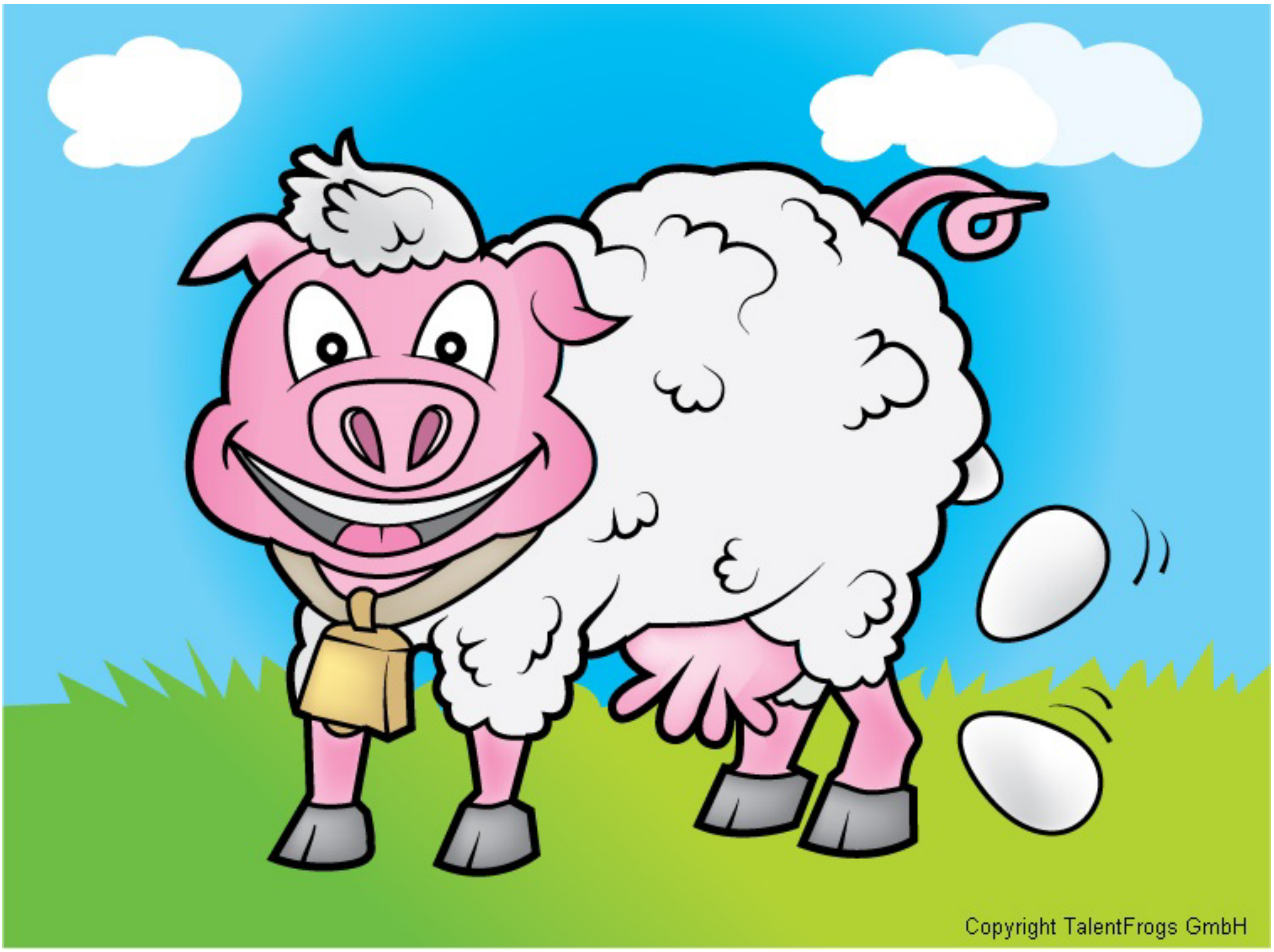
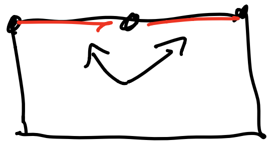


Def A cylinder is *simple* if it has only one saddle connection on each side.

