r + b$ , and maps injectively to  $\mathcal{M}_{g,n}$ . Let  $\mathbb{R}S$  be the subspace spanned by  $S$ , so  $\mathbb{R}S$  is a vector subspace of dimension  $r + b + 1$ . (Recall from the statement of Theorem 2.1 that  $S$  is the translate of a certain subspace  $L$  by  $\omega$ . Every element of  $L$  is zero on the core curves of the horizontal cylinders of  $\omega$ , so  $\omega \notin L$  and hence  $0 \notin S$ .)

If  $r + b + 1 \leq 2d$ , extend  $\mathbb{R}S$  to a subspace  $S' \subset \Omega\bar{N}(Y, \omega)$  of dimension  $2d$ . By the Constant Rank Theorem (a corollary of the Inverse Function Theorem), since the map  $S \rightarrow \mathcal{M}_{g,n}$  is injective, the derivative must have rank at least  $r + b$  at some point. But since  $S' \in R'$  by supposition, we get  $r + b \leq d$ . This contradicts  $2r + b > 2d$ .

If  $r + b + 1 \geq 2d$ , pick a subspace  $S'$  of  $\mathbb{R}S$  of dimension  $2d$  and containing  $\omega$ . As above, we get a point where the derivative of the map has rank at least  $2d - 1$ , and since  $S' \in R'$  we get  $2d - 1 \leq d$ . We get a contradiction since  $d > 1$ .

The second statement follows similarly since by Lemma 2.2 we can choose  $S$  so that it contains  $\ker(p)$  and so that  $p(S)$  contains a Lagrangian.  $\square$

We now give the result of [EFW] that we will use, phrased in a way to suit our present purpose. It can be viewed as a black box. Define  $G(Y, \omega)$  to be the subgroup of  $GL(\Omega\bar{N}(Y, \omega))$  that acts trivially on  $\ker(p) \cap \Omega\bar{N}(Y, \omega)$ , preserves the tautological plane  $\text{span}(\text{Re}(\omega), \text{Im}(\omega))$ , and induces a symplectic linear transformation of  $p(\Omega\bar{N}(Y, \omega))$ .

*Remark 3.10.* There exists a basis for  $\Omega\bar{N}(Y, \omega)$  beginning with a basis for  $\ker(p) \cap \Omega\bar{N}(Y, \omega)$  followed by  $\text{Re}(\omega), \text{Im}(\omega)$  with respect to which

$G(Y, \omega)$  can be informally specified as

$$\begin{pmatrix} I & 0 & * \\ 0 & SL(2, \mathbb{R}) & 0 \\ 0 & 0 & Sp(2r - 2, \mathbb{R}) \end{pmatrix},$$

where  $r$  is the rank and  $I$  is an identity matrix.

**Theorem 3.11** (Eskin-Filip-Wright). *Let  $\Omega\bar{N}$  be any affine invariant submanifold, and let  $\mathcal{T}$  be its tangent bundle. Let  $V$  be a measurable equivariant vector subbundle of any tensor power construction of  $\mathcal{T}$  and its dual. Then, for almost every  $(Y, \omega) \in \Omega\bar{N}$ , the fiber  $V(Y, \omega)$  is invariant under  $G(Y, \omega)$ .*

Note by definition  $\Omega\bar{N}(Y, \omega)$  is the fiber of  $\mathcal{T}$  at  $(Y, \omega)$ , and hence any linear transformation of  $\Omega\bar{N}(Y, \omega)$  induces a linear transformation of any tensor power of this vector space or its dual.

**Corollary 3.12.** *Let  $R'(Y, \omega)$  be a subvariety of  $R(Y, \omega)$  for all  $(Y, \omega) \in \Omega\bar{N}$  that is equivariant and that is defined in the Plücker embedding as the set of zeros of a (possibly infinite) set of polynomials that vary analytically. Then at almost every  $(Y, \omega)$ , the fiber  $R'(Y, \omega)$  is invariant under  $G(Y, \omega)$ .*

*Proof of Corollary.* Recall that the Plücker embedding of the Grassmannian of  $2d$  dimensional subspaces in  $\Omega\bar{N}(Y, \omega)$  maps each such subspace to a line in  $\mathbb{P}(\Lambda^{2d}\Omega\bar{N}(Y, \omega))$ . Degree  $D$  homogeneous polynomials on this projective space are elements of the  $D$ -th symmetric power of the dual of  $\Lambda^{2d}\Omega\bar{N}(Y, \omega)$ . Both exterior and symmetric powers of a vector space are subspaces of tensor powers of that vector space.

Let  $R'_D(Y, \omega)$  be the subvariety of  $R(Y, \omega)$  defined by those homogeneous polynomials of degree  $D$  that vanish on  $R'(Y, \omega)$ . On the complement of a invariant analytic subvariety, the span of these polynomials has constant dimension. We thus get that the equivariant subbundle defined by these polynomials is invariant under  $G(Y, \omega)$ , and hence  $R'_D(Y, \omega)$  is invariant under  $G(Y, \omega)$ .

