

NOTES ON FILIP'S PROOF THAT ORBIT CLOSURES ARE ALGEBRAIC

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These are my personal notes on the work of Filip. Reader beware: I may have introduced misleading or incorrect statements while trying to summarize Filip's argument. Corrections welcome.

1. CONTEXT

Previous results. The setting for Filip's work was as follows. Every known orbit closure was a locus of pairs (X, ω) defined by conditions of the following two types.

- $\text{Jac}(X)$ has certain endomorphisms (projection to a factor, real multiplication on a factor) with ω as an eigenform.
- The difference of two zeros of ω is torsion in (a factor of) $\text{Jac}(X)$.

One should expect any loci cut out by such algebro-geometric conditions to be a variety by basic reasons, and indeed to prove orbit closures are varieties Filip explains that it suffices to show that they are cut out by such conditions. (Filip expands the list slightly.)

By two deep theorems of Möller, every closed orbit is known to be described by such conditions, which you might think would give hope that verifying a similar result for all orbit closures might be possible. But on the other hand, Möller's proof concerning torsion, although short, is only accessible to those with a truly formidable background in algebraic geometry, and it is not even clear what the statement of a generalization of this result to all orbit closures would look like. And perhaps most importantly, Möller's work in fact used that closed orbits are varieties! (Smillie's Theorem gives that closed orbits are line bundles over algebraic curves.)

Linear equations coming from algebraic geometry. Eskin-Mirzakhani-Mohammadi recently showed that all orbit closures are defined by linear equations in period coordinates. Both conditions above imply linear equations on the period coordinates of ω .

Each endomorphism of $\text{Jac}(X)$ gives a map $A : H_1(X, \mathbb{Z}) \rightarrow H_1(X, \mathbb{Z})$. If ω is an eigenform for this endomorphism with eigenvalue r , then

$$\int_{A(\gamma)} \omega = r \int_{\gamma} \omega$$

for all $\gamma \in H_1(X, \mathbb{Z})$. If the orbit closure consists of surfaces with such an endomorphism, then these linear equations are some of the linear equations, guaranteed by Eskin-Mirzakhani-Mohammadi, that define the orbit closure in period coordinates.

The condition that a map $A : H_1(X, \mathbb{Z}) \rightarrow H_1(X, \mathbb{Z})$ gives an endomorphism of the Jacobian is that its real linear extension to an endomorphism of $H_1(X, \mathbb{R})$ is in fact complex linear. The complex structure comes from the real linear isomorphism $H_1(X, \mathbb{R}) \simeq H^{1,0}(X)^*$. This complex structure varies with X in a complicated way, which can be recorded with the period mapping and reflects the variation of Hodge structure. Given that the linear equations defining an orbit closure only involve a single line in $H^{1,0}(X)$ (the one spanned by ω), there is no obvious reason to expect that the linear equations defining an orbit closure would have anything whatsoever to do with this complex structure. (The exception is genus two, where due to the smallness of the space, McMullen observed that certain endomorphism stabilizing a complex line are automatically complex linear.)

If p and q are points on X , and $p - q$ is torsion in $\text{Jac}(X)$, this exactly means that for any path $\gamma_{p,q}$ from p to q , there is a rational homology class $\gamma \in H_1(X, \mathbb{Q})$ such that

$$\int_{\gamma_{p,q}} \eta = \int_{\gamma} \eta$$

for all $\eta \in H^{1,0}(X)$. Thus, taking $\eta = \omega$, we see that the torsion condition does imply a linear equations on periods, but again we should be surprised to see the linear equations guaranteed by Eskin-Mirzakhani-Mohammadi arise in this way, since these equations should really only have to do with ω , and not the other holomorphic one-forms η on X .

Nonetheless, for closed orbits, Möller's work shows that all the linear equations on absolute periods come from endomorphisms, and all the linear equations relating relative to absolute periods come from torsion. In sum, for closed orbits, all the linear equations defining the orbit (closure) arise from endomorphisms and torsion. Again we emphasize that this would constitute a proof that closed orbits are varieties, except that in fact Möller's work in fact uses as an input that closed orbits are varieties.

