THE ALGEBRAIC HULL OF THE KONTSEVICH–ZORICH COCYCLE

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ABSTRACT. We compute the algebraic hull of the Kontsevich–Zorich cocycle over any $\mathrm{GL}_2^+(\mathbb{R})$ invariant subvariety of the Hodge bundle, and derive from this finiteness results on such subvarieties.

1. INTRODUCTION

The space of Riemann surfaces equipped with a holomorphic 1-form carries a natural action of $\mathrm{GL}_2^+(\mathbb{R})$. The group of diagonal matrices corresponds to the Teichmüller geodesic flow and can be viewed as a renormalization process for certain flows on surfaces. This renormalization process has applications to a class of dynamical systems including interval exchange transformations and flows on surfaces. For an introduction and survey of these topics, see Forni–Matheus [**FM14**], Masur–Tabachnikov [**MT02**], Wright [**Wri15b**], and Zorich [**Zor06**].

Topological and measure rigidity results for the $\mathrm{GL}_2^+(\mathbb{R})$-action due to McMullen [**McM07**] and Eskin, Mirzakhani, and Mohammadi [**EM13, EMM15**] show many similarities with locally homogeneous spaces and Ratner's Theorems. In particular, if (X, ω) is a Riemann surface with holomorphic 1-form, the closure in the stratum of Abelian differentials of the orbit $\overline{\mathrm{GL}_2^+(\mathbb{R}) \cdot (X, \omega)}$ is an immersed orbifold given in certain local coordinates by linear equations. Such immersed sub-orbifolds, usually called "affine invariant submanifolds", are subvarieties of strata of Abelian differentials [**Fil16**].

Our main results analyze the Kontsevich–Zorich cocycle, which encodes parallel translation of cohomology classes along $\mathrm{GL}_2^+(\mathbb{R})$ orbits. When translation surfaces are described by polygons in the plane, the $\mathrm{GL}_2^+(\mathbb{R})$-action distorts the polygons linearly. The Kontsevich–Zorich cocycle encodes the procedure of cutting and regluing the polygons to a less distorted shape, and hence carries the mysterious part of the dynamics of the $\mathrm{GL}_2^+(\mathbb{R})$-action. It has been studied extensively, see e.g. [**For02, FMZ14a**].

The algebraic hull of a cocycle is, informally speaking, the smallest algebraic group into which the cocycle can be conjugated [**Zim84**].

We analyze the algebraic hull of the Kontsevich–Zorich cocycle over an arbitrary affine invariant submanifold $\mathcal{M}$. Theorem 1.1 computes it in terms of monodromy, and further results on monodromy determine precisely some parts of the algebraic hull. Theorem 1.3 shows that for any sequence of affine manifolds $\mathcal{M}_i \subset \mathcal{M}$ that equidistribute inside $\mathcal{M}$, the algebraic hulls of $\mathcal{M}$ and $\mathcal{M}_i$ eventually agree, up to finite index and compact factors.

The main applications of these results is to finiteness of orbit closures. Theorem 1.5 implies that in any genus, there are only finitely many orbit closures, with only two kinds of exceptions. First, there are always the square-tiled surfaces, which generate Teichmüller curves. Second, there could be finitely many families of orbit closures of a very special kind, and such families are themselves contained in finitely many higher-dimensional orbit closures. Theorem 1.5 is one of many results in mathematics stating that when infinitely many "special" subvarieties of a given dimension exist it is because they are contained in a larger dimensional special subvariety, whose existence implies the existence of the smaller special subvarieties. Compare to the André–Oort and Zilber–Pink Conjectures. (For us the "special" subvarieties are the $\mathrm{GL}_2^+(\mathbb{R})$ invariant ones.)

Methods related to those in the current paper were used by Matheus–Wright [**MW15**] and Lanneau–Nguyen–Wright [**LNW**] to prove finiteness results for Teichmüller curves. Very different methods, which unlike ours are in principle effective, have been used by McMullen, Möller, Brainbridge, and Habegger to prove finiteness results for Teichmüller curves [**McM05b, McM05a, Möl08, BM12, BHM16**]. McMullen classified all orbit closures in genus 2 [**McM07**].