$R'(Y, \omega)$  is the intersection of all the  $R'_D(Y, \omega)$ . The intersection of  $G(Y, \omega)$  invariant sets must be  $G(Y, \omega)$  invariant.  $\square$

**Lemma 3.13.** *Suppose  $R'$  is a nonempty closed  $GL(2, \mathbb{R})$  invariant subset of  $R$  whose fibers are defined by a (possibly infinite) collection of polynomials that vary real analytically. Then every fiber  $R'(Y, \omega)$  must contain all subspaces  $S'$  in  $R(Y, \omega)$  for which  $\ker(p) \cap S'$  has maximal dimension and such that the symplectic form restricted to  $p(S')$  is as degenerate as possible.*

*Proof.* This follows from Corollary 3.12, because any nonempty closed subset of  $R(Y, \omega)$  invariant under  $G(Y, \omega)$  must contain all such subspaces  $S'$ . Since  $R'$  is closed, if this is true almost everywhere then in fact it is true everywhere.  $\square$

*Proof of Theorem 1.1.* As indicated at the beginning of this section, in order to find a contradiction, we assume  $Q\bar{N} \neq QN$  and get that  $Q\bar{N}$  has dimension greater than  $2d$ . By Lemma 3.6 and Corollary 3.8, fibers of  $R'$  are non-empty varieties.

Lemma 3.9 gives that fibers of  $R'$  cannot contain all subspaces  $S'$  in  $R(Y, \omega)$  for which  $\ker(p) \cap S'$  has maximal dimension and such that the symplectic form restricted to  $p(S')$  is as degenerate as possible. This contradicts Lemma 3.13.  $\square$

## REFERENCES

- [AEM17] Artur Avila, Alex Eskin, and Martin Möller, *Symplectic and isometric  $SL(2, \mathbb{R})$ -invariant subbundles of the Hodge bundle*, J. Reine Angew. Math. **732** (2017), 1–20.
- [Ant17a] Stergios M. Antonakoudis, *Isometric disks are holomorphic*, Invent. Math. **207** (2017), no. 3, 1289–1299.
- [Ant17b] ———, *Teichmüller spaces and bounded symmetric domains do not mix isometrically*, Geom. Funct. Anal. **27** (2017), no. 3, 453–465.
- [EFW] Alex Eskin, Simion Filip, and Alex Wright, *The algebraic hull of the Kontsevich-Zorich cocycle*, preprint, arXiv: 1702.02074 (2017), to appear in Ann. Math.
- [EMM15] Alex Eskin, Maryam Mirzakhani, and Amir Mohammadi, *Isolation, equidistribution, and orbit closures for the  $SL(2, \mathbb{R})$  action on moduli space*, Ann. of Math. (2) **182** (2015), no. 2, 673–721.
- [EMMW] Alex Eskin, Curtis McMullen, Ronen Mukamel, and Alex Wright, *Billiards, quadrilaterals and moduli spaces*, preprint.
- [Fil16] Simion Filip, *Splitting mixed Hodge structures over affine invariant manifolds*, Ann. of Math. (2) **183** (2016), no. 2, 681–713.
- [FM12] Benson Farb and Dan Margalit, *A primer on mapping class groups*, Princeton Mathematical Series, vol. 49, Princeton University Press, Princeton, NJ, 2012.
- [Gar77] Frederick P. Gardiner, *The existence of Jenkins-Strebel differentials from Teichmüller theory*, Amer. J. Math. **99** (1977), no. 5, 1097–1104.
- [HM79] John Hubbard and Howard Masur, *Quadratic differentials and foliations*, Acta Math. **142** (1979), no. 3-4, 221–274.
- [Jen57] James A. Jenkins, *On the existence of certain general extremal metrics*, Ann. of Math. (2) **66** (1957), 440–453.
- [Liu04] Jinsong Liu, *On the existence of Jenkins-Strebel differentials*, Bull. London Math. Soc. **36** (2004), no. 3, 365–377.
- [Mas82] Howard Masur, *Interval exchange transformations and measured foliations*, Ann. of Math. (2) **115** (1982), no. 1, 169–200.

- [McM14] Curtis T. McMullen, *Moduli spaces of isoperiodic forms on Riemann surfaces*, Duke Math. J. **163** (2014), no. 12, 2271–2323.
- [MMW17] Curtis T. McMullen, Ronen E. Mukamel, and Alex Wright, *Cubic curves and totally geodesic subvarieties of moduli space*, Ann. of Math. (2) **185** (2017), no. 3, 957–990.
- [Möl08] Martin Möller, *Linear manifolds in the moduli space of one-forms*, Duke Math. J. **144** (2008), no. 3, 447–487.
- [Roy74] H. L. Royden, *Invariant metrics on Teichmüller space*, Contributions to analysis (a collection of papers dedicated to Lipman Bers), Academic Press, New York, 1974, pp. 393–399.
- [Str84] Kurt Strebel, *Quadratic differentials*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 5, Springer-Verlag, Berlin, 1984.
- [Vee82] William A. Veech, *Gauss measures for transformations on the space of interval exchange maps*, Ann. of Math. (2) **115** (1982), no. 1, 201–242.
- [Wol95] Michael Wolf, *On the existence of Jenkins-Strebel differentials using harmonic maps from surfaces to graphs*, Ann. Acad. Sci. Fenn. Ser. A I Math. **20** (1995), no. 2, 269–278.
- [Wri14] Alex Wright, *The field of definition of affine invariant submanifolds of the moduli space of abelian differentials*, Geom. Topol. **18** (2014), no. 3, 1323–1341.
- [Wri15] ———, *Cylinder deformations in orbit closures of translation surfaces*, Geom. Topol. **19** (2015), no. 1, 413–438.