Deligne semi-simplicity and endomorphisms of the Jacobian.

Let \mathcal{M} be an orbit closure of translation surfaces, and let H^1 denote the flat vector bundle over \mathcal{M} whose fiber over (X, ω) is $H^1(X, \mathbb{C})$. There is closely related bundle H_{rel}^1 with fibers $H^1(X, \Sigma, \mathbb{C})$, and a natural map $p : H_{rel}^1 \rightarrow H^1$. The tangent bundle $T(\mathcal{M})$ of \mathcal{M} is naturally a flat subbundle of H_{rel}^1 , and hence $p(T(\mathcal{M}))$ is a flat subbundle of H^1 . Let $\mathbf{k}(\mathcal{M})$ be the affine field of definition of \mathcal{M} , i.e., the smallest subfield of \mathbb{R} so that \mathcal{M} can be cut by linear equations in period coordinates with coefficients in \mathbb{R} . The author showed that $p(T(\mathcal{M}))$ is defined over $\mathbf{k}(\mathcal{M})$, and that there is a decomposition of flat bundles

$$H^1 = \bigoplus_{\iota} p(T(\mathcal{M}))^{\iota} \oplus E,$$

where the ι are the different field embeddings $\mathbf{k}(\mathcal{M}) \rightarrow \mathbb{C}$, the $p(T(\mathcal{M}))^{\iota}$ are ‘‘Galois conjugates’’ to $p(T(\mathcal{M}))$, and E is some unknown flat bundle. (A consequence of Filip’s work is that all the field embeddings of $\mathbf{k}(\mathcal{M})$ have image in \mathbb{R} , i.e., $\mathbf{k}(\mathcal{M})$ is totally real.)

Suppose it was known that \mathcal{M} was a quasi-projective variety. Then a theorem of Deligne would say that the above decomposition of flat bundles is compatible with the Hodge decomposition, i.e. that each summand is the sum of its intersections with $H^{1,0}(X)$ and $H^{0,1}(X)$ at each point $(X, \omega) \in \mathcal{M}$. Furthermore, for each $r \in \mathbf{k}(\mathcal{M})$, one then obtains an endomorphism on $\text{Jac}(X)$ whose linear action on $H^1(X, \mathbb{C})$ is given by multiplication by the scalar $\iota(r)$ on $p(T(\mathcal{M}))^{\iota}$, and multiplication by 0 on E . (Actually, these are in general only in $\text{End}(\text{Jac}(X)) \otimes \mathbb{Q}$, and are in $\text{End}(\text{Jac}(X))$ when r is in some order in $\mathbf{k}(\mathcal{M})$. That is, in general one gets an action on $H_1(X, \mathbb{Q})$ by $\mathbf{k}(\mathcal{M})$, and there is an order in $\mathbf{k}(\mathcal{M})$ which preserves the integer lattice.)

The key step in Deligne’s proof is the following result of Schmid.

Theorem 1.1 (Theorem of the fixed part). *Suppose that W is a Variation of Hodge Structure (VHS) over a quasi-projective variety, and suppose that ϕ is a flat global section of W . Then the (p, q) parts of ϕ are again flat.*

Recall that a VHS of weight n is a flat bundle W with a direct sum decomposition into subbundles $W^{p,q}$ with $p + q = n$, subject to some elementary conditions, such as the requirement that $\bigoplus_{p \leq p_0} W_{p, n-p}$ be a holomorphic subbundle of W for all p_0 .

The theorem of the fixed part is proved in the following way. Using Hodge norm, one cooks up a function which measures the failure of some specific (p, q) part to be flat. Using computations of a differential-geometric nature, one shows that this function is subharmonic, and that

if it is constant then the given (p, q) part is actually flat. Using the structure at infinity of quasi-projective varieties one shows this function is bounded. Since bounded subharmonic functions on quasi-projective varieties are constant, one concludes the result.

