FULL RANK AFFINE INVARIANT SUBMANIFOLDS

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1. INTRODUCTION

General context. A translation surface is a pair (X, ω), where X is a Riemann surface and ω is a nonzero Abelian differential on X. The moduli space of translation surfaces of fixed genus is the complement of the zero section of the Hodge bundle over the moduli space of Riemann surfaces, and admits a $GL(2, \mathbb{R})$ action. The study of this action grew out of problems in Teichmüller theory, and dynamical systems arising in physics such as billiards in a polygon, and has been enriched in the past decades by connections to ergodic theory on homogeneous spaces and algebraic geometry.

The relation to billiards is given via a procedure called unfolding, which associates a translation surface to each polygon whose angles are rational multiples of π. However, many concrete applications to billiards have been elusive, because the set of unfoldings of polygons is measure zero. Thus, despite the fact that almost every translation surface has dense $GL(2, \mathbb{R})$ orbit, the $GL(2, \mathbb{R})$ orbit closures of unfoldings have remained mysterious.

One application of our main result will be to show that infinitely many triangles unfold to translation surfaces with dense $GL(2, \mathbb{R})$ orbit. Besides polygons all of whose angles are multiples of $\frac{\pi}{2}$ and polygons that unfold to translation surfaces of low genus or with closed orbit, these are the first examples where the orbit closure of the unfolding of a polygon has been computed. Our proof uses new results explained below, and builds upon a wide array of technology that has been developed over the past decades by a large and diverse group of mathematicians.

Recently discovered examples of $GL(2, \mathbb{R})$ orbit closures have lead to the absence even of conjectures on what types of $GL(2, \mathbb{R})$ orbit closures exist. Our intuition is that very large non-trivial orbit closures should not exist, and largeness should be measured by rank, an integer between 1 and the genus. Our main result, which classifies maximal

rank orbit closures, is a first step in this direction. The proof proceeds by inductive arguments, and marked points produced using degenerations are remembered and made use of. We hope these techniques will be applicable in future work.

Statement of main results. The Hodge bundle is stratified according to the number and order of the zeros, and this stratification is preserved by the $GL(2, \mathbb{R})$ action. Following [**EM, EMM15**], orbit closures for the $GL(2, \mathbb{R})$ action are also known as affine invariant submanifolds. We say that an affine invariant submanifold has full rank if its rank has maximal value, that is, the rank is equal to the genus. Rank is defined as one half of the dimension of the image of the tangent space in absolute cohomology (see Section 2). So $\mathcal{M}$ has full rank if and only if the absolute periods of Abelian differentials are locally unconstrained in $\mathcal{M}$.

We prove that non-trivial full rank affine invariant submanifolds do not exist.

Theorem 1.1. *Let $\mathcal{M}$ be a full rank affine invariant submanifold. Then $\mathcal{M}$ is either a connected component of a stratum, or the locus of hyperelliptic translation surfaces therein.*

By work of Filip [**Fil**, Theorem 1.6], this is equivalent to the following.

Theorem 1.2. *If $\mathcal{M}$ is any affine invariant submanifold other than a connected component of a stratum or a hyperelliptic locus, then the Jacobian of every translation surface in $\mathcal{M}$ has non-trivial endomorphisms.*

Theorem 1.1 has applications to unfoldings of polygons.

Theorem 1.3. *Given any polygon all of whose angles are multiples of $\frac{\pi}{3}$, almost every polygon with the same angles unfolds to a translation surface whose orbit is dense in a connected component of a stratum.*

Theorem 1.4. *There are infinitely many rational triangles whose unfoldings have dense $GL(2, \mathbb{R})$ orbit in a connected component of a stratum.*

Theorem 1.4 is in contrast to the fact that unfoldings of triangles have many properties in common with translation surfaces with *closed* $GL(2, \mathbb{R})$ orbits. For example, their Jacobians have a factor with real (and even complex) multiplication, and the difference of any two zeros of the Abelian differential is torsion in the Jacobian, compare to [**Möl06b, Möl06a**].

The greatest common denominator k of the angles divided by π plays a prominent role in the study of the unfolding of a rational polygon. The methods used to prove Theorem 1.4 show that at least 74 percent of the 1436 non-isosceles triangles with k odd and less than 50 unfold to translation surfaces with dense orbit. See Figure B.1 for a list when k is odd and at most 25. When k is even the unfolding has an involution, and in some cases when this involution is hyperelliptic the methods can show the orbit is dense in the hyperelliptic locus.

The proof of Theorem 1.4 also uses work of Filip on variation of Hodge structures, and often gives non-trivial lower bounds for the rank of the orbit closure of the unfolding even in instances where the methods do not show the orbit closure has full rank. (See Theorem 7.5.) Since closed orbits are rank 1 orbit closures, our methods are applicable to the open problem of classifying obtuse triangles that unfold to translation surfaces with closed orbits [**Hoo13, KS00, Puc01**].

A new result on cylinder deformations. Every translation surface has infinitely many cylinders [**Mas86**], and the cylinder deformations introduced in [**Wri15**] can be used to study affine invariant submanifolds. A cylinder on a translation surface in an affine invariant submanifold $\mathcal{M}$ is called $\mathcal{M}$-free if the result of stretching or shearing this cylinder remains in $\mathcal{M}$. The following is a key ingredient in the proof of Theorem 1.1. It says that the only affine invariant submanifolds that are unconstrained in terms of cylinder deformations are connected components of strata.

Theorem 1.5. *Suppose that $\mathcal{M}$ is an affine invariant submanifold for which every cylinder on every surface in $\mathcal{M}$ is $\mathcal{M}$-free. Then $\mathcal{M}$ is a connected component of a stratum.*

This implies in particular that if a surface has no pairs of parallel cylinders then its orbit is dense.

Theorems 1.1 and 1.5 are false for quadratic differentials. For any connected component $\mathcal{Q}$ of a stratum of quadratic differentials, consider the locus $\tilde{\mathcal{Q}}$ of Abelian differentials obtained via double covers of all quadratic differentials in $\mathcal{Q}$. In analogy with Theorems 1.1 and 1.5, one can ask the following questions.

(A) Is every affine invariant submanifold $\mathcal{M} \subset \tilde{\mathcal{Q}}$ with $\operatorname{rank} \mathcal{M} = \operatorname{rank} \tilde{\mathcal{Q}}$ equal to $\tilde{\mathcal{Q}}$?

(B) Suppose $\mathcal{M} \subset \tilde{\mathcal{Q}}$ is an affine invariant submanifold, and every cylinder deformation of every surface in $\mathcal{M}$ that remains in $\tilde{\mathcal{Q}}$ in fact remains in $\mathcal{M}$. Must $\mathcal{M} = \tilde{\mathcal{Q}}$?

Both questions have negative answers. The first will be addressed in forthcoming work of Eskin-McMullen-Mukamel-Wright, which gives examples of unfoldings of polygons with unexpected orbit closures. The existence of hyperelliptic square-tiled surfaces with exactly two cylinders (which are homologous) in any cylinder direction implies a negative answer to the second question [**MY10**, Remark 3.1].

Relation to previous results. McMullen classified orbit closures of translation surfaces in genus 2 [**McM07**]. (One open problem remains concerning square-tiled surfaces.) Apisa classified higher rank (rank larger than 1) orbit closures in hyperelliptic connected components of strata [**Apia**]. In hyperelliptic connected components of strata Theorem 1.1 follows from their work. In strata of Abelian differentials with only one zero, Theorem 1.1 is immediate from the definition of rank.

Rank 2 affine invariant submanifolds in strata of genus 3 Abelian differentials with 2 zeros were classified in [**AN**], and so Theorem 1.1 completes the classification of higher rank affine invariant submanifolds in these strata. Higher rank affine invariant submanifolds of genus 3 Abelian differentials with only one zero were classified in [**NW14, ANW16**]. We hope Theorem 1.1 can play a role in ongoing efforts to classify higher rank affine invariant submanifolds in the remaining strata in genus 3.

Theorems 1.1 and 1.2 were claimed in the special case of strata of Abelian differentials with $2g-2$ simple zeros in [**Möl08**], however there was a gap in the proof.[1]

Athreya-Eskin-Zorich studied unfoldings of right angled polygons, and in particular showed that the generic such unfolding has dense orbit in the hyperelliptic locus [**AEZ**]. Their methods do not apply to polygons whose angles are not all multiples of $\frac{\pi}{2}$, but also give stronger conclusions.

In general, unfoldings of triangles may have rank 1 orbit closure [**Vee89**]. Moreover, at first glance it may seem plausible that all unfoldings of triangles have rank 1 orbit closure. To apply Theorem 1.1 to unfoldings of triangles, we rely on the connection between $GL(2, \mathbb{R})$ orbit closures and Jacobians with extra endomorphisms, due to McMullen, Möller, and Filip [**McM03, Möl06b, Fil**]. We were inspired by McMullen's proof that the locus of eigenforms for real multiplication is not $GL(2, \mathbb{R})$ invariant in genus 3, and we use Filip's recent results on the variation of Hodge structure over an affine invariant submanifold.

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[1]According to Möller, the surjectivity claim in the second paragraph of the proof of [**Möl08**, Theorem 4.1] is not justified.

These results of Filip were a key component of his proof that affine invariant submanifolds are algebraic varieties [**Fil16**].

New orbit closures related to unfoldings of quadrilaterals are given in [**MMW**] and in forthcoming work of Eskin-McMullen-Mukamel-Wright.

Outline of the paper. Section 2 recalls necessary background on cylinder deformations and affine invariant submanifolds.

Theorem 1.1 asserts that an affine invariant submanifold $\mathcal{M}$ of full rank that is not equal to a connected component of a stratum must be the hyperelliptic locus. We assume by induction that this result has been proven for all strata of surfaces with smaller genus or same genus but with fewer zeros. The base case is genus 2.

Section 3 prepares for the proof of Theorem 1.1 by using Theorem 1.5 to show that there is a collection of homologous cylinders on a surface (X, ω) in $\mathcal{M}$ such that not all these cylinders are free. This collection is later shown to consist of exactly two homologous cylinders, so for simplicity we will assume this now. We think of this pair of cylinders as a certificate that $\mathcal{M}$ is not a connected component of a stratum.

Section 4 completes the proof of Theorem 1.1. The two homologous cylinders divide the surface into two halves, and a degeneration supported in each gives rise to two affine invariant submanifolds $\mathcal{M}_1, \mathcal{M}_2$ that lie in the boundary of $\mathcal{M}$ described in [**MW**]. It is not obvious that these $\mathcal{M}_i$ are full rank (see Figure 4.2), but we are able to verify this and moreover show that neither is a non-hyperelliptic connected component of a stratum. The inductive hypothesis then gives that the $\mathcal{M}_i$ are hyperelliptic, and we conclude by combining the hyperelliptic involutions on the two degenerations to create a hyperelliptic involution on $(X, \omega) \in \mathcal{M}$.

Both the proof that the $\mathcal{M}_i$ are not non-hyperelliptic connected component of strata as well as the step which combines the hyperelliptic involutions make use of results of Apisa and Apisa-Wright on marked points. Some technical difficulties complicate the proof of Theorem 1.1, so that the above rough outline of the idea of the proof is not completely accurate.

Section 5 proves Theorem 1.5 by making use of deformations arising from cylinders on different surfaces to obtain horizontally periodic surfaces with maximally many horizontal cylinders.

Section 6 proves that for any tuple of rational angles there is an affine invariant submanifold which is equal to the orbit closure of the

unfolding of a generic polygon with those angles, and establishes Theorem 1.3. Section 7 makes use of variation of Hodge structures to prove Theorem 1.4.

Open problems. In Section 6, we associate an affine invariant submanifold $\mathcal{M}(\boldsymbol{\theta})$ to any tuple $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_n)$ of angles of a rational polygon; this can be defined as the orbit closure of the unfolding of the generic polygon with these angles. One long term goal is to compute these $\mathcal{M}(\boldsymbol{\theta})$ for all $\boldsymbol{\theta}$.

Question 1.6. If $n = 4$, are there infinitely many $\mathcal{M}(\boldsymbol{\theta})$ such that $\boldsymbol{\theta}$ has more than two distinct angles and $\mathcal{M}(\boldsymbol{\theta})$ is neither a connected component of a stratum nor a locus $\tilde{\mathcal{Q}}$ of double covers of all quadratic differentials in a given stratum? If $n > 4$, are there any at all?

In relation to Theorem 1.5 and Question B (the analogue of Theorem 1.5 for quadratic differentials) we have the following.