Detailed statements. Let $\mathcal{M}$ be an affine invariant submanifold, equipped with the flat bundles of absolute cohomology H^1 or relative cohomology H^1_{rel}. The fibers of these bundles are the cohomology groups of the Riemann surfaces parametrized by $\mathcal{M}$. The topological trivializations of the bundles lead to a flat connection, and parallel transport along the orbits of $\mathrm{GL}_2^+(\mathbb{R})$ gives the Kontsevich–Zorich cocycle.

Suppose now that V is a subbundle of H^1 or H^1_{rel}, for instance V could be the entire bundle. The algebraic hull of the Kontsevich–Zorich cocycle on V is defined to be the smallest linear algebraic group such that there exists a measurable choice of basis in each fiber of V such that all matrices obtained by parallel translation along $\mathrm{GL}_2^+(\mathbb{R})$ orbits in $\mathcal{M}$ give matrices in that group. It is a nontrivial fact that the algebraic hull is well-defined up to conjugacy. We denote the algebraic hull

by $A_V(\mathcal{M})$, or, when $\mathcal{M}$ is clear from context, just A_V, and similarly we denote the Zariski closure of monodromy as $G_V(\mathcal{M})$ or just G_V.

Let $p : H^1_{rel} \to H^1$ denote the forgetful map from relative to absolute cohomology. For any flat subbundle $V \subset H^1_{rel}$, it is known that $p(V)$ is the direct sum of simple subbundles [**AEM14**].

Consider the tautological bundle $T \subset H^1_{rel}$, defined as the span of the real and imaginary parts of the holomorphic 1-form. The bundle T is a fundamental example of a bundle which is $\mathrm{GL}_2^+(\mathbb{R})$-invariant, but not flat (unless $\mathcal{M}$ is a Teichmüller curve). We call $p(T)$ the tautological bundle of H^1.

Theorem 1.1. *Let $\mathcal{M}$ be an affine invariant submanifold and let V be any flat subbundle of H^1 or H^1_{rel}. The algebraic hull A_V is the stabilizer of the tautological plane in the Zariski closure of monodromy G_V.*

If V does not contain the tautological plane, then the algebraic hull coincides with the Zariski closure of monodromy.

Recall that any affine invariant submanifold $\mathcal{M}$ has a tangent bundle $T\mathcal{M} \subset H^1_{rel}$, and we may also consider its image $p(T\mathcal{M}) \subset H^1$ in absolute cohomology. If the equations defining $\mathcal{M}$ have entries in some number field larger than $\mathbb{Q}$, then the Galois conjugates of the above bundles give additioanal flat subbundles.

The next result follows from Theorem 1.1 and additional results on Zariski closure of monodromy. Even for strata of abelian differentials, the result gives new, nontrivial information on the algebraic hull.

Theorem 1.2. *Let $\mathcal{M}$ be an affine invariant submanifold.*

(1) The algebraic hulls $A_{p(T\mathcal{M})}$ and $A_{T\mathcal{M}}$ are the full group of endomorphisms that respect the symplectic form, the tautological plane, and, in the case of $T\mathcal{M}$, the kernel of the map p from relative to absolute cohomology.

(2) The algebraic hull of a nontrivial Galois conjugate of $p(T\mathcal{M})$ or $T\mathcal{M}$ is the full group of endomorphisms that respect the symplectic form, and, in the case of $T\mathcal{M}$, the kernel of the map p from relative to absolute cohomology.

An endomorphism respects $\ker(p)$ if it acts as the identity on $\ker(p)$; in that case it respects the symplectic form if the induced map on the image in H^1 preserves the symplectic form.

The next results will be essential for proving finiteness statements on orbit closures.