2. FILIP'S PROOF OF SEMI-SIMPLICITY

The first step in Filip's work is to prove Deligne's semi-simplicity result for orbit closures. Since we do not yet know that orbit closures are varieties, we may not simply invoke Deligne's work. Instead the idea is to mimic the proof of Deligne's result, using the dynamics of the $GL(2, \mathbb{R})$ -action to compensate for the fact that we do know ahead of time that the base is a variety. We will begin by proving the Theorem of the Fixed Part. That is, we will assume that we have a global flat section ϕ , and prove that its (p, q) parts are also flat.

The necessary differential-geometric calculations can be nicely black-boxed.

Theorem 2.1 (Black Box). *Suppose that ϕ is a holomorphic section of a VHS over an orbit closure \mathcal{M} . Suppose the VHS has weight n , and $0 \leq q \leq n$, and that $\phi^{n-q', q'} = 0$ for $q' > q$.*

Then $\log \|\phi^{n-q, q}\|$ is subharmonic, and if it is constant, then $\phi^{n-q, q}$ is flat.

Here $\|\cdot\|$ is the Hodge norm. Note that when the base is \mathbb{C} , and the VHS is weight 0 and dimension 1, this recovers the fact that if $\phi : \mathbb{C} \rightarrow \mathbb{C}$ is a holomorphic function, then $\log |\phi(z)|$ is subharmonic, and if $\log |\phi(z)|$ is constant, then $\phi(z)$ is constant.

A version of the Black Box is used in the standard proof of Schmid's Theorem of the Fixed Part, but with $\|\phi^{n-q, q}\|$ instead of $\log \|\phi^{n-q, q}\|$.

The only two VHS that will be needed in the proof of algebraicity are H^1 and $\text{End}(H^1)$. The Hodge decomposition of $\text{End}(H^1)$ is into the $(1, -1)$ endomorphisms, which map $H^{0,1}$ to $H^{1,0}$ and annihilate $H^{1,0}$, the $(0, 0)$ endomorphisms, which preserve both $H^{1,0}$ and $H^{0,1}$, and the $(-1, 1)$ endomorphisms, which map $H^{1,0}$ to $H^{0,1}$ and annihilate $H^{0,1}$.

Define

$$g_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \quad \text{and} \quad r_\theta = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}.$$

Lemma 2.2. *Suppose f is a subharmonic function on \mathcal{M} invariant under r_θ . Suppose f does not grow too fast, in that $f(g_t(X, \omega))$ is $O(e^{ct})$ for some $0 < c < 1$, where the implied constant can be taken to be uniform on compact sets. Then f is constant.*

This exact lemma does not appear in Filip's work; we are choosing to prove the subharmonic functions are constant in a different way than Filip did. This way requires a hard fact about translation surfaces and Filip's doesn't, but we find this way easier to remember. Furthermore, Filip's approach requires a linear growth bound and an "almost sure sub-linear growth bound", whereas the approach we give here just needs not too fast exponential growth. This weakened assumption doesn't seem to make anything easier, but helps to understand what is truly required for the proof of algebraicity.

Proof. The fact is that there exists a compact subset K of \mathcal{M} such the sets

$$B(T) = \{(X, \omega) \in \mathcal{M} : g_t r_\theta(X, \omega) \notin K \text{ for all } \theta \text{ and } 0 \leq t \leq T\}$$

have measure $O(e^{-c'T})$ for all $0 < c' < 1$. (This statement is closely related to Athreya's thesis, and it appears in Avila-Gouezel-Yoccoz as Theorem 2.15.)

The growth bound gives that on $B(T-1) - B(T)$, the function f is at most $O(e^{cT})$. If we choose $c' > c$, we have that $\sum_{T=1}^{\infty} e^{-c'T} e^{cT} < \infty$, so this gives that f is in L^1 .