Question 1.7. Suppose that $\mathcal{M}$ is a higher rank affine invariant submanifold, and any pair of $\mathcal{M}$-parallel cylinders on any $(X, \omega) \in \mathcal{M}$ are homologous. Must $\mathcal{M}$ be a connected component of a stratum or a hyperelliptic locus?

Question 1.8. Suppose $\mathcal{M} \subset \tilde{\mathcal{Q}}$ is a higher rank affine invariant submanifold, and every cylinder deformation of every surface in $\mathcal{M}$ that remains in $\tilde{\mathcal{Q}}$ in fact remains in $\mathcal{M}$. Must $\mathcal{M} = \tilde{\mathcal{Q}}$?

2. Background and notation

Definitions. The stratum $\mathcal{H}(k_1, \ldots, k_s)$ is defined to be the set of translation surfaces (X, ω), where ω has s zeros, of orders $k_1, \ldots, k_s$. Connected components of strata are classified in [**KZ03**]. We allow marked points, which are considered to be zeros of order zero. However, unless otherwise indicated, surfaces in this paper will not have marked points.

An affine invariant submanifold is a properly immersed submanifold of a stratum whose image is given in local period coordinates near any point by a finite union of real linear subspaces.

The rank of an affine invariant submanifold $\mathcal{M}$ is defined as

$$\text{rank}(\mathcal{M}) = \frac{1}{2} \dim_{\mathbb{C}} p(T_{(X,\omega)}\mathcal{M}).$$

Here (X, ω) is any surface in $\mathcal{M}$, and $T_{(X,\omega)}\mathcal{M} \subset H^1(X, \Sigma)$ is the tangent space to $\mathcal{M}$ at (X, ω). As usual, Σ denotes the collection of marked points and zeros of ω, and $p : H^1(X, \Sigma) \to H^1(X)$ is the usual map from relative to absolute cohomology with complex coefficients. Rank is an integer because a result of Avila-Eskin-Möller gives that $p(T_{(X,\omega)}\mathcal{M})$ is symplectic [**AEM**]. Note that $\mathcal{M}$ has full rank if and only if $p(T_{(X,\omega)}\mathcal{M}) = H^1(X)$.

Cylinder deformations. A cylinder on a translation surface is the image of an isometric embedding of $\mathbb{R}/(c\mathbb{Z}) \times (0, h)$ that does not have marked points in its interior and is not a proper subset of any other cylinder. We call c the circumference, h the height, and h/c the modulus.

Let $\mathcal{M}$ be an affine invariant submanifold, and consider a surface $(X, \omega) \in \mathcal{M}$. Two cylinders on (X, ω) are said to be $\mathcal{M}$-parallel if they are parallel and remain so on any sufficiently nearby surface in $\mathcal{M}$. The ratio of circumferences of two $\mathcal{M}$-parallel cylinders remains constant on all nearby surfaces in $\mathcal{M}$. (See [**LNW**] for details on the case when $\mathcal{M}$ has self-crossings.) Indeed, if α_1, α_2 are the core curves and c is the ratio of circumferences, then $\int_{\alpha_1} \omega = c \int_{\alpha_2} \omega$ is one of the linear equations that locally define $\mathcal{M}$.

Define

$$u_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \qquad a_t = \begin{pmatrix} 1 & 0 \\ 0 & e^t \end{pmatrix}.$$

Given a cylinder C on a translation surface (X, ω) with a choice of orientation for the core curve, the cylinder shear (or twist) $u_t^C(X, \omega)$ is defined to be the result of rotating (X, ω) so that C becomes horizontal with core curve oriented in the positive real direction, applying the matrix u_t only to C, and then applying the inverse rotation. The cylinder stretch $a_t^C(X, \omega)$ is defined similarly. Using the Poincare duality isomorphism between $H^1(X, \Sigma)$ and $H_1(X \setminus \Sigma)$, these deformations are dual to scalar multiples of the core curve of C [**MW**, Section 4].

A collection of parallel cylinders is called consistently oriented if an orientation for each of their core curves is chosen and the integral of ω over any two of these curves are positive multiples of each other. If $\mathcal{C} = \{C_1, \ldots, C_k\}$ is a collection of parallel consistently oriented cylinders, we define $u_t^{\mathcal{C}}(X, \omega) = u_t^{C_1} \cdots u_t^{C_k}(X, \omega)$ and $a_t^{\mathcal{C}}(X, \omega) = a_t^{C_1} \cdots a_t^{C_k}(X, \omega)$. The main result of [**Wri15**] is the following.

Theorem 2.1 (Cylinder Deformation Theorem). *If $\mathcal{C}$ is an equivalence class of $\mathcal{M}$-parallel cylinders on (X, ω), then $u_t^{\mathcal{C}} a_s^{\mathcal{C}}(X, \omega) \in \mathcal{M}$ for all $s, t \in \mathbb{R}$.*

In particular, the derivative $u^{\mathcal{C}}$ of $u_t^{\mathcal{C}}(X, \omega)$ lies in $T_{(X,\omega)}\mathcal{M}$. A key ingredient in the proof is the following result of [**SW04**], which builds upon the work of Minsky-Weiss [**MW02**].

Theorem 2.2 (Smillie-Weiss). *Every closed, horocycle flow invariant subset of a stratum contains a horizontally periodic surface.*

Marked points. Degenerating a surface without marked points may produce a surface with marked points. Because of this, the study of orbit closures of surfaces with marked points cannot be separated from the case with no marked points.

Let $\mathcal{M}$ be an affine invariant submanifold of a stratum $\mathcal{H}$. Let $\mathcal{H}^{n*}$ be the stratum of surfaces in $\mathcal{M}$ together with a collection of n distinct marked points, none of them equal to a zero of positive order, and let $\pi : \mathcal{H}^{n*} \to \mathcal{H}$ be the natural forgetful map.

We define an n-point marking over $\mathcal{M}$ to be an affine invariant submanifold $\mathcal{N}$ of the stratum $\mathcal{H}^{*n}$ such that $\pi(\mathcal{N})$ is equal to a dense subset of $\mathcal{M}$, equivalently, $\pi(\mathcal{N})$ is equal to $\mathcal{M}$ minus a finite, possibly empty, union of smaller dimensional affine invariant submanifolds.

The proof of Theorem 1.1 uses the following result of Apisa [**Apia**].

Theorem 2.3 (Apisa). *Suppose that $\mathcal{H}$ is a non-hyperelliptic connected component of a stratum. Any n-point marking $\mathcal{N}$ over $\mathcal{H}$ is given by n unconstrained points, i.e., $\dim \mathcal{N} = \dim \mathcal{H} + n$.*

The following extension of this result, established by Apisa [**Apib**] for hyperelliptic connected components of strata and in general in forthcoming work, states that there are no non-trivial point markings over the hyperelliptic locus.

Theorem 2.4 (Apisa, Apisa-Wright). *Suppose that $\mathcal{N}$ is an n-point marking over the hyperelliptic locus $\mathcal{M}$ in some stratum. Then there are non-negative integers $f, w,$ and v such that $n = f + w + 2v$ and $\mathcal{N}$ is the set of all (X, ω, S) where $|S| = n$, at least w points of S are fixed by the hyperelliptic involution, and at least v pairs of points of S are exchanged by the hyperelliptic involution.*

Note $\dim \mathcal{N} = \dim \mathcal{H} + f + v$, and f of the points of S are unconstrained.

3. Preparation for the proof of Theorem 1.1

The results of this section and the next assume Theorem 1.5, which is proven in Section 5.

Lemma 3.1. *If $\mathcal{M}$ has full rank and $(X, \omega) \in \mathcal{M}$, then two cylinders on (X, ω) are $\mathcal{M}$-parallel if and only if they are homologous.*

Proof. It is trivial that homologous cylinders are $\mathcal{M}$-parallel (for any affine invariant submanifold $\mathcal{M}$).

If α_1 and α_2 are core curves of $\mathcal{M}$-parallel cylinders, then the equation $\int_{\alpha_1} \omega = c \int_{\alpha_2} \omega$ must hold locally on $\mathcal{M}$, where c is the ratio of circumferences [**Wri15**, Lemma 4.7]. However, no non-trivial linear equations on absolute periods hold for a full rank affine invariant submanifold, so α_1 and α_2 must be homologous. □

Lemma 3.2. *Theorem 1.1 holds in genus 2.*

This is a consequence of [**McM07**], and also follows from Theorem 1.5 as follows.

Proof. A genus 2 translation surface never has a pair of homologous cylinders, so the Cylinder Deformation Theorem and Lemma 3.1 imply that every cylinder on every surface is free. □

Remark 3.3. Similarly, Theorem 1.1 follows from Theorem 1.5 in strata where the Abelian differentials have two zeros, both of odd order, because such Abelian differentials may not have homologous cylinders.

Proposition 3.4. *Assume $\mathcal{M}$ is full rank but not a connected component of a stratum. Then there exists $(X, \omega) \in \mathcal{M}$ such that:*

(1) *(X, ω) has a collection $\mathcal{C} = \{C_1, \ldots, C_k\}$ of horizontal homologous cylinders, such that the heights of the cylinders in $\mathcal{C}$ are locally constrained in $\mathcal{M}$ to satisfy a non-trivial linear equation, and such that $\mathcal{C}$ includes any cylinder homologous to C_1;*

(2) *(X, ω) is horizontally periodic, and the core curves of the horizontal cylinders span a Lagrangian subspace of $H_1(X)$;*

(3) *Every connected component of the complement of $\cup_{i=1}^k C_i$ contains a horizontal cylinder that is not homologous to any other cylinder.*

In (1), non-trivial means that not all coefficients are zero.

Remark 3.5. The following notation will be used throughout this section and the next. Recall that a cylinder is an isometric embedding of $\mathbb{R}/(c\mathbb{Z}) \times (0, h)$. This always extends to a map from $\mathbb{R}/(c\mathbb{Z}) \times [0, h]$ to

the surface. If the cylinder is horizontal, the map can be chosen to be locally of the form $(x + c\mathbb{Z}, y) \mapsto x + iy$, where $\omega = d(x + iy)$. The top of a horizontal cylinder is defined to be the image of $\mathbb{R}/(c\mathbb{Z}) \times \{h\}$ and the bottom is defined to be the image of $\mathbb{R}/(c\mathbb{Z}) \times \{0\}$.

Given any collection of k pairwise homologous disjoint simple closed curves on a surface, none of which is null-homologous, removing these curves gives k connected components, each with exactly two boundary circles.

We will index the C_i so that the connected component of the complement of $\cup_{i=1}^{k} C_i$ that shares a boundary saddle connection with the top of C_i also shares a boundary saddle connection with the bottom of C_{i+1}. We will adopt the convention that $C_0 = C_k$ and $C_{k+1} = C_1$. For each $i = 1, \ldots, k$, we will pick once and for all one horizontal cylinder D_i in the connected component of the complement of $\cup_{i=1}^{k} C_i$ that shares boundary with the top of C_i, such that D_i is not homologous to any other cylinder.

Lemma 3.6. *Let (X', ω') be a surface in an affine invariant submanifold $\mathcal{M}$ and let $\mathcal{D}$ be the set of horizontal cylinders on (X', ω'). Then there is a path in $\mathcal{M}$ from (X', ω') to a surface (X, ω) such that all the cylinders in $\mathcal{D}$ persist and remain horizontal along this path, and such that (X, ω) is horizontally periodic and the core curves of the horizontal cylinders span a subspace of $H_1(X)$ of dimension at least $\mathrm{rank}(\mathcal{M})$.*

Proof. See [**Wri15**, Section 8]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Proof of Proposition 3.4. By Theorem 1.5, there is a surface $(X', \omega') \in \mathcal{M}$ with a cylinder C_1 that is not free. Let $\mathcal{C}'$ be the set of cylinders on (X', ω') homologous to C_1.

Rotating the surface, we can assume the cylinders of $\mathcal{C}'$ are horizontal. Let (X, ω) denote a surface as in Lemma 3.6, and let $\mathcal{C}$ be the set of all cylinders on (X, ω) homologous to cylinders that persist from cylinders in $\mathcal{C}'$ on (X', ω').

We first claim that not every cylinder in $\mathcal{C}$ is free. Otherwise, for each cylinder C' in $\mathcal{C}'$, the corresponding cylinder C in $\mathcal{C}$ is free. Hence the Poincare dual of the core curve of C is in $T_{(X,\omega)}\mathcal{M}$. Parallel transporting this along the path to (X', ω') gives that the Poincare dual of the core curve of C' is in $T_{(X',\omega')}\mathcal{M}$, and hence that C' is free. Thus, because not every cylinder in $\mathcal{C}'$ is free, not every cylinder in $\mathcal{C}$ is free.