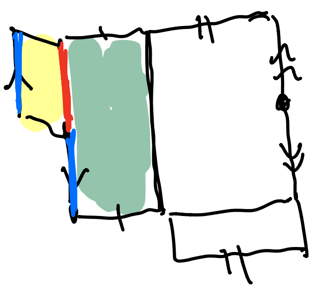
Lemma: On almost every Ab. diff, every cylinder is simple

Hint

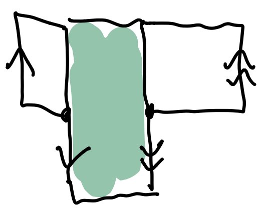


Masur-Zorich: On a typical quad. diff, every cylinder is one of 5 type

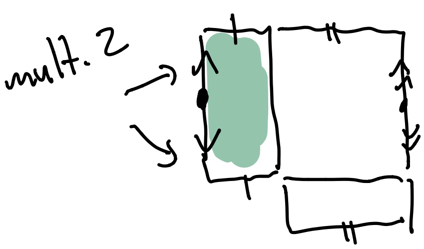
1. simple cylinder
2. half simple cylinder.



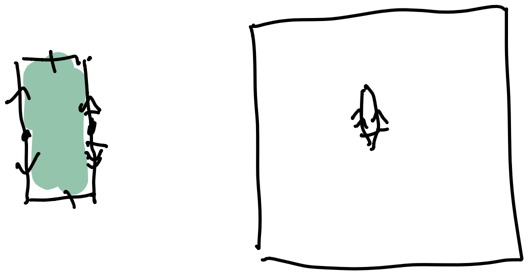
3. complex cylinder



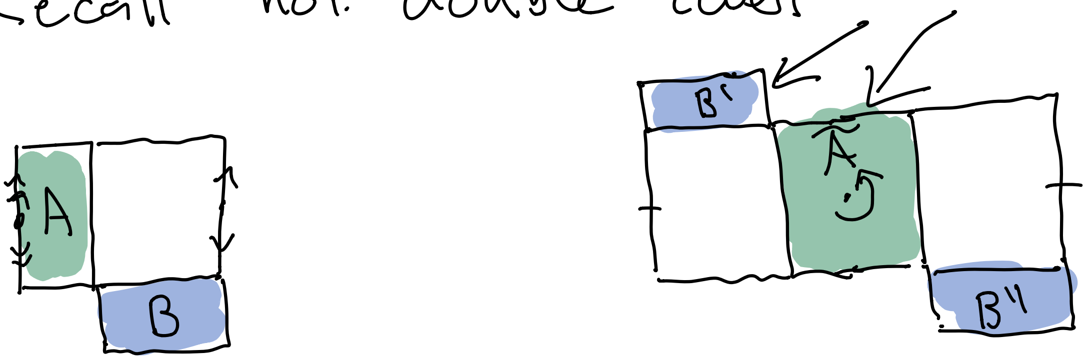
4. simple envelope



5. complex envelope



Recall hol. double covers



can deform $A \longleftrightarrow$ deform the lift $\tilde{A}$ (single cylinder)

deform $B \longleftrightarrow$ deform <u>both</u> lifts B' & B'' equally

---

<u>Def</u> Let C be a cylinder on a surface (X, ω) in an orbit closure $\mathcal{M}$.

- C is <u>$\mathcal{M}$-free</u>, or just <u>free</u>, if deformations of C stay in $\mathcal{M}$
- A pair of cylinders are <u>$\mathcal{M}$-twins</u>, or just <u>twins</u>, if <u>they are isometric on all deformations</u> in $\mathcal{M}$, and

deforming both equally stays in $\mathcal{M}$.

Def Say $\mathcal{M}$ is *geminal* if every cylinder on every surface in $\mathcal{M}$ is free or has a twin.

Thm (Apisa-W) Every geminal orbit closure not consisting of torus covers.

1) a stratum of Ab diffs
2) a double (quad or Ab)

* 3) a full locus of covers of surfaces in a genus 0 stratum of QD.

Proved using diamonds

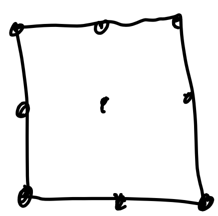
Good news

- boundary is "obviously" geminal.

Bads news

- need version for tori, more complicated
-

$$\begin{tikzcd} & \mathcal{M} \arrow[dl, dash] \arrow[dr, dash] & \\ \mathcal{M}_{c_1} \arrow[dr, dash] & & \mathcal{M}_{c_2} \arrow[dl, dash] \\ & \mathcal{M}_{c_1, c_2} & \end{tikzcd}$$



use "optimal maps"

- need to consider diamonds where M_{a_1} is an Ab. double
  M_{a_2} is a quad double

Say $\mathcal{M}$ is a <u>fake stratum</u> if every cylinder is free, & $\mathcal{M}$ is not a stratum.