Now, the value of f at any point (X, ω) is at most the average value over the disk

$$\{g_t r_\theta(X, \omega), 0 \leq \theta \leq 2\pi, 0 \leq t \leq T\}.$$

For almost every $(X, \omega) \in \mathcal{M}$, as $T \rightarrow \infty$, these disks equidistribute, so we see that the value of f at (X, ω) is at most the average value of f on \mathcal{M} . We conclude that f is constant. (In fact this equidistribution is true for all $(X, \omega) \in \mathcal{M}$ by E-M-M. But for almost every (X, ω) it is much easier, and follows just from ergodicity of g_t .) \square

Note that for the VHS we are interested in, $\|\phi(r_\theta(X, \omega))\| = \|\phi(X, \omega)\|$. Work of Forni gives the following growth bound.

$$\|\phi(g_t(X, \omega))\| \leq e^t \|\phi(X, \omega)\|.$$

Thus we get a much better than necessary growth bound on log of Hodge norm (after taking logs, we get linear growth, and we need only not too fast exponential). The Black Box, used repeatedly after subtracting off the parts that we already know are flat, now shows the the theorem of the fixed part.

We will not give the general proof of the semi-simplicity result using the theorem of the fixed part. Rather, in the next section we will derive a particular case, which is sufficient for algebraicity and gives the idea of the general result.

3. THE EQUATIONS ON ABSOLUTE PERIODS AND ENDOMORPHISMS

Now return to the decomposition

$$H^1 = \bigoplus_{\iota} p(T(\mathcal{M}))^{\iota} \oplus E,$$

and let $\phi \in \text{End}(H^1)$ denote the projection onto one of the factors. Using the previous section, we see that the $(1, -1)$, $(0, 0)$ and $(-1, 1)$ parts of ϕ are also flat sections.

Flat sections of $\text{End}(H^1)$ are the same as endomorphisms of $\text{End}(H^1(X, \Sigma))$ that commute with monodromy, so using irreducibility of monodromy and the Schur Lemma we see that each $\phi^{p,q}$ must be a scalar on each $p(T(\mathcal{M}))^{\iota}$. Hence only one of $\phi^{1,-1}$, $\phi^{0,0}$, $\phi^{1,-1}$ is non-zero on our chosen factor $p(T(\mathcal{M}))^{\iota}$ (since scalar multiplication on this factor can't be simultaneous of two different (p, q) types). It must be $\phi^{0,0}$, since both $\phi^{1,-1}$ and $\phi^{1,-1}$ are nilpotent. Hence $\phi = \phi^{0,0}$. By definition, $(0, 0)$ endomorphisms send $H^{1,0}$ to itself and send $H^{0,1}$ to itself. Hence we get that

$$p(T(\mathcal{M}))^{\iota} = \phi(H^{1,0}) + \phi(H^{0,1}) = p(T(\mathcal{M}))^{\iota} \cap H^{1,0} + p(T(\mathcal{M}))^{\iota} \cap H^{0,1}$$

as desired. This shows that the splitting of H^1 is compatible with the Hodge structure, which in turns shows that we have the desired endomorphisms of Jacobians.

Note, the endomorphisms we have produced do not necessarily all come from real multiplication on a factor of $\text{Jac}(X)$ in the strictest sense. Projection onto $\bigoplus_{\iota} p(T(\mathcal{M}))^{\iota}$ gives one endomorphism and a corresponding factor of the Jacobian, and we have also produced an action of $\mathbf{k}(\mathcal{M})$ on $\bigoplus_{\iota} p(T(\mathcal{M}))^{\iota}$ by endomorphisms. When $\mathbf{k}(\mathcal{M})$ has degree smaller than the dimension of $\bigoplus_{\iota} p(T(\mathcal{M}))^{\iota}$, classically speaking this action would not be called real multiplication, because the eigenspaces for the endomorphisms will have dimension greater than one.

4. THE EQUATIONS ON REL

The above endomorphisms are easily seen to give all linear equations satisfied by absolute periods on \mathcal{M} . In particular, if there are no equations on rel, i.e., if $\ker(p) \subset T(\mathcal{M})$, the above suffices to prove algebraicity.