Because the cylinders of $\mathcal{C}$ are not all free, [**Wri15**, Corollary 3.4] gives that some linear equation on the moduli of the cylinders of $\mathcal{C}$ holds on all surfaces sufficiently close to (X, ω). See also [**MW**, Lemma 4.6].

Since the cylinders of $\mathcal{C}$ all have equal circumference, this gives a linear equation on the heights.

Since $\mathcal{M}$ has full rank, this establishes the first two statements of the proposition. We will now show that there is a cylinder in every connected component of the complement of $\cup_{i=1}^{k} C_i$. Otherwise, because (X, ω) is horizontally periodic, the top of a cylinder of $\mathcal{C}$, say C_1, must be glued to the bottom of another cylinder of $\mathcal{C}$, say C_2, as in Figure 3.1.
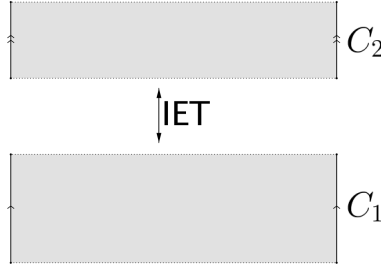

FIGURE 3.1. Proof of Proposition 3.4.

The region above a core curve of C_1 and below a core curve of C_2 is a subsurface of positive genus, which contradicts the fact that the core curves of the horizontal cylinders span a Lagrangian subspace.

Thus there is a horizontal cylinder D_i in each connected component.

We must now show that we can pick D_i to be not homologous to any other horizontal cylinder. If the given connected component of the complement of $\cup_{i=1}^{k} C_i$ does not contain homologous cylinders, we are done. Otherwise, pick E_1, E_2 homologous cylinders in this connected component, such that the connected component of the complement of $E_1 \cup E_2$ that does not contain the C_i does not contain any pair of homologous horizontal cylinders. (That is, pick E_1 and E_2 to be "innermost", or "as close together as possible".) The above argument shows that the top of E_1 cannot be glued as in Figure 3.1 to the bottom of E_2 or vice versa, and hence we can find a cylinder D_i in the connected component of the complement of $E_1 \cup E_2$ that does not contain the C_i. This D_i cannot be homologous to any other horizontal cylinder. $\square$

4. Proof of Theorem 1.1

We will prove Theorem 1.1 by induction on genus and the dimension of the stratum. The base case is given by Lemma 3.2.

That is, we will consider a full rank affine invariant submanifold $\mathcal{M}$ of genus $g > 2$ surfaces, and we will assume $\mathcal{M}$ is not a connected component of a stratum. By induction, we assume the following.

Assumption 4.1. *Theorem 1.1 is true in genus at least 2 and strictly less than g, as well as in all strata of genus g surfaces where the Abelian differentials have strictly fewer zeros than those in $\mathcal{M}$.*

Our goal is to prove that $\mathcal{M}$ is a hyperelliptic locus. Throughout the proof, we will refer to the surface $(X, \omega) \in \mathcal{M}$ produced by Proposition 3.4 and the notation given in Remark 3.5. We will assume that (X, ω) has two additional properties.

Assumption 4.2. *The relative cohomology class of ω does not sit in any proper subspace of $T_{(X,\omega)}\mathcal{M}$ defined over $\overline{\mathbb{Q}} \cap \mathbb{R}$.*

Assumption 4.3. *Each D_i contains a vertical saddle connection.*

The first assumption can be arranged by generically perturbing the real parts of the period coordinates of (X, ω), and the second assumption, which is made solely for concreteness of exposition, can be arranged by additionally performing cylinder deformations.

Let (X_i, ω_i, S_i) denote the limit of $a_t^{D_i}(X, \omega)$ as $t \to -\infty$ in the partial compactification of strata discussed in [**MW**]. This "what you see is what you get" partial compactification can be described formally by taking limits in Deligne-Mumford with all zeros marked, and then deleting zero area components and separating nodes. It should be contrasted with larger compactifications, see for example [**BCG**⁺] and the references therein.

In other words, for each i, the surface (X_i, ω_i, S_i) is obtained by vertically "squashing" D_i, i.e. letting the height go to zero while keeping the circumference constant. Because we have assumed that each D_i contains a vertical saddle connection, this results in the collision of zeros.

Let (X_i, ω_i) be the same limit surface with marked points forgotten. On (X_i, ω_i, S_i) all the cylinders in $\mathcal{C}$, as well as all $D_j, j \neq i$, persist and have the same area and circumference as on (X, ω). Note also that squashing D_i does not create new horizontal cylinders, because squashing D_i does not change the horizontal straight line flow outside of D_i. Hence, because of Assumption 4.2, squashing D_i does not create any new cylinders homologous to those in $\mathcal{C}$.

Lemma 4.4. *On each (X_i, ω_i) there is a unique component that contains all the cylinders in $\mathcal{C}$ as well as all $D_j, j \neq i$.*

Proof. (X, ω) minus the core curves of C_i and C_{i+1} has two connected components, one of which contains all $D_j, j \neq i$, as well as all $C_j, j \neq i, i+1$ and "half" of the two cylinders C_i and C_{i+1} that were cut in two by the removal of their core curves. Thus all the cylinders in question
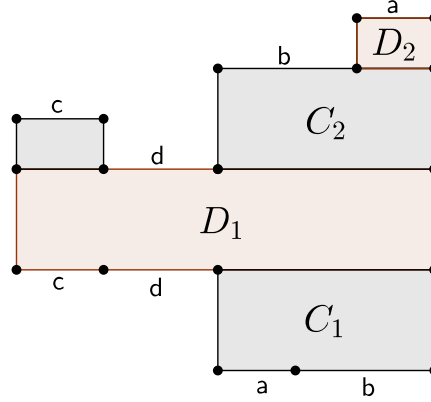
FIGURE 4.1. In this example, if D_1 is squashed, the surface becomes disconnected. (One connected component is the horizontal cylinder bounded above by c, and the other component is $C_1 \cup C_2 \cup D_2$.) If either D_1 or D_2 are squashed then C_1 and C_2 "merge" to form a larger cylinder.

other than D_i are contained in a connected set disjoint from D_i, and so will remain in a connected set after squashing D_i. $\square$

Let Σ denote the set of zeros of ω, and let $V_i \subset H_1(X, \Sigma)$ denote the kernel of the induced map on relative homology of the natural collapse map from X to X_i (see [**MW**, Section 9] for a formal definition). Note that V_i always includes all vertical saddle connections in D_i. (In Figure 4.1, if D_1 is squashed, V_1 additionally includes the saddle connection labelled d. Indeed, (X_1, ω_1) consists of one component corresponding to the horizontal cylinder below c, and one component corresponding to $C_1 \cup C_2 \cup D_2$, glued together at a point, and the collapse map sends d to this point. Note that the two components may be separated by "ungluing" at the point, and the map remains well defined on relative cohomology.)

The tangent space to the stratum of (X_i, ω_i, S_i) can be naturally identified with $\mathrm{Ann}(V_i) \subset H^1(X, \Sigma)$, the space of cohomology classes that are zero on all relative homology classes in V_i [**MW**, Section 9]. The space $\mathrm{Ann}(V_i) \cap T_{(X,\omega)}\mathcal{M}$ can be thought of as the space of deformations of (X, ω) that stay in $\mathcal{M}$ and do not change the part of the surface that is crushed by squashing D_i. This space should be

the tangent space to the component of the boundary of $\mathcal{M}$ that contains (X_i, ω_i, S_i), and this has been verified after projecting to each connected component [**MW**, Theorem 2.9].

Lemma 4.5. *Let π_i be the map that forgets all but the component of (X_i, ω_i) given by Lemma 4.4. Then $\pi_i(X_i, \omega_i, S_i)$ is contained in an affine invariant submanifold $\mathcal{M}_i^*$ whose tangent space at $\pi_i(X_i, \omega_i, S_i)$ is given by $(\pi_i)_*(\mathrm{Ann}(V_i) \cap T_{(X,\omega)}\mathcal{M})$.*

$\pi_i(X_i, \omega_i)$ is contained in an affine invariant submanifold $\mathcal{M}_i$ whose tangent space at $\pi_i(X_i, \omega_i)$ is given by the image of $T_{\pi_i(X_i, \omega_i, S_i)}\mathcal{M}_i^*$ under the forgetful map.

If Σ_i is the set of zeros of ω_i, the forgetful map is the restriction map from linear functionals on $H_1(X, \Sigma_i \cup S_i)$ to linear functionals on $H_1(X, \Sigma_i) \subset H_1(X, \Sigma_i \cup S_i)$. One may define $(\pi_i)_* : \mathrm{Ann}(V_i) \to H^1(X_i, S_i)$ to be the inverse of the isomorphism $\pi_i^* : H^1(X_i, S_i) \to \mathrm{Ann}(V_i) \subset H^1(X, \Sigma)$.

Proof. The first statement is an application of [**MW**, Theorem 2.9]. This second statement is a corollary of the first by [**LMW16**, Lemma 6]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Lemma 4.6. *Each $\mathcal{M}_i$ has full rank.*

$$\int_\alpha \omega = 2 \int_\beta \omega \qquad\qquad\qquad \int_\alpha \omega = 2 \int_\beta \omega$$
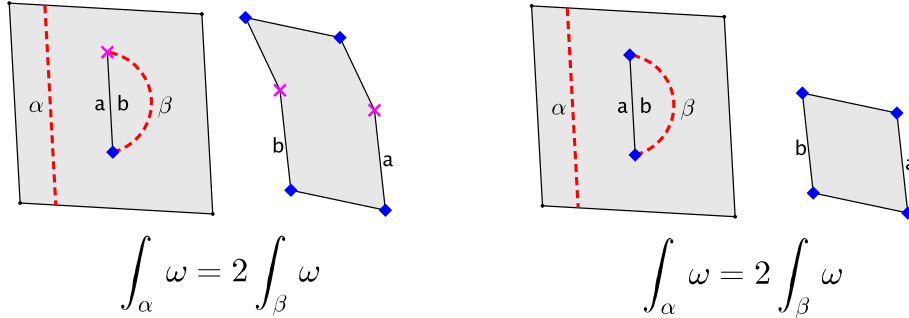
FIGURE 4.2. Left: A surface in $\mathcal{H}(1,1)$ with an equation on relative periods. Right: A degeneration of the surface to $\mathcal{H}(2)$.

Remark 4.7. The cautionary example illustrated in Figure 4.2 shows that outside of the context of affine invariant submanifolds, linear equations on periods that do not restrict absolute periods can in fact restrict absolute periods after a degeneration.
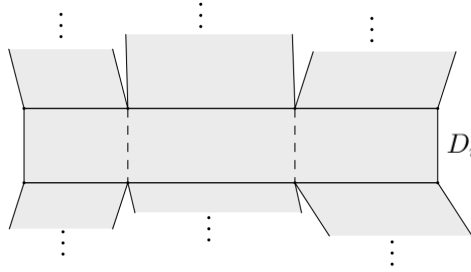
FIGURE 4.3. This figure illustrates one situation that is avoided by Assumption 4.2. In this highly ungeneric picture, the cylinders above D_i have exactly the same circumferences as those below D_i. If D_i is vertically squashed, the genus of the degeneration is 2 less than that of the original surface.

Proof. By the second statement of Proposition 3.4, there are horizontal cylinders $F_1, \ldots, F_{g-1}$ so that $F_1, \ldots, F_{g-1}, D_i$ are disjoint and have core curves that are linearly independent in homology. Let $\mathcal{F}_j$ denote the set of all cylinders homologous to F_j. (The example in Figure 4.1 shows that it is not always possible to assume $\mathcal{F}_j = \{F_j\}$.)

We consider the twists $u^{\mathcal{F}_j}, j = 1, \ldots, g-1$ and their projections via the map $\pi_i : X \to X_i$. The $u^{\mathcal{F}_j}$ are symplectically orthogonal, and remain so after applying π_i, because they are Poincare dual to disjoint curves. Furthermore, the core curves of the F_j are homologically independent on (X_i, ω_i) since by Assumption 4.2 their periods are linearly independent over $\mathbb{Q}$. Hence the $\pi_i(u^{\mathcal{F}_j}), j = 1, \ldots, g-1$ give an isotropic subspace of dimension $g-1$ in the absolute cohomology of (X_i, ω_i).

If (X_i, ω_i) has genus at most $g-1$, then this gives a Lagrangian subspace, and so $\mathcal{M}_i$ has full rank.