Theorem 1.3. *Fix an affine invariant submanifold $\mathcal{M}$, and $V \subset H^1$ or $V \subset H^1_{rel}$ a flat subbundle over $\mathcal{M}$. For any affine invariant submanifold $\mathcal{M}' \subset \mathcal{M}$ we have the containment of algebraic hulls $A_V(\mathcal{M}') \subset A_V(\mathcal{M})$.*

Furthermore, there is a finite union $\mathcal{B}$ of proper affine invariant submanifolds of $\mathcal{M}$ such that if $\mathcal{M}'$ is not contained in $\mathcal{B}$, then $A_V(\mathcal{M}')$ and $A_V(\mathcal{M})$ are equal up to finite index and compact factors.

The second statement implies eventual agreement (up to finite index and compact factors) of algebraic hulls for infinite sequences of manifolds $\mathcal{M}_i$ equidistributing inside $\mathcal{M}$.

The locus $\mathcal{B}$ is contained in a locus analogous to one where the second fundamental form of the Hodge bundle fails to have full rank. If the algebraic hull of $\mathcal{M}$ is connected and has no compact factors, then the theorem gives that the algebraic hulls of $\mathcal{M}$ and $\mathcal{M}'$ are exactly equal. This is the case when $\mathcal{M}$ is a stratum, and gives the following consequence, where we restrict to square-tiled surfaces because they are abundant and much studied.

Corollary 1.4. *Let $\mathcal{M}_i$ be a sequence of closed $\mathrm{GL}_2^+(\mathbb{R})$ orbits of genus g square-tiled surfaces, and assume $\mathcal{M}_i$ equidistributes to a stratum. Then $A_{H^1}(\mathcal{M}_i) = Sp(2) \times Sp(2g-2)$ for all i sufficiently large.*

The key finiteness statement below involves the notions of rank and degree of an affine invariant manifold. Recall that an affine manifold is cut out by linear equations in period coordinates of an ambient stratum, and has a tangent space. There is a smallest field such that the coefficients of the linear equations can be chosen in it, and it is called the field of affine definition. It is a number field and its degree over $\mathbb{Q}$ is called for brevity the degree of the manifold. The rank of the manifold is defined as half the dimension of the projection of its tangent space to absolute cohomology. For strata, the rank is the genus of the parametrized Riemann surfaces, and for Teichmüller curves the rank is 1. More details on the field of affine definition can be found in [**Wri14**].

Theorem 1.5. *In each stratum of Abelian differentials, all but finitely many affine invariant submanifolds have rank 1 and degree at most 2.*

In each genus there is a finite union of rank 2 degree 1 affine invariant submanifolds $\mathcal{M}$ such that all but finitely many of the affine invariant submanifolds of rank 1 and degree 2 are a codimension 2 subvariety of one of these $\mathcal{M}$.

A special case of of the first statement in Theorem 1.5 is the following.

Corollary 1.6. *In each genus, there are only finitely many Teichmüller curves with trace field of degree greater than 2.*

Affine invariant submanifolds of rank 1 and degree 1 consist of branched covers of tori, and these are dense in every stratum.

The strata in genus 2 and Prym loci in genus 3, 4, 5 are examples of rank 2 degree 1 affine invariant submanifolds; these are known to contain dense sets of codimension 2 affine invariant submanifolds of rank 1 and degree 2 by independent work of McMullen and Calta for the case of genus 2 and work of McMullen for the Prym loci [**Cal04, McM03, McM06**]. A new example of a rank 2 degree 1 affine invariant submanifold was discovered and shown to contain infinitely many rank 1 degree 2 affine invariant submanifolds in [**MMW16**], and one additional example is forthcoming in [**EMMW**].

Theorem 1.7. *Any rank 2 degree 1 affine invariant submanifold $\mathcal{M}$ contains a dense set of codimension 2 affine invariant submanifolds of rank 1 and degree 2.*

Theorem 1.7 is a consequence of the phenomenon discovered by McMullen in [**McM03**].

For some results related to ours, see [**Ham**].