The remaining equations are of the form $\int_{\gamma} \omega = 0$, where now $\gamma \in H_1(X, \Sigma, \mathbf{k}(\mathcal{M}))$. There is a natural homomorphism

$$c : H_1(X, \Sigma, \mathbf{k}(\mathcal{M})) \rightarrow \mathbf{k}(\mathcal{M})^{s-1},$$

where $s = |\Sigma|$. Here c stands for “count”, and indeed this homomorphism merely sums the coefficients of relative cycles that start and end at different points of Σ . (According to taste, one might prefer to define the codomain of c to be $\mathbf{k}(\mathcal{M})^s$, and note that the image is contained in the set where the sum of the coordinates is 0.) This map can also be thought of as the quotient map to $H_1(X, \Sigma, \mathbf{k}(\mathcal{M}))/H_1(X, \mathbf{k}(\mathcal{M}))$. Note that, passing to a finite cover where points of Σ are marked, the map c is globally defined over all of \mathcal{M} .

We will assume that $c(\gamma) \neq 0$, since otherwise γ is in fact an absolute homology class, and we have already taken care of equations coming from absolute homology classes.

For each γ , there is a $\gamma' \in H_1(X, \Sigma, \mathbb{C})$ such that $c(\gamma) = c(\gamma')$ and $\int_{\gamma'} \eta = 0$ for all holomorphic one forms $\eta \in H^{1,0}(X)$. This γ is unique up to adding something in the annihilator of $H^{1,0}(X)$ inside of $H_1(X, \mathbb{C})$.

The difference $\gamma - \gamma'$ is an absolute homology class. Define

$$\pi : H_1(X, \mathbb{C}) \rightarrow \text{Ann}(H^{0,1}) \cap p(T(\mathcal{M}))^*$$

as follows. First, project to the “dual” of $p(T(\mathcal{M}))$, which is a flat subbundle of $H_1(X, \mathbb{C})$. Next, using the decomposition of this into the annihilator of $p(T(\mathcal{M}))^{1,0}$ plus the annihilator of $p(T(\mathcal{M}))^{0,1}$, project to the second coordinate.

We now claim that $\pi(\gamma - \gamma')$ depends only on $c(\gamma)$, and hence gives a global section. Indeed, first note that γ is well defined given $c(\gamma)$, up to the additional of absolute homology classes orthogonal to the “dual” of $p(T(\mathcal{M}))$. Next note that γ' is well defined given γ , up to the addition of an absolute homology class in the annihilator of $H^{1,0}(X)$, and there are no absolute homology classes that annihilate both $H^{0,1}(X)$ and $H^{1,0}(X)$.

$\pi(\gamma - \gamma')$ is not in fact holomorphic when viewed as a section of the bundle H_1 , but it is when viewed as a section of $p(T(\mathcal{M}))^*/\text{Ann}(H^{1,0})$. This is because it can be computed as follows. Define $\text{Ann}(H^{1,0})_\Sigma$ to be the bundle of *relative* homology classes which integrate to zero against all holomorphic one forms, and continue to let $\text{Ann}(H^{1,0})$ denote the bundle of absolute homology classes which integrate to zero against all holomorphic one forms.

Consider $H_1(X, \Sigma, \mathbb{C})$ modulo $\text{Ann}(H^{1,0})_\Sigma$. Note $\text{Ann}(H^{1,0})_\Sigma$ and $\text{Ann}(H^{1,0})$ vary holomorphically, because $H^{1,0}$ varies holomorphically. Note also that $H_1(X, \mathbb{C})/\text{Ann}(H^{1,0})$ and $H_1(X, \Sigma, \mathbb{C})/\text{Ann}(H^{1,0})_\Sigma$ are isomorphic. Thus, first we take the image of γ in

$$H_1(X, \Sigma, \mathbb{C})/\text{Ann}(H^{1,0})_\Sigma = H_1(X, \mathbb{C})/\text{Ann}(H^{1,0}).$$

Then we project to the “dual” of $p(T(\mathcal{M}))$ viewed in this space. (Because the direct sum decomposition of H_1 respects the Hodge structure, we get a corresponding direct sum decomposition of $H_1(X, \mathbb{C}) / \text{Ann}(H^{1,0})$.) The result is the image of $\pi(\gamma - \gamma')$ in $p(T(\mathcal{M}))^* / \text{Ann}(H^{1,0})$, and it is now holomorphic by construction.