So suppose the genus of (X_i, ω_i) is g. Hence no zero appears on both the top and bottom of D_i, and $\mathrm{Ann}(V_i)$ is spanned by the vertical saddle connections in D_i.

Because $\mathcal{M}$ has full rank, there is a real cohomology class $v \in T_{(X,\omega)}\mathcal{M}$ that is nonzero on the core curve of D_i and that is zero on the core curves of the $F_j, j = 1, \ldots, g-1$. By Assumption 4.2, the vertical saddle connections in D_i must all give rise to parallel saddle connections on the surface obtained by adding a small real multiple of v to the period coordinates of (X, ω). Hence we may add to v a multiple of u^{D_i} to assume that v is zero on all vertical saddle connections in D_i, and conclude that $v \in \mathrm{Ann}(V_i) \cap T_{(X,\omega)}\mathcal{M}$.

To show that $\mathcal{M}_i$ has full rank, it now suffices to show that the twists in $\mathcal{F}_j$ together with v spans a Lagrangian subspace.

Let $\alpha_j, j = 1, \ldots, g$ be the core curves of $F_1, \ldots, F_{g-1}, D_i$, and pick β_j so $\{\alpha_j, \beta_j\}$ is a symplectic basis of $H_1(X, \mathbb{Z})$. The symplectic pairing on cohomology classes $v, w \in H^1(X, \mathbb{R})$ is given by

$$\sum_{j=1}^{g} v(\alpha_j)w(\beta_j) - v(\beta_j)w(\alpha_j).$$

The twist in $\mathcal{F}_j$ is nonzero on β_j and zero on all other elements of the basis. The deformation v is zero on all $\alpha_j, j < g$, so v is symplectically orthogonal to the twist in each $\mathcal{F}_j$. $\qquad \square$

Lemma 4.8. *Each $\mathcal{M}_i$ is contained in a stratum of surfaces of genus at least 2.*

Proof. D_{i+1} and a cylinder from $\mathcal{C}$ are disjoint on $\pi_i(X_i, \omega_i)$, and their core curves have different holonomy by Assumption 4.2. $\qquad \square$

Lemma 4.9. *Each $\mathcal{M}_i$ is a hyperelliptic locus or a hyperelliptic connected component of a stratum.*

Proof. By the inductive hypothesis and Lemma 4.6, it suffices to show that each $\mathcal{M}_i$ is not a connected component of a non-hyperelliptic stratum.

A cylinder on an unmarked surface that contains a marked point in its interior is considered to be two cylinders on the marked surface. Hence the cylinders of $\mathcal{C}$ remain well defined on (X_i, ω_i, S_i). However it is possible that some cylinders in $\mathcal{C}$ "merge" on (X_i, ω_i), as in Figure 4.1, because the marked points are forgotten.

If none of the cylinders in $\mathcal{C}$ merge on (X_i, ω_i), then because their moduli must remain related, $\mathcal{M}_i$ cannot be a stratum.

If to the contrary C_i and C_{i+1} merge to create a single larger cylinder on (X_i, ω_i), then there must be marked points separating C_i and C_{i+1} on (X_i, ω_i, S_i).

The heights of the cylinders in $\mathcal{C}$ on (X_i, ω_i, S_i) satisfy some linear equation. Let γ_i be the core curve of C_i. Consider the relative cohomology class Poincare dual to $\gamma_i - \gamma_{i+1}$, which is a multiple of the derivative of a deformation which increases the height of C_i and decreases the height of C_{i+1}. We now consider two cases.

In the first case, suppose the dual of $\gamma_i - \gamma_{i+1}$ is in $T\mathcal{M}_i^*$. In this case the equation on the heights of the cylinders in $\mathcal{C}$ on (X_i, ω_i, S_i) depends only on the sum of the heights of C_i and C_{i+1} and not on the individual heights of C_i and C_{i+1}, so when the marked points are forgotten there

is still a linear equation that holds on the heights. This shows $\mathcal{M}_i$ is not connected component of a stratum.

In the second case, suppose the dual of $\gamma_i - \gamma_{i+1}$ is not in $T\mathcal{M}_i^*$. This implies that the marked points separating C_i from C_{i+1} are not all unconstrained, since if they were all unconstrained then moving them all upwards at a constant rate would realize the deformation dual to $\gamma_i - \gamma_{i+1}$. So by Theorem 2.3 we conclude that $\mathcal{M}_i$ is not a non-hyperelliptic connected component of a stratum. □

Lemma 4.10. *If a translation surface (Y, η) is hyperelliptic, then each cylinder is homologous to at most one other. Homologous pairs of cylinders are exchanged by the hyperelliptic involution.*

Proof. Say $\mathcal{C}$ is a collection of homologous cylinders. Let ι be the hyperelliptic involution. Because ι acts as -1 on homology, each of the components of (Y, η) minus the core curves of $\mathcal{C}$ must be preserved, and ι must interchange the two boundary components. □

Lemma 4.11. *The number k of cylinders in $\mathcal{C}$ is 2.*

Proof. Lemmas 4.9 and 4.10 imply that (X_i, ω_i) has at most two homologous cylinders. Since at most two cylinders (namely, C_i and C_{i+1}) can merge on (X_i, ω_i), this shows $k \le 3$. Furthermore, for this same reason exactly two cylinders must always merge on (X_i, ω_i) for each i.

Suppose $k = 3$, and suppose without loss of generality that the area of $C_1 \cup C_2$ is strictly larger than the area of C_3. On (X_i, ω_i), the cylinders C_1 and C_2 merge to give a single cylinder homologous to C_3 but of larger area. This contradicts Lemmas 4.9 and 4.10, so we conclude that $k = 2$. □

Lemma 4.12. *The set of marked points on every surface in $\mathcal{M}_i^*$, $i = 1, 2$, is preserved by the hyperelliptic involution.*

Proof. By Theorem 2.4, it suffices to show that $\mathcal{M}_i^*$ does not have unconstrained marked points.

Suppose to the contrary that $\mathcal{M}_1^*$ has unconstrained marked points. Then so must the boundary affine invariant submanifold obtained by squashing D_2, forgetting marked points, and then squashing D_1 and remembering marked points (and in each case applying the projection to the component containing the cylinders in $\mathcal{C}$). However, since $\mathcal{M}_2$ is hyperelliptic, the second step cannot produce unconstrained marked points, because the set of marked points must be invariant under the hyperelliptic involution.

Squashing D_1 and remembering marked points, and then squashing D_2 and forgetting new marked points, gives the same affine invariant

submanifold as doing it in the other order. This is true since the boundary affine invariant submanifolds are locally the set of limits of degenerations [**MW**, Sections 7, 9], and in both cases the set of limits is equal to the result of squashing both D_1 and D_2, projecting to a connected component, and forgetting the marked points on the D_2 side of $C_1 \cup C_2$.   □

Lemma 4.13. $\mathcal{M}$ *is contained in the hyperelliptic locus.*
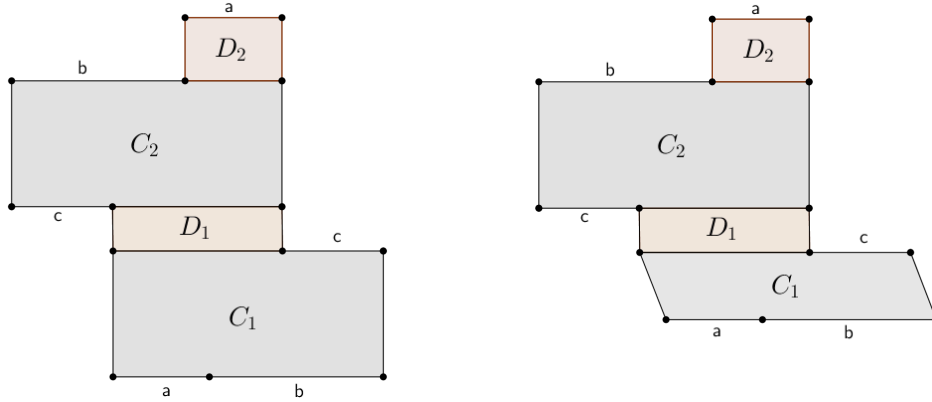

FIGURE 4.4. Left: A hyperelliptic surface. Right: deforming C_1 gives a surface without an involution.

Remark 4.14. The cautionary example illustrated on the right of Figure 4.4 shows that outside of the context of affine invariant submanifolds, a non-hyperelliptic surface with cylinders D_1, D_2 may become hyperelliptic after either D_i is squashed and marked points are forgotten.

Proof. By Lemmas 4.9 and 4.12, both $\pi_1(X_1, \omega_1, S_1)$ and $\pi_2(X_2, \omega_2, S_2)$ have involutions τ_i respecting marked points.

For $i = 1, 2$, let R_i be $\overline{C_1 \cup C_2}$, union the connected component of $X \setminus (C_1 \cup C_2)$ that contains D_{i+1}. τ_i induces an involution on R_i that preserves zeros of ω.

By Assumption 4.2, the saddle connections on the bottom of C_2 have distinct lengths. Since any involution of $\overline{C_1 \cup C_2}$ must send each saddle connection on the top of C_1 to a saddle connection on the bottom of C_2 of the same length, and since there is a unique such saddle connection on the bottom of C_2, $\overline{C_1 \cup C_2}$ has at most one involution exchanging C_1 and C_2 and preserving the set of zeros of ω. Hence $\tau_1|_{R_1}$ and $\tau_2|_{R_2}$ agree on $R_1 \cap R_2 = \overline{C_1 \cup C_2}$, and we can glue these involutions to get an involution τ on (X, ω).

τ is a hyperelliptic involution since X/τ has genus 0. Indeed, X/τ can be obtained by gluing together the two genus 0 surfaces obtained as the quotient of R_i by $\tau_i|_{R_i}$.

By [**Wri14**, Corollary 1.3] and Assumption 4.2, (X, ω) has dense orbit in $\mathcal{M}$. Hence every surface in $\mathcal{M}$ is hyperelliptic. $\square$

Lemma 4.15. $\mathcal{M}$ *is equal to the hyperelliptic locus in the ambient stratum.*

Proof. It remains only to show that $\mathcal{M}$ is not properly contained in the hyperelliptic locus. Let $v \in H^1(X, \Sigma)$ be supported on one component of X minus the core curves of C_1 and C_2 and suppose v is contained in the -1 eigenspace for the hyperelliptic involution. By Lemma 4.9 and Lemma 4.5, $v \in T\mathcal{M}$.

Consider a generic hyperelliptic surface $(X, \omega) \in \mathcal{M}$ with a pair of homologous cylinders C_1 and C_2, and let (X', ω') be any nearby hyperelliptic surface. Then (X', ω') can be obtained from (X, ω) as follows. First, apply $g \in GL(2, \mathbb{R})$ to get $g(X, \omega)$, where g is chosen so there is an affine map with derivative 1 taking $\overline{C_1 \cup C_2}$ on $g(X, \omega)$ to $\overline{C_1 \cup C_2}$ on (X', ω') and that maps zeros to zeros. (Note that since (X, ω) is generic, C_1 and C_2 are simple cylinders.) Then, apply a deformation supported on one side of $g(X, \omega) \setminus (C_1 \cup C_2)$, and then a deformation supported on the other side, to obtain (X', ω').

This shows that the deformations from the first paragraph, together with $\mathrm{Re}(\omega)$ and $\mathrm{Im}(\omega)$, span the tangent space to the hyperelliptic locus. $\square$

This concludes the proof of Theorem 1.1.

5. Proof of Theorem 1.5

A digraph is a pair $\Gamma = (V, E)$, where V is a finite set of vertices, and E is a finite multi-set of ordered pairs $(x, y) \in V \times V$. An element $(x, y) \in E$ is called a directed edge from x to y. We say that x is the tail of (x, y) and that y is the tip. A directed graph is allowed to have multiple edges between two vertices or from a vertex to itself.

A directed path in a digraph is a nonempty sequence of edges such that the tip of each is the tail of the next. The path is said to go from the tail of the first edge to the tip of the last. A directed loop is a directed path from a point to itself.

A digraph $\Gamma(V, E)$ is called strongly connected if, for all $x, y \in V$, there is a directed path from x to y and a directed path from y to x.

A directed loop is said to be embedded if it passes through each edge and each vertex at most once. Let $\mathbb{Q}^E$ denote the vector space with

basis E. Each directed loop gives an element of $\mathbb{Q}^E$ by taking the sum of edges that appear in the loop. Define the loop space $L(\Gamma) \subset \mathbb{Q}^E$ to be the span of all directed embedded loops in $\mathbb{Q}^E$.