<u>Thm</u> (Mirzakhani-W) Fake strata of ab. diffs do not exist.

Counterexamples know for quad. diffs, but only using square tiled surfaces

- Ornythorynque. (Forni-Matheus)
- More Matheus-Yoccoz.

If $\mathcal{F}$ is a fake stratum, the locus $\tilde{\mathcal{F}}$ of hol. double covers is geminal.

Build examples using cyclic covers.

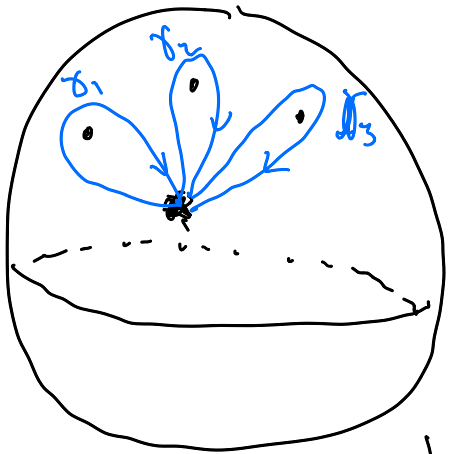
$(X, q) \in Q(k_1, k_2, \dots, k_s)$ $\qquad \sum k_i = 4g-4 = -4.$

Σ = set of singularities.

Fix $l > 1$, $a = (a_1, \dots, a_s) \in \mathbb{Z}/l$ with

① $\sum a_i = 0$

② $\langle a_i \rangle = \mathbb{Z}/\ell$ generate (gcd)



Consider

$$\pi_1(X - \Sigma) \to \mathbb{Z}/\ell$$

$$\gamma_i \longmapsto a_i$$

Let $F(K, a)$ be the locus of all covers

$GL(2,\mathbb{R})$- inv.

Thm (Apisa-W) Suppose that $\forall I \subset \{1, 2, \dots, s\}$ with $\sum_I K_i = -2$, $\gcd(\sum_I a_i, \ell) = 1$

$$\langle \sum_I a_i \rangle = \mathbb{Z}/\ell$$

Then $F(K, a)$ is a fake stratum.

Ex $K = (\underbrace{-1, \dots, -1}_{s-1}, s-5)$ $\quad I = \{i, j\}$

$1 \le i < j \le s-1$

$a = (1, \dots, 1, \ell + 1 - s)$ $\quad a_i + a_j = 2$

ℓ odd.

Note All closed curves on spheres are separating

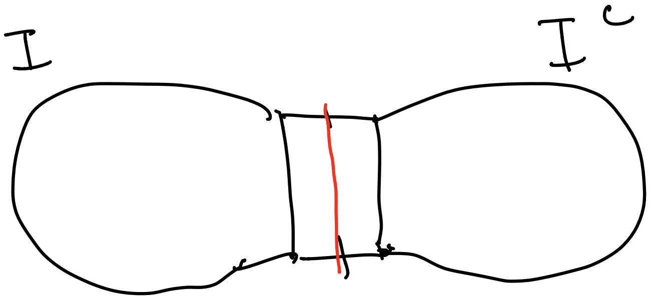
Lemma If $I \subset \{1, \dots, s\}$ corresponds to the set of sing. on one side of a cylinder

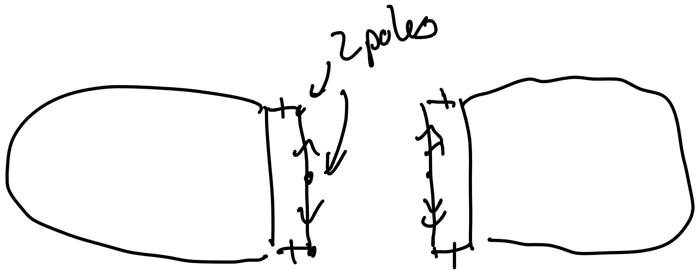
$$\sum_I k_i = -2$$

Note $\sum k_i = -4$

$\Rightarrow \sum_{I^c} k_i = -2$

Pf





$$\sum_{i \in I} k_i - 1 - 1 = -4 \qquad \square$$

Proof If γ is a core curve of a cyl. C $\gamma \longmapsto$ a generator of $\mathbb{Z}/\ell$

$\Rightarrow$ C has only one lift $\tilde{C}$ (ℓ-times the circ)

deforming $C \longleftrightarrow$ deform $\tilde{C}$ $\square$