This holomorphicity in $p(T(\mathcal{M}))^* / \text{Ann}(H^{1,0})$ is in fact good enough for the Black Box. (In fact, had we stated the Black Box differently, this would have been explicitly allowed, but instead we opted to start with a simpler statement.)

In the next section we will explain the key step in showing $\log \|\pi(\gamma - \gamma')\|$ satisfies an appropriate growth bound along g_t orbits. Hence this subharmonic function is constant and $\pi(\gamma - \gamma')$ is in fact flat. But there are no flat sections of $p(T(\mathcal{M}))^*$, so we get that $\gamma = \gamma'$ as functionals on $p(T(\mathcal{M}))^{1,0}$. Hence γ annihilates $p(T(\mathcal{M}))^{1,0}$. In other words, the same equation that hold for ω actually holds for all holomorphic one forms in $p(T(\mathcal{M}))^{1,0}$!

Similarly, for each field embedding ι , Filip considers a section obtained from the Galois conjugate $\iota(\gamma)$ of γ , and finds that this Galois conjugate equation holds for all holomorphic one forms in $(p(T(\mathcal{M})))^\iota$.

If γ is rational, we conclude that

$$\int_{\gamma} \eta = 0 \quad \text{for all } \eta \in \left(\bigoplus_{\iota} p(T(\mathcal{M}))^\iota \right)^{1,0}.$$

If $c(\gamma) = np_1 - np_2$, this exactly says that $p_1 - p_2$ is n -torsion in this factor of the Jacobian. If $c(\gamma) = \sum a_i p_i$ where p_i are the points in Σ and $a_i \in \mathbb{Q}$, this exactly says that $\sum a_i p_i$ is zero in the corresponding factor of the Jacobian, which is just another kind of torsion relation.

Now, suppose that $c(\gamma) = \sum_{i=1}^s a_i p_i$ where the $a_i \in \mathbf{k}(\mathcal{M})$. (This is the most general case that needs to be considered, because the linear equations defining $T(\mathcal{M})$ are defined over $\mathbf{k}(\mathcal{M})$.) Since the a_i sum to zero, this can be rewritten as $\sum_{i=2}^s a_i (p_i - p_1)$. Pick γ_i to be a path from p_i to p_0 . Write

$$\gamma = \sum a_i \gamma_i - \alpha,$$

where $\alpha \in H_1(X, \mathbf{k}(\mathcal{M}))$. Note that α can be assumed to be in the “dual” of $p(T(\mathcal{M}))$, since the Galois conjugates of the dual of $p(T(\mathcal{M}))$ annihilate $p(T(\mathcal{M}))$. Now, define α' to be the sum of α and its Galois conjugates, so α is rational, and note that α and α' in fact define the same linear functional on $p(T(\mathcal{M}))$. Thus

$$\sum a_i \int_{\gamma_i} \omega = \int_{\alpha'} \omega$$

is one of the equations defining $T(\mathcal{M})$. Now, using the above work, we see that this equation holds for all holomorphic one forms in $p(T(\mathcal{M})^{1,0})$, and that the Galois conjugate equations hold for all holomorphic one forms in $(p(T(\mathcal{M})')^{1,0})$. Multiplying by an integer, we can assume that α' is integral, and this gives that $\sum a_i(p_i - p_1)$ is zero in the factor of the Jacobian corresponding to $\bigoplus p(T(\mathcal{M})')$. Here the $a_i \in \mathbf{k}(\mathcal{M})$ act on the Jacobian via endomorphisms, and for each i we have that $p_i - p_1$ gives a point in the Jacobian in the usual way.