Lemma 5.1. *If Γ is strongly connected, then* $\dim L(\Gamma) = |E| - |V| + 1$.

Compare to [**BJG01**, Theorem 10.1.4]; since we haven't been able to find a precise reference, we provide a proof.

Proof. We induct on $|V|$. If $|V| = 1$, the result is obvious. So assume $|V| > 1$.

Let ℓ be an embedded directed loop in Γ, and let $\Gamma' = (V', E')$ be the digraph resulting from collapsing ℓ to a point. One can check that Γ' is strongly connected.

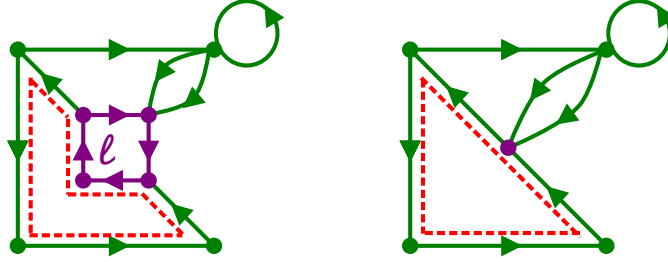FIGURE 5.1. Left: A digraph Γ. Right: Contracting a loop ℓ in Γ gives Γ'. The dashed line on the right is an embedded directed loop; and on the left, its lift.

An embedded loop in Γ does not necessarily map to an embedded loop in Γ', because it may pass through a vertex more than once. However, the image of an embedded loop will map to a sum of embedded loops in $L(\Gamma')$. Hence we get a map $L(\Gamma) \to L(\Gamma')$.

Each embedded loop in Γ' lifts to an embedded directed loop in Γ, so in fact there is a short exact sequence

$$0 \to \mathbb{Q} \cdot \ell \to L(\Gamma) \to L(\Gamma') \to 0.$$

If ℓ has k edges, then $|V'| = |V| - k + 1$ and $|E'| = |E| - k$, and so the result follows by induction. $\square$

Let (X, ω) be a horizontally periodic translation surface. We define its cylinder digraph to be the digraph with one vertex for each horizontal cylinder, and one directed edge for each horizontal saddle connection. The directed edge associated to a horizontal saddle connection goes from the cylinder below the saddle connection to the cylinder above it. The cylinder digraph is strongly connected.
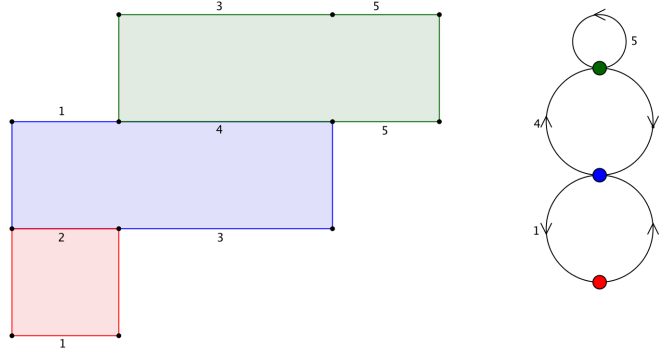
FIGURE 5.2. A horizontally periodic surface and its cylinder digraph. The colors and labels are for illustrational purposes only.

Given a horizontally periodic surface $(X, \omega) \in \mathcal{M}$, the cylinder preserving space is defined in [**Wri15**] to be the subspace of $T\mathcal{M}$ that is zero on core curves of cylinders, and the twist space of (X, ω) is the subspace of $T\mathcal{M}$ that is zero on all horizontal saddle connections. The following summarizes [**Wri15**, Lemma 8.6, Corollary 8.11].

Lemma 5.2. *Let $(X, \omega) \in \mathcal{M}$ be horizontally periodic. The cylinder preserving space has codimension at most* $\operatorname{rank} \mathcal{M}$ *in $T\mathcal{M}$. If the twist space is not equal to the cylinder preserving space, then there is a horizontally periodic surface in $\mathcal{M}$ with more horizontal cylinders than* (X, ω).

Lemma 5.3. *Let $\mathcal{M}$ be an affine invariant submanifold as in Theorem 1.5. Let $(X, \omega) \in \mathcal{M}$ be horizontally periodic in $\mathcal{M}$, and let Γ be the cylinder digraph. If the cylinder preserving space is equal to the twist space, then* $\dim L(\Gamma) \leq \operatorname{rank} \mathcal{M}$.

Proof. For each embedded simple loop ℓ in the cylinder digraph, we may twist the horizontal cylinders of (X, ω) to obtain a surface with a vertical cylinder that passes through exactly the saddle connections and cylinders of ℓ. Stretching this vertical cylinder gives an element of $T\mathcal{M}$ which is equal to a nonzero constant on the horizontal saddle connections of ℓ, and is zero on on all other horizontal saddle connections. The cohomology classes obtained in this way from different loops ℓ are linearly independent, so their span $L \subset T\mathcal{M}$ has dimension $\dim L(\Gamma)$.

Since the cylinder preserving space has codimension at most $\operatorname{rank} \mathcal{M}$, any subspace of $T\mathcal{M}$ of dimension strictly greater than $\operatorname{rank} \mathcal{M}$ must have non-trivial intersection with it. Thus, if $\dim L(\Gamma) > \operatorname{rank} \mathcal{M}$, some nonzero element of L is contained in the cylinder preserving space.

Since every nonzero element of L is nonzero on some horizontal saddle connection, no nonzero element of L is in the twist space. We conclude that if $\dim L(\Gamma) > \operatorname{rank} \mathcal{M}$ the cylinder preserving space is strictly larger than the twist space. $\qquad\square$

Lemma 5.4. *Suppose $\mathcal{M}$ is an affine invariant submanifold of genus g Abelian differentials with s distinct zeros. If $\mathcal{M}$ contains a surface with $g + s - 1$ horizontal free cylinders, then $\mathcal{M}$ is a connected component of a stratum.*

Proof. The twists in the cylinders span a subspace of $T\mathcal{M}$ of dimension $g + s - 1$ whose image under p is isotropic. Since $\ker(p)$ has dimension $s - 1$, the image under p has dimension g. Hence $\mathcal{M}$ has full rank and $p(T\mathcal{M}) = H^1(X)$.

The intersection of the span of these twists with $\ker(p)$ has dimension $s - 1$, which is the dimension of $\ker(p)$. Hence $\ker(p) \subset T\mathcal{M}$. We conclude that $T\mathcal{M} = H^1(X, \Sigma)$. $\qquad\square$

Proof of Theorem 1.5. Consider a surface $(X, \omega) \in \mathcal{M}$ with the maximum number of horizontal cylinders. We consider the cylinder digraph (V, E). Suppose X has genus g and ω has s distinct zeros, so $|E| = 2g - 2 + s$.

Since every cylinder is free, the dimension of the twist space is equal to $|V|$. By Lemma 5.1, $|V| = |E| - \dim L(\Gamma) + 1$. By Lemma 5.3, $|V| \geq |E| - g + 1 = g - 1 + s$, since rank is always at most genus. The result now follows from Lemma 5.4. $\qquad\square$

6. Unfoldings of polygons

In this section, we give some foundational results on affine invariant submanifolds associated to rational polygons and prove Theorem 1.3.

Fix numbers $\theta_1, \ldots, \theta_n \in (0, \pi) \cup (\pi, 2\pi)$ such that $\theta_i/\pi \in \mathbb{Q}$ for each i and $\sum_{i=1}^{n} \theta_i = (n-2)\pi$. Let $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_n)$. Define k to be the least common denominator of $\theta_i/\pi, i = 1, \ldots, n$, and set $q_i = k\theta_i/\pi$. For each tuple $\mathbf{z} = (z_1, \ldots, z_n) \in \mathbb{C}^n$ of distinct complex numbers, define $X_{\mathbf{z}}$ to be the normalization of the plane algebraic curve

$$y^k = \prod_{i=1}^{n} (z - z_i)^{q_i}.$$

This Riemann surface has an automorphism T given by $T(y, z) = (\xi y, z)$, where $\xi = \exp(2\pi I/k)$. This automorphism generates the deck group for the covering map $(y, z) \mapsto z$. We will refer to $X_{\mathbf{z}}$ as a cyclic cover of $\mathbb{P}^1$. Let $\Sigma_{\mathbf{z}} \subset X_{\mathbf{z}}$ be the union of the preimages of each z_i such that q_i does not divide k.

For $\ell \in \mathbb{Z}/k = \{0, 1, \ldots, k-1\}$, define $H_\ell^{1,0}(X_{\mathbf{z}})$, $H_\ell^1(X_{\mathbf{z}})$ and $H_\ell^1(X_{\mathbf{z}}, \Sigma_{\mathbf{z}})$ to be the ξ^ℓ-eigenspaces for the induced action of T on $H^{1,0}(X_{\mathbf{z}})$, $H^1(X_{\mathbf{z}})$ and $H^1(X_{\mathbf{z}}, \Sigma_{\mathbf{z}})$ respectively.

Let $\{x\} \in [0, 1)$ denote the fractional part of $x \in \mathbb{R}$, and let $\lfloor x \rfloor = x - \{x\}$ denote the greatest integer less than or equal to x. For $\ell \in \mathbb{Z}/k$, define

$$t_i(\ell) = \left\{ \frac{\ell q_i}{k} \right\} \quad \text{and} \quad t(\ell) = \sum_{i=1}^{n} \left\{ \frac{\ell q_i}{k} \right\}.$$

Lemma 6.1. *$H_{-\ell}^{1,0}(X_{\mathbf{z}})$ is spanned by*

$$p(z) \prod_{i=1}^{n} (z - z_i)^{-t_i(\ell)} dz$$

where $p(z)$ is a polynomial of degree at most $t(\ell) - 2$. In particular, $H_\ell^{1,0}(X_{\mathbf{z}})$ has dimension $t(-\ell) - 1$.

This lemma is standard and can be found in many sources. One reference that uses the same notation is [**Wri12**, Lemma 2.6] (for the case $n = 4$, which is identical to the general case).

Here $\prod_{i=1}^{n} (z - z_i)^{-t_i(\ell)}$ is shorthand for $y^{-\ell} \prod_{i=1}^{n} (z - z_i)^{\lfloor \frac{\ell q_i}{k} \rfloor}$. Since $H_\ell^1(X_{\mathbf{z}})$ is the direct sum of $H_\ell^{1,0}(X_{\mathbf{z}})$ and the complex conjugate of $H_{-\ell}^{1,0}(X_{\mathbf{z}})$, we also have the following.

Corollary 6.2. *$H_\ell^1(X_{\mathbf{z}})$ has dimension $t(\ell) + t(-\ell) - 2$.*

Lemma 6.1 gives in particular that

$$\omega_{\mathbf{z}} = \prod_{i=1}^{n} (z - z_i)^{\frac{\theta_i}{\pi} - 1} dz \in H_1^{1,0}(X_{\mathbf{z}}).$$

Let $\mathcal{M}_{cyc}(\boldsymbol{\theta})$ denote the set of all $(X_{\mathbf{z}}, \omega_{\mathbf{z}})$ for n-tuples $\mathbf{z}$ of distinct complex numbers. Define $\mathcal{M}(\boldsymbol{\theta})$ to be the smallest affine invariant submanifold containing $\mathcal{M}_{cyc}(\boldsymbol{\theta})$.

Recall that the set of polygons with given angles is a smooth manifold with a well defined Lebesgue measure class.

Lemma 6.3. *For almost every n-gon P with interior angles $\theta_1, \ldots, \theta_n$, the unfolding of P has $GL(2, \mathbb{R})$ orbit closure $\mathcal{M}(\boldsymbol{\theta})$.*

Proof. Let $\mathcal{P}$ be a connected component of the space of n-gons with angles $\theta_1, \ldots, \theta_n$. The theory of Schwarz-Christoffel mappings gives that the set of unfoldings of polygons in $\mathcal{P}$ is equal to the set of $(X_{\mathbf{z}}, \omega_{\mathbf{z}})$ where the z_i range over an open subset of $\mathbb{R}^n$, see for example [**DT02**].

Recall that a holomorphic function on a connected open subset of $\mathbb{C}^n$ that vanishes on a nonempty open subset of $\mathbb{R}^n \subset \mathbb{C}^n$ must be

identically zero. The relative periods of $\omega_{\mathbf{z}}$ are holomorphic functions of the z_i, so any linear equation that holds on the periods of all unfoldings of polygons in $\mathcal{P}$ also holds for all $(X_{\mathbf{z}}, \omega_{\mathbf{z}})$. We conclude that $\mathcal{M}(\boldsymbol{\theta})$ is the smallest affine invariant submanifold containing all unfoldings of polygons in $\mathcal{P}$.

There are only countably many affine invariant submanifolds [**EMM15, Wri14**], so in particular there are only countably many proper affine invariant submanifolds of $\mathcal{M}(\boldsymbol{\theta})$. Any $P \in \mathcal{P}$ not in their preimage has unfolding with orbit closure equal to $\mathcal{M}(\boldsymbol{\theta})$. $\qquad\square$

$\mathcal{M}_{cyc}(\boldsymbol{\theta})$ is locally defined by the ξ eigenspace for T:

Lemma 6.4. *Every $(X_{\mathbf{z}}, \Sigma_{\mathbf{z}})$ in $\mathcal{M}_{cyc}(\boldsymbol{\theta})$ has a neighbourhood in $\mathcal{M}_{cyc}(\boldsymbol{\theta})$ whose image under the period coordinate map $(X', \omega') \mapsto H^1(X_{\mathbf{z}}, \Sigma_{\mathbf{z}})$ is an open set in $H^1_1(X_{\mathbf{z}}, \Sigma_{\mathbf{z}})$.*

We thank Curtis McMullen for helpful discussions regarding this lemma. Compare to [**McM13**, Corollary 6.8].

Proof. Note that $\xi^{-1}T^*$ fixes $(X_{\mathbf{z}}, \omega_{\mathbf{z}})$. We can consider $\xi^{-1}T$ as an element of the mapping class group of the surface with zeros of the Abelian differential marked. In the Teichmüller space of the stratum (the set of surfaces in the stratum, equipped with an isotopy class of homeomorphisms from a fixed translation surface taking zeros to zeros), a connected component of the pre-image of $\mathcal{M}_{cyc}(\boldsymbol{\theta})$ is given by a connected component of the set of fixed points of $\xi^{-1}T$, which is evidently linear because the action of this mapping class is of course linear in period coordinates. $\qquad\square$

Corollary 6.5. *The tangent space to $\mathcal{M}(\boldsymbol{\theta})$ at a point of $\mathcal{M}_{cyc}(\boldsymbol{\theta})$ contains $H^1_1(X_{\mathbf{z}}, \Sigma_{\mathbf{z}}) + H^1_{-1}(X_{\mathbf{z}}, \Sigma_{\mathbf{z}})$.*

Note that while eigenspaces of T define subbundles of H^1 over $\mathcal{M}_{cyc}(\boldsymbol{\theta})$, they are not defined on $\mathcal{M} \setminus \mathcal{M}_{cyc}(\boldsymbol{\theta})$.

Corollary 6.6. *If all θ_i are multiples of $\pi/3$, then $\mathcal{M}(\boldsymbol{\theta})$ is full rank.*

Proof. Since T^* has order 3 and does not fix any nonzero Abelian differentials, $H^1(X_{\mathbf{z}})$ is the direct sum of the ξ and $\xi^{-1} = \xi^2$ eigenspaces. $\qquad\square$

Lemma 6.7. *Suppose k is odd and either $n > 4$ or $\boldsymbol{\theta}$ has more than two distinct angles. Then $\mathcal{M}(\boldsymbol{\theta})$ is not contained in the hyperelliptic locus.*

Proof. Suppose $(X, \omega) \in \mathcal{M}_{cyc}(\boldsymbol{\theta})$, and τ is a hyperelliptic involution on X. The assumption that k is odd guarantees that $\langle T \rangle$ does not contain an involution, and so in particular it does not contain τ. Since

τ and T commute, τ descends to an involution of $(X, \omega)/\langle T \rangle$ that is a local isometry for the flat metric. If (X, ω) is the unfolding of a polygon P, then $(X, \omega)/\langle T \rangle$ is the "pillowcase double" of P.

The generic element of $\mathcal{M}_{0,n}$ does not have an involution when $n > 4$. If $n = 3$ or $n = 4$, the generic element of $\mathcal{M}_{0,n}$ does have an involution, which exchanges two of the marked points when $n = 3$ and exchanges two pairs of marked points when $n = 4$. $\qquad\square$

Proof of Theorem 1.3. By Corollary 6.6, $\mathcal{M}(\boldsymbol{\theta})$ has full rank whenever the angles of P are all multiples of $\pi/3$. By Lemma 6.7, it is not contained in the hyperelliptic locus if the genus is greater than 2. Hence Theorem 1.1 gives that $\mathcal{M}(\boldsymbol{\theta})$ is a connected component of a stratum, and Corollary 6.3 concludes the proof. $\qquad\square$

7. THE INSTABILITY OF EIGENFORM LOCI

The goal of this section is to improve the following "trivial rank bound" in some cases. We introduce the ideas gradually as they are required for applications rather than proceeding immediately to the strongest possible statements.

Lemma 7.1. *If at least one θ_i is not a multiple of $\frac{\pi}{2}$, the rank of $\mathcal{M}(\boldsymbol{\theta})$ is at least $n - 2$. If all the θ_i are multiples of $\frac{\pi}{2}$, the rank is at least $(n - 2)/2$.*

Proof. By Corollary 6.5, the tangent space of $\mathcal{M}(\boldsymbol{\theta})$ at a point of $\mathcal{M}_{cyc}(\boldsymbol{\theta})$ contains the ξ and ξ^{-1} eigenspaces for T. If $k = 2$, these are equal. By Corollary 6.2, the ξ eigenspace has dimension $t(1) + t(-1) - 2$. If ℓ is relatively prime to k, then $t(\ell) + t(-\ell) = n$. $\qquad\square$

For the remainder of this section we assume $k > 2$. We have written the section to illustrate the relevance of the techniques for arbitrary $n \geq 3$, however at key places we have assumed $n = 3$ to avoid technical difficulties, and so ultimately we only give applications to orbit closures of unfoldings of triangles. The results of this section are partially inspired by [**McM03**, Theorem 7.5], which exhibits a locus of genus 3 eigenforms for real multiplication by an order in a totally real cubic field that is not $GL(2, \mathbb{R})$ invariant.

7.1. **Showing rank bigger than $n - 2$.** For any affine invariant submanifold $\mathcal{N}$, let H^1 denote the flat bundle over $\mathcal{N}$ whose fiber over (X, ω) is $H^1(X)$. This bundle can be decomposed as a direct sum of symplectically orthogonal flat subbundles

$$H^1 = \bigoplus_{\rho} p(T\mathcal{N})^{\rho} \bigoplus_{s} \mathbb{W}_s,$$

where ρ runs over the different field embeddings of the affine field of definition $\mathbf{k}(\mathcal{N})$, the $p(T\mathcal{N})^\rho$ are the Galois conjugates of the bundle $p(T\mathcal{M})$, and the $\mathbb{W}_s$ are the remaining isotypic components for the monodromy representation [**Wri14**]. By work of Filip, this decomposition is a decomposition of variations of Hodge structures, which means that the fibers of each bundle are equal to the direct sum of their intersections with $H^{1,0}(X)$ and $H^{0,1}(X)$ [**Fil**].

Define, for any triple of Abelian differentials ω, α, β on a Riemann surface X,

$$B_\omega(\alpha, \beta) = \frac{I}{2} \int_X \alpha\beta\frac{\overline{\omega}}{\omega}.$$

The expression computes the second fundamental form for the Hodge bundle $H^{1,0}$ and the derivative of the period matrix [**FMZ14**, Lemma 2.1, Lemma 2.2].

Lemma 7.2. *Let $\mathcal{N}$ be an affine invariant submanifold, and let $(X, \omega) \in \mathcal{N}$. If $\alpha, \beta \in H^{1,0}(X)$ lie in different summands of the above decomposition of H^1, then $B_\omega(\alpha, \beta) = 0$.*

This follows from the fact that B computes the second fundamental form of $H^{1,0}$, and that the above decomposition respects the Hodge filtration. See [**FMZ14**, Section 2.3] for the definition of the second fundamental form.

Recall that by Lemma 6.1, the eigenspace H_a^1 contains nonzero holomorphic one-forms exactly when $t(-a) > 1$.

Proposition 7.3. *Let $0 < a, b < k$ with $t(-a) > 1$ and $t(-b) > 1$.*

(1) If $a + b \neq 2 \mod k$, then for any $(X, \omega) \in \mathcal{M}_{cyc}(\boldsymbol{\theta})$, $\alpha \in H_a^{1,0}(X)$, and $\beta \in H_b^{1,0}(X)$ we have $B_\omega(\alpha, \beta) = 0$.
(2) If $n = 3$ and $a + b = 2 \mod k$ there exists $(X, \omega) \in \mathcal{M}_{cyc}(\boldsymbol{\theta})$, $\alpha \in H_a^{1,0}(X)$, and $\beta \in H_b^{1,0}(X)$ such that $B_\omega(\alpha, \beta) \neq 0$.

The proof is deferred to Subsection 7.3. It is plausible that the second statement is also true when $n > 3$, and we have checked this in some but not all cases.

Problem 7.4. Determine precisely when the second statement of Proposition 7.3 is true when $n > 3$.

Because of Proposition 7.3, we will be interested in pairs of eigenspaces H_a^1, H_{2-a}^1 when $t(-a) > 1$ and $t(-(2 - a)) > 1$, i.e. when both of these eigenspaces contain nonzero holomorphic one-forms. We will generally want to assume that the eigenspaces H_a^1 and H_{2-a}^1 are distinct, i.e. that $a \neq 2 - a \mod k$.

Theorem 7.5. *Suppose $n = 3$, and suppose there exists $a \in (\mathbb{Z}/k)^*$ with $2a \neq 2 \mod k$ and $t(-a) > 1$ and $t(-(2-a)) > 1$. Then $\mathcal{M}(\boldsymbol{\theta})$ has rank strictly greater than $n - 2$.*

If Proposition 7.3 is true when $n > 3$, then so is Theorem 7.5.

Proof. If $\mathcal{M}(\boldsymbol{\theta})$ has rank $n - 2$, then the proof of Lemma 7.1 gives that $p(T_{(X,\omega)}\mathcal{M}(\boldsymbol{\theta}))$ is equal to $H_1^1(X) \oplus H_{-1}^1(X)$ at any $(X, \omega) \in T\mathcal{M}_{cyc}(\boldsymbol{\theta})$. Similarly, each Galois conjugate of $p(T_{(X,\omega)}\mathcal{M}(\boldsymbol{\theta}))$ is of the form $H_a^1(X) \oplus H_{-a}^1(X)$ for some $a \in (\mathbb{Z}/k)^*$.

Since $t(-a) > 1$ and $t(-(2 - a)) > 1$, we can pick nonzero $\omega_a \in H_a^{1,0}(X)$ and $\omega_{2-a} \in H_{2-a}^{1,0}(X)$.

Since $2a \neq 2 \mod k$, the forms ω_a and ω_{2-a} lie in different eigenspaces for T. Since by assumption $k > 2$, we have $2 - a \neq -a \mod k$, so in fact ω_{2-a} does not lie in the Galois conjugate of $p(T_{(X,\omega)}\mathcal{M}(\boldsymbol{\theta}))$ that contains ω_a.

By Proposition 7.3, we get that $B_\omega(\omega_a, \omega_{2-a}) \neq 0$, and so Lemma 7.2 gives a contradiction. $\square$

The proof can be viewed as showing that the $SL(2, \mathbb{R})$ orbit of a surface in $\mathcal{M}_{cyc}$ is not even tangent to the locus of (X, ω) admitting the endomorphisms which would be present if $\mathcal{M}(\boldsymbol{\theta})$ had rank $n - 2$. These endomorphisms are described in [**Fil**, Theorem 7.3].

7.2. **Full rank examples.** We proceed to the proof of Theorem 1.4, for which we may pick the triangles to be rather special. Afterwards we continue our analysis so that it applies to as many triangles as possible, leading to a simple algorithm which can often show that the unfolding of a triangle has full rank. We continue to assume $n = 3$, even though much of our analysis also applies when $n > 3$.

Lemma 7.6. *Consider $(X, \omega) \in \mathcal{M}_{cyc}(\boldsymbol{\theta})$. Then there exists a multiplicative subgroup $A \subset (\mathbb{Z}/k)^*$ containing -1 such that $p(T_{(X,\omega)}\mathcal{M}(\boldsymbol{\theta})) = V \oplus V'$, where*

$$V = \bigoplus_{a \in A} H_a^1(X),$$

and V' is orthogonal to the direct sum $\bigoplus_{a \in (\mathbb{Z}/k)^*} H_a^1(X)$ of the primitive eigenspaces. The Galois conjugates of V are given by the direct sums of $H_a^1(X)$ over cosets of A in $(\mathbb{Z}/k)^*$.