This condition on $\sum a_i(p_i - p_1)$ can be thought of as a “twisted torsion” condition, i.e., it is similar to the usual torsion condition, but “twisted” by the use of the endomorphisms a_i . This is an algebraic condition, and concludes the proof of algebraicity.

One final remark is that when $\ker(p) \cap T(\mathcal{M}) = \{0\}$, as is the case for Teichmüller curves, we can consider $c(\gamma) = p_i - p_1$ one at a time, and so all the a_i are rational (actually 0 or 1) and we get honest torsion, rather than twisted torsion.

5. GROWTH BOUNDS FOR RELATIVE HOMOLOGY

For the previous section to work, it suffices to show the following: for any compact set $K \subset \mathcal{M}$, there are constants $C > 0$ and $0 < c < 1$ such that

$$\log \|P(g_t(\alpha))\| \leq C + e^{ct}.$$

Here P is the projection from relative homology to absolute homology whose kernel is the annihilator of the space of harmonic 1 forms. We will in fact show the desired inequality without the log and for some $c > 0$.

First idea. Consider the norm on relative homology given by picking a constant norm on the quotient $H_1(X, \Sigma)/H_1(X)$, and letting the norm of a relative homology class α be equal to the Hodge norm of $P(\alpha)$ plus the norm of its image in $H_1(X, \Sigma)/H_1(X)$. This is called relative Hodge norm, and we will denote it $\|\alpha\|_{relHodge}$. By construction, $\|\alpha\|_{relHodge} \geq \|P(\alpha)\|$, so it is sufficient to bound the growth of $\|g_t(\alpha)\|_{relHodge}$. However, the required growth bounds on relative Hodge norm are not known.

Second idea. Eskin-Mirzakhani-Mohammadi gave a definition of modified Hodge norm $\|\beta\|_{mod}$ on absolute homology. As in the first idea, gives rise to a relative version $\|\alpha\|_{relmod}$ on relative homology .

This norm grows at most exponentially on g_t orbits, but $\|\beta\|_{mod}$ is not an upper bound for $\|\beta\|$, and hence $\|\alpha\|_{relmod}$ is not an upper

bound for $\|P(\alpha)\|$. (Modified Hodge norm does dominant Hodge norm on cohomology, but this does not hold true for homology.)

Final idea. The failure of $\|\alpha\|_{mod}$ to be an upper bound for $\|\alpha\|$ is mild in comparison to the very weak bounds that we need. Namely, if

$$r(X, \omega) = \sup \frac{\|\alpha\|}{\|\alpha\|_{mod}},$$

then $r(X, \omega)$ itself grows at most exponential along g_t orbits. That is, $r(g_t(X, \omega)) \leq e^{c't}r(X, \omega)$ for some c' . This is simply a consequence of the fact that both modified and usual Hodge norm can increase and decay at most by an exponential factor along g_t orbits (see Lemma 7.5 in EMM). Thus,

$$\begin{aligned} \|P(g_t(\alpha))\| &\leq r(g_t(X, \omega))\|P(g_t(\alpha))\|_{relmod} \\ &\leq r(g_t(X, \omega))\|g_t(\alpha)\|_{relmod} \end{aligned}$$

grows at most exponentially, since each of the two factors grows at most exponentially.

Concluding remark. The final idea shows that relative Hodge norm grows exponentially, in that if (X, ω) is in a compact set K , then there are constants $C, c > 0$ so that

$$\|g_t(\alpha)\|_{relHodge} \leq Ce^{ct}.$$

Here C is related to the max of $r(X, \omega)$ over K . (But c does not depend on K .)

In the first idea, we said that growth bounds for relative Hodge norm are not known. By this we meant that it is unknown if there is a constant $c > 0$ so that $\|g_t(\alpha)\|_{relHodge} \leq e^{ct}\|\alpha\|_{relHodge}$, for all (X, ω) in the stratum. The arguments in “final idea” do not resolve this question.