Proof. Let A be the stabilizer for the action of the Galois group $(\mathbb{Z}/k)^*$ of $\mathbb{Q}[\exp(2\pi I/k)]$ on the affine field of definition $\mathbf{k}$ of $\mathcal{M}(\boldsymbol{\theta})$. Note A contains -1 since $\mathbf{k} \subset \mathbb{R}$.

The projection of the tangent space of $\mathcal{M}(\boldsymbol{\theta})$ to absolute cohomology contains $V = \bigoplus_{a \in A} H^1_a(X)$, because any subspace defined over $\mathbf{k}$ is stable under any field automorphism fixing $\mathbf{k}$.

Since $p(T_{(X,\omega)}\mathcal{M}(\boldsymbol{\theta}))$ is orthogonal to all its Galois conjugates, it is orthogonal to $\bigoplus_{a \notin A} H^1_a(X)$. It follows that $p(T_{(X,\omega)}\mathcal{M}(\boldsymbol{\theta}))$ is the direct sum of $\bigoplus_{a \in A} H^1_a(X)$ and its intersection with $\bigoplus_{a \in (\mathbb{Z}/k) \setminus (\mathbb{Z}/k)^*} H^1_a(X)$. $\qquad \square$

Proof of Theorem 1.4. Pick $k > 2$ to be prime. Let r be a primitive root mod k, and let a be the solution to $a(2-a)^{-1} = r \mod k$. It follows that $2a \neq 2 \mod k$. Set

$$
\begin{aligned}
q_1 &= (k-1)(-a)^{-1} & \mod k, \\
q_2 &= (k-1)(-(2-a))^{-1} & \mod k, \\
q_3 &= k - q_1 - q_2 & \mod k.
\end{aligned}
$$

The number of primitive roots mod k goes to infinity as $k \to \infty$, so r can be chosen so that q_i are nonzero and distinct. (The equation $q_3 = 0$, and each equation $q_i = q_j$, determines only boundedly many r values, since these are polynomial equations of bounded degree over a field.) Since the q_i are distinct, by Lemma 6.7 the unfolding (X, ω) of the triangle with angles $\frac{q_i}{k}\pi$ is not hyperelliptic.

Note $t(-a) > 1$ since $t_1(-a) = \frac{k-1}{k}$, and all $t_i(-a) \geq \frac{1}{k}$. Similarly $t(-(2-a)) > 1$.

Let A be as in Lemma 7.6. Pick nonzero $\omega_a \in H^{1,0}_a(X)$ and $\omega_{2-a} \in H^{1,0}_{2-a}(X)$. By Proposition 7.3, $B_\omega(\omega_a, \omega_{2-a}) \neq 0$, so by Lemma 7.2 it follows that $r \in A$, since otherwise $H^1_a(X)$ and $H^1_{2-a}(X)$ would belong to different Galois conjugates of $p(T_{(X,\omega)}\mathcal{M}(\frac{q_1}{k}\pi, \frac{q_2}{k}\pi, \frac{q_3}{k}\pi))$.

Since r is a primitive root, we get that $A = (\mathbb{Z}/k)^*$ and hence that $\mathcal{M}(\frac{q_1}{k}\pi, \frac{q_2}{k}\pi, \frac{q_3}{k}\pi)$ has full rank. Since (X, ω) is not hyperelliptic, Theorem 1.1 gives that $\mathcal{M}(\frac{q_1}{k}\pi, \frac{q_2}{k}\pi, \frac{q_3}{k}\pi)$ must be a connected component of a stratum. $\qquad \square$

We continue our analysis so that it will apply more broadly.

Lemma 7.7. *For any $(X, \omega) \in \mathcal{M}_{cyc}(\boldsymbol{\theta})$, $p(T_{(X,\omega)}\mathcal{M}(\boldsymbol{\theta}))$ respects the direct sum decomposition of cohomology as $\bigoplus_{a \in \mathbb{Z}/k} H^1_a(X)$, in that it is equal to the direct sum of its intersections with the summands.*

The idea of the proof can be seen already for the unfolding of the right angled triangle with smallest angle $\frac{\pi}{8}$, which is the regular octagon. Gradually rotating the octagon by $\frac{2\pi}{8}$ gives a path in the stratum from the octagon to itself, whose monodromy is a deck transformation that generates the cyclic deck group of the map to $\mathbb{P}^1$.

Proof. Recall that $\mathcal{M}(\boldsymbol{\theta})$ was defined to be the smallest affine invariant submanifold containing $\mathcal{M}_{cyc}(\boldsymbol{\theta})$. Since the self crossing locus of any affine invariant submanifold is a smaller dimensional affine invariant submanifold, we may conclude that $\mathcal{M}_{cyc}(\boldsymbol{\theta})$ is not contained in the self crossing locus of $\mathcal{M}(\boldsymbol{\theta})$. Pick $(X, \omega) \in \mathcal{M}_{cyc}(\boldsymbol{\theta})$ that is not contained in the self crossing locus of $\mathcal{M}(\boldsymbol{\theta})$.

Recall that $\omega \in H^1_1(X)$, which means exactly that $T^*(\omega) = \exp(2\pi I/k)\omega$. For any translation surface (X, ω), the surface $(X, \exp(2\pi I/k)\omega)$ is obtained by rotating (X, ω) by angle $2\pi/k$.

$p(T_{(X,\omega)}\mathcal{M}(\boldsymbol{\theta}))$ is invariant under the monodromy of any path in $\mathcal{M}(\boldsymbol{\theta})$. The path $(X, \exp(I\phi)\omega), \phi \in [0, 2\pi/k]$ has monodromy T, and so $p(T_{(X,\omega)}\mathcal{M}(\boldsymbol{\theta}))$ is invariant under T.

The result now follows from the fact that any subspace of a vector space invariant under a finite order linear transformation T is equal to the direct sum of its intersections with the eigenspaces of T.   □

Remark 7.8. It was necessary to pick (X, ω) not in the self crossing locus because if (X, ω) is in the self crossing, then the image of $\mathcal{M}(\boldsymbol{\theta})$ in the stratum is locally defined at (X, ω) by a finite union of vector subspaces, which might be permuted by T. Note that, although X is an orbifold point of $\mathcal{M}_g$, (X, ω) is typically not an orbifold point of the stratum, because the automorphisms T^i of X typically do not satisfy $(T^i)^*(\omega) = \omega$.

Corollary 7.9. *When $n = 3$, $p(T_{(X,\omega)}\mathcal{M}(\boldsymbol{\theta}))$ is the direct sum of eigenspaces for any $(X, \omega) \in \mathcal{M}_{cyc}(\boldsymbol{\theta})$.*

Proof. When $n = 3$, nonzero eigenspaces are 1 dimensional.   □

The monodromy of the flat bundle H^1_a over $\mathcal{M}_{cyc}(\boldsymbol{\theta})$ is typically (and possibly always) irreducible ([**Roh09**, Theorems 3.3.4, 5.1.1, Proposition 5.5.1], see also [**Loo07**, Section 4], [**McM13**, Corollary 5.3]), which would imply that Corollary 7.9 applies equally well when $n > 3$. Since we haven't found a reference stating explicitly that the monodromy is always irreducible, and since we are happy to continue to assume $n = 3$, we haven't attempted to check whether there are exceptional cases where the monodromy isn't irreducible.

Together with Corollary 7.9, the following slight extension of Lemma 7.6 will underlie all of our remaining analysis.

Lemma 7.10. *Let $A \subset (\mathbb{Z}/k)^*$ be as in Lemma 7.6. Let $(X, \omega) \in \mathcal{M}_{cyc}(\boldsymbol{\theta})$.*

Suppose $H^1_a(X) \subset p(T_{(X,\omega)}\mathcal{M}(\boldsymbol{\theta}))$, and $b \in (\mathbb{Z}/k)^*$. If $b \in A$ then $H^1_{ab}(X) \subset p(T_{(X,\omega)}\mathcal{M}(\boldsymbol{\theta}))$, and otherwise $B_\omega(\omega_a, \omega_{ab}) = 0$ for all $\omega_a \in$

$H_a^{1,0}(X)$ and $\omega_{ab} \in H_{ab}^{1,0}(X)$.

Proof. The first statement follows as in the proof of Lemma 7.6 because $p(T\mathcal{M})$ is invariant under field automorphisms of the affine field of definition $\mathbf{k}$, and the second statement follows from Lemma 7.2. $\square$

Lemma 7.11. *Suppose* $(X, \omega) \in \mathcal{M}_{cyc}(\boldsymbol{\theta})$ *and* d_1, d_2 *are divisors of* k *such that*

(1) $p(T_{(X,\omega)}\mathcal{M}(\boldsymbol{\theta}))$ *contains* H_a^1 *for some* a *with* $\gcd(a, k) = d_1$ *and* H_a^1 *nonzero, and*

(2) $p(T_{(X,\omega)}\mathcal{M}(\boldsymbol{\theta}))$ *does not contain* H_b^1 *for any* b *with* $\gcd(b, k) = d_2$ *and* H_b^1 *nonzero.*

Then for any a', b' *with* $\gcd(a', k) = d_1, \gcd(b', k) = d_2$, *and any* $\alpha \in H_{a'}^1, \beta \in H_{b'}^1$, $B_\omega(\alpha, \beta) = 0$.

Proof. By the same arguments as in the proof of Lemma 7.10, in this case $\bigoplus_{\gcd(a,k)=d_1} H_a^1$ is contained in the sum of $p(T_{(X,\omega)}\mathcal{M}(\boldsymbol{\theta}))$ and its Galois conjugates, and $\bigoplus_{\gcd(b,k)=d_2} H_b^1$ is contained in the complement, so the result follows by Lemma 7.2. $\square$

It is helpful to group the eigenspaces H_a^1 according to how primitive the associated roots of unity are. This gives the following direct sum decomposition of H^1 defined over $\mathbb{Q}$

$$H^1 = \bigoplus_{d|k} \left(\bigoplus_{\gcd(a,k)=d} H_a^1 \right).$$

By Corollary 6.2, the subspace $\bigoplus_{\gcd(a,k)=d} H_a^1$ is nonzero if and only if there exists an a with $\gcd(a, k) = d$ and $t(-a) > 1$.

Define $\mathcal{D}$ to be the set of divisors d of k such that there exists an a with $\gcd(a, k) = d$ and $t(-a) > 1$. Consider the equivalence relation $\sim$ on $\mathcal{D}$ generated by $d_1 \sim d_2$ if there is exists a with $\gcd(a, k) = d_1, \gcd(2 - a, k) = d_2$ and $t(-a) > 1, t(-(2-a)) > 1$.

Lemma 7.12. *If* $d \sim 1$ *then there is an* b *with* $\gcd(k, b) = d$ *and* $t(-b) > 1$ *and* $H_b^1 \subset p(T_{(X,\omega)}\mathcal{M}(\boldsymbol{\theta}))$. *Furthermore* A *contains*

$$\{a \in (\mathbb{Z}/k)^* : a = 1 \mod k/d\}.$$

Proof. Suppose $a \in \mathbb{Z}/k$ with $\gcd(a, k) = d_1, \gcd(2 - a, k) = d_2$ and $t(-a) > 1, t(-(2-a)) > 1$, and suppose there exists an $a' \in \mathbb{Z}/k$ with $t(-a') > 1$ and $\gcd(a', k) = d_1$ and $H_{a'}^1 \subset p(T_{(X,\omega)}\mathcal{M}(\boldsymbol{\theta}))$. Let $\omega_a \in H_a^{1,0}$ and $\omega_{2-a} \in H_{2-a}^{1,0}$. By Proposition 7.3, $B_\omega(\omega_a, \omega_{2-a}) \neq 0$, so Lemma 7.11 gives that $p(T_{(X,\omega)}\mathcal{M}(\boldsymbol{\theta}))$ contains H_b^1 for some b with $\gcd(b, k) = d_2$ and H_b^1 nonzero.

Using the fact that $H_1^1 \subset p(T_{(X,\omega)}\mathcal{M}(\boldsymbol{\theta}))$ and the definition of the equivalence relation, the first claim follows.

The second statement follows from the first statement and Lemma 7.10, because for $a \in (\mathbb{Z}/k)^*$ with $a = 1 \mod k/d$ and any b with $\gcd(b,k) = d$, $H_{ab}^1 = H_b^1$. $\qquad\square$

Lemma 7.13. *Suppose $1 \sim d$ and $\gcd(a,k) = d$, $2 - a = wa \mod k$ for some $w \in (\mathbb{Z}/k)^*$, and $t(-a) > 1, t(-(2-a)) > 1$. Then $w \in A$.*

The assumptions imply $\gcd(2-a,k) = d$ and that d equals 1 or 2.

Proof. Pick $\omega_a \in H_a^{1,0}$ and $\omega_{2-a} \in H_{2-a}^{1,0}$ nonzero. By Proposition 7.3, $B_\omega(\omega_a, \omega_{2-a}) \neq 0$.

In order to find a contradiction, assume $w \notin A$. Since $1 \sim d$, there is a b with $\gcd(b,k) = d$ and H_b^1 nonzero and contained in $p(T\mathcal{M})$. This is a contradiction, since in this case H_a^1 and H_{2-a}^1 are in different Galois conjugates of $p(T\mathcal{M})$. $\qquad\square$

Theorem 7.14. *Suppose $1 \sim d$ for all $d \in \mathcal{D}$, and $A = (\mathbb{Z}/k)^*$. Then $\mathcal{M}$ is full rank.*

See Appendix A for an algorithm that determines when Theorem 7.14 applies.

Proof. By Corollary 7.9, over $\mathcal{M}_{cyc}(\boldsymbol{\theta})$ the bundle $p(T\mathcal{M}(\boldsymbol{\theta}))$ is a direct sum of eigenbundles H_a^1. By Lemma 7.12, for each d with $\bigoplus_{\gcd(a,k)=d} H_a^1$ nonzero, there is some $b \in \mathbb{Z}/k$ with $\gcd(b,k) = d$ and H_b^1 nonzero and contained in $p(T\mathcal{M}(\boldsymbol{\theta}))$. By Lemma 7.10, we get that H_{ab}^1 for all $a \in A = (\mathbb{Z}/k)^*$. It follows that $p(T\mathcal{M}(\boldsymbol{\theta}))$ contains all eigenbundles H_a^1. $\qquad\square$

7.3. **Proof of Proposition 7.3.** For $0 < a < k$ with $t(-a) > 1$ and $(X_\mathbf{z}, \omega_\mathbf{z}) \in \mathcal{M}_{cyc}(\boldsymbol{\theta})$, recall Lemma 6.1 and define

$$\omega_a = \prod_{i=1}^n (z - z_i)^{-t_i(-a)} dz \in H_a^{1,0}(X_\mathbf{z}).$$

Lemma 6.1 states that any element of $H_a^{1,0}(X_\mathbf{z})$ can be written as $p_a(z)\omega_a$, where $p_a(z)$ is a polynomial of degree at most $t(-a) - 2$. Recall also that $\omega_\mathbf{z} \in H_1^{1,0}(X_\mathbf{z})$.

Proof of Proposition 7.3. We are considering expressions of the form $B_{\omega_\mathbf{z}}(p_a(z)\omega_a, p_b(z)\omega_b)$. If we define $p(z) = p_a(z)p_b(z)$, this is equal to

$$\int p(z)\omega_a\omega_b \frac{\overline{\omega_\mathbf{z}}}{\omega_\mathbf{z}} = \frac{1}{k} \int p(z) \sum_{\ell=1}^k (T^*)^\ell \left(\omega_a\omega_b \frac{\overline{\omega_\mathbf{z}}}{\omega_\mathbf{z}}\right).$$

If $a + b \neq 2 \mod k$, this integral is zero because T acts on $\omega_a \omega_b \frac{\overline{\omega_{\mathbf{z}}}}{\omega_{\mathbf{z}}}$ by a root of unity other than 1. This proves the first claim.

If $a + b = 2 \mod k$, the integrand is T invariant, and hence the integral descends to an integral on $X_{\mathbf{z}}/\langle T \rangle = \mathbb{P}^1$, which we will now give an expression for.

For the remainder of the proof we assume $\theta_i < \pi$ for all i. At the final stage of the proof we will assume that $n = 3$, which implies $\theta_i < \pi$ since $\theta_1 + \theta_2 + \theta_3 = \pi$ when $n = 3$. Assume $b = 2 - a \mod k$. Compute

$$\omega_a \omega_b = \prod_{i=1}^{n} (z - z_i)^{-t_i(-a) - t_i(-b)} (dz)^2 = \omega_{\mathbf{z}}^2 \prod_{i=1}^{n} (z - z_i)^{\varepsilon_i},$$

where

$$\begin{aligned} \varepsilon_i &= 2(1 - \theta_i/\pi) - t_i(-a) - t_i(-b) \\ &= 2 - 2t_i(1) - t_i(-a) - t_i(-b). \end{aligned}$$

With this notation,

$$\omega_a \omega_b \frac{\overline{\omega_{\mathbf{z}}}}{\omega_{\mathbf{z}}} = \prod_{i=1}^{n} (z - z_i)^{\varepsilon_i} \prod_{i=1}^{n} |z - z_i|^{2\theta_i/\pi - 2} dz d\overline{z}.$$

We require the following numerical observations.

(1) $\varepsilon_i \in \{-1, 0, 1\}$ for all i. This follows since $0 \leq t_i(\ell) < 1$ and $0 < t_i(1)$, using that $\varepsilon_i \in \mathbb{Z}$.

(2) $\varepsilon_i = -1$ implies $\theta_i \geq \frac{\pi}{2}$. This follows similarly, using that $2t_i(1) = t_i(2)$ if $\theta_i < \frac{\pi}{2}$ and $2t_i(1) = 1 + t_i(2)$ otherwise.

(3) $\sum \varepsilon_i = 4 - t(-a) - t(-b)$. Indeed, sum the definition of ε_i, and note that the sum of the angles in an n-gon is $(n - 2)\pi$.

(4) $4 - t(-a) - t(-b) = 0$. Indeed, by assumption, $t(-a) > 1, t(-b) > 1$, which implies $t(-a) = 2, t(-b) = 2$ since $n = 3$.

Finally, we assume $n = 3$. The remainder of the proof follows from the next lemma, which applies after moving one of the z_i to infinity. If the triangle is obtuse, then the z_i corresponding to the obtuse angle should be the one moved to infinity. $\qquad \square$

Lemma 7.15. *Assume* $0 < \frac{q_1}{k}, \frac{q_2}{k} < \frac{1}{2}$. *With* $\varepsilon_1, \varepsilon_2$ *equal to 0 or 1,*

$$\int_{\mathbb{C}} \frac{z^{\varepsilon_1} (z - 1)^{\varepsilon_2}}{|z|^{2 - 2\frac{q_1}{k}} |z - 1|^{2 - 2\frac{q_2}{k}}} dA$$

is nonzero (assuming the integrand is integrable, in particular that at most one ε_i *is nonzero).*

Here dA is the usual area form, which is proportional to $dz d\overline{z}$.

Proof. If $\varepsilon_1, \varepsilon_2$ are both zero, the result is obvious. If $\varepsilon_1 = 1, \varepsilon_2 = 0$, then the absolute value of the real part of the integrand

$$\frac{z}{|z|^{2-2\frac{q_1}{k}}|z-1|^{2-2\frac{q_2}{k}}}$$

at z with $\mathrm{Re}(z) > 0$ is greater than that at $-z$. Thus the positive contribution, from when $\mathrm{Re}(z) > 0$, dominates the negative contribution from when $\mathrm{Re}(z) < 0$. The argument is similar when $\varepsilon_1 = 0, \varepsilon_2 = 1$. $\square$

Appendix A. Algorithm for Theorem 7.14

input : $q_1, q_2, q_3, k = q_1 + q_2 + q_3$, such that $\gcd(q_1, q_2, q_3) = 1$
output: true if Theorem 7.14 implies $\mathcal{M}(\frac{q_1}{k}\pi, \frac{q_2}{k}\pi, \frac{q_3}{k}\pi)$ has full
            rank, false otherwise

$\mathcal{D} \leftarrow \{1\}$
$\mathcal{E} \leftarrow \{\}$
$\mathsf{A} \leftarrow \{1, -1\}$

for $a \leftarrow 2$ **to** $k - 1$ **do**
    if $t(-a) > 1$ **then**
        $d_1 \leftarrow \gcd(a, k)$
        $\mathcal{D} \leftarrow \mathcal{D} \cup \{d_1\}$
        if $t(-(2 - a)) > 1$ **then**
            $d_2 \leftarrow \gcd(2 - a, k)$
            if $d_1 = d_2$ **then**
                $\mathsf{A} \leftarrow \mathsf{A} \cup \{w \in (\mathbb{Z}/k)^* : wa = 2 - a\}$
            else
                $\mathcal{E} \leftarrow \mathcal{E} \cup \{(d_1, d_2)\}$
            end
        end
    end
end

if *the equivalence relation on* $\mathcal{D}$ *generated by* $\mathcal{E}$ *has more than one*
 equivalence class **then**
  | **return** *false*
end

for d *in* $\mathcal{D}$ **do**
 | $\mathsf{A} \leftarrow \mathsf{A} \cup \{a \in (\mathbb{Z}/k)^* : a = 1 \mod (k/d)\}$
end

if A *generates* $(\mathbb{Z}/k)^*$ **then**
 | **return** *true*
else
 | **return** *false*
end

Appendix B. List of triangles

$(1, 2, 8)$	$(1, 3, 7)$	$(2, 4, 5)$			
$(1, 4, 8)$	$(2, 3, 8)$	$(3, 4, 6)$			
$(4, 5, 6)$					
$(1, 2, 14)$	$(1, 4, 12)$	$(1, 5, 11)$	$(2, 3, 12)$	$(2, 4, 11)$	$(2, 6, 9)$
$(2, 7, 8)$	$(3, 4, 10)$	$(3, 6, 8)$	$(4, 6, 7)$		
$(1, 2, 16)$	$(1, 3, 15)$	$(1, 4, 14)$	$(1, 5, 13)$	$(1, 6, 12)$	$(1, 7, 11)$
$(2, 3, 14)$	$(2, 4, 13)$	$(2, 5, 12)$	$(2, 6, 11)$	$(2, 7, 10)$	$(2, 8, 9)$
$(3, 4, 12)$	$(3, 5, 11)$	$(3, 6, 10)$	$(3, 7, 9)$	$(4, 6, 9)$	$(4, 7, 8)$
$(5, 6, 8)$					
$(1, 4, 16)$	$(1, 5, 15)$	$(3, 5, 13)$	$(4, 7, 10)$	$(6, 7, 8)$	
$(1, 2, 20)$	$(1, 3, 19)$	$(1, 4, 18)$	$(1, 5, 17)$	$(1, 6, 16)$	$(1, 7, 15)$
$(1, 8, 14)$	$(1, 9, 13)$	$(2, 3, 18)$	$(2, 4, 17)$	$(2, 5, 16)$	$(2, 6, 15)$
$(2, 7, 14)$	$(2, 8, 13)$	$(2, 9, 12)$	$(2, 10, 11)$	$(3, 4, 16)$	$(3, 5, 15)$
$(3, 6, 14)$	$(3, 7, 13)$	$(3, 8, 12)$	$(3, 9, 11)$	$(4, 5, 14)$	$(4, 6, 13)$
$(4, 7, 12)$	$(4, 8, 11)$	$(4, 9, 10)$	$(5, 6, 12)$	$(5, 7, 11)$	$(5, 8, 10)$
$(6, 7, 10)$	$(6, 8, 9)$				
$(1, 3, 21)$	$(1, 4, 20)$	$(1, 6, 18)$	$(1, 7, 17)$	$(1, 8, 16)$	$(1, 10, 14)$
$(2, 3, 20)$	$(2, 4, 19)$	$(2, 5, 18)$	$(2, 7, 16)$	$(2, 8, 15)$	$(2, 10, 13)$
$(3, 4, 18)$	$(3, 5, 17)$	$(3, 6, 16)$	$(3, 7, 15)$	$(3, 8, 14)$	$(3, 9, 13)$
$(3, 10, 12)$	$(4, 5, 16)$	$(4, 6, 15)$	$(4, 8, 13)$	$(4, 9, 12)$	$(4, 10, 11)$
$(5, 6, 14)$	$(6, 7, 12)$	$(6, 8, 11)$	$(6, 9, 10)$	$(7, 8, 10)$	

Figure B.1. A list of the triples (q_1, q_2, q_3) with $q_1 < q_2 < q_3$ and $k = q_1 + q_2 + q_3 \leq 25$ odd for which Theorem 7.14 implies that the triangle with angles $(\frac{q_1}{k}\pi, \frac{q_2}{k}\pi, \frac{q_3}{k}\pi)$ unfolds to a surface with dense orbit.

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