Remark. Mirzakhani conjectured that the only invariant subvarieties of $\Omega\mathcal{M}_g$ of rank two or more, in the sense of [**Wri15a**], are those coming from strata of 1–forms or quadratic differentials [**ANW16**, Conj. 1.3]. Some rank two counterexamples to the conjecture were constructed in [**MMW16**] and [**EMMW**]. It follows immediately from Theorem 1.5 that there can be only finitely many counterexamples to the conjecture in any fixed stratum.

Additional applications. Forthcoming work will use results of this paper to study totally geodesic submanifolds of Teichmüller space [**Wri**] as well as marked points and the illumination and security problems [**AW**].

A more general context for Thm. 1.3. Although the proof of Thm. 1.3 is presented in the context of the $\mathrm{GL}_2\,\mathbb{R}$-action on strata of abelian differentials, the result holds more generally in the context of cocycles over group actions, and is based on the following two assumptions. First, for any invariant measure, the measurable and continuous algebraic hulls of the cocycle should coincide, i.e. any measurable reduction of the structure group of the cocycle in fact agrees a.e. with a continuous one (over the support of the measure). Second, any sequence of ergodic invariant probability measures should equidistribute,

along a subsequence, to another such measure, and the supports of the subsequence should be contained in the support of the limit measure.

On the other hand, the cohomology groups that determine the cocycle also provide coordinate systems for the underling strata and affine submanifolds. Both Thm. 1.1 and Thm. 1.2 use in an essential way this structure.

Organization. Section 2 gives general background on algebraic hulls, and Section 3 proves Theorem 1.1. Section 4 gives results on Zariski closure of monodromy, which together with Theorem 1.1 imply Theorem 1.2. Section 5 proves Theorem 1.3, which is applied in Section 6 to prove Theorem 1.5. Section 6 also establishes Theorem 1.7. The two appendices extend results of [**Fil15**] and show the equivalence of two different definitions of algebraic hull.

2. Algebraic hulls

This section introduces algebraic hulls, in the form that they will be used in this paper. Our definition is slightly different from Zimmer's [**Zim84**, Ch. 4]; in the appendix (see §B.1) we will show it is equivalent.

Chevalley's Theorem. The motivation for the definition of algebraic hull used in this paper is provided by the following result [**Zim84**, Prop. 3.1.4], sometimes called Chevalley's Theorem. Let G be an affine algebraic group, acting faithfully on a vector space V. This gives an inclusion $G \subset \mathrm{GL}(V)$. Below, a *standard tensor operation* refers to taking direct sums, duals, or tensor products, in any order and any finite number of times. That is, a standard tensor operation is any finite direct sum of representations of the form $V^{\otimes i} \otimes (V^\vee)^{\otimes j}$ for $i, j \geq 0$. (Since exterior and symmetric products are contained in such representations, they could also be used below.)

Theorem 2.1. *For any algebraic subgroup $H \subset G$, there exists a representation $\mathcal{S}(V)$ constructed from V by standard tensor operations, and a line $l \subset \mathcal{S}(V)$, such that H is the stabilizer of l in G.*

Similarly, for any reductive *algebraic subgroup $H \subset G$, there exists a representation $\mathcal{S}(V)$ constructed from V by standard tensor operations, and a vector $v \in \mathcal{S}(V)$, such that H is the stabilizer of v in G.*

For example, the diagonal subgroup of $\mathrm{SL}_2(\mathbb{R})$ stabilizes the quadratic form $dx \cdot dy$, and the upper-triangular group stabilizes the horizontal axis in $\mathbb{R}^2$. Statements of Chevalley's Theorem do not always include that V is constructed by standard tensor operations, but this is implicit in the proof, which can be found, for example, in [**Hum75**, Section 11.2]. To connect to the statement in [**Zim84**, Prop. 3.1.4], first note that for an algebraic group $G \subset \mathrm{GL}(V)$, the tensor products $V^{\otimes i} \otimes (V^\vee)^{\otimes j}$ contain all the regular (i.e. algebraic) functions on G, and therefore any other algebraic representation $G \to \mathrm{GL}(W)$ will be contained in some direct sum of standard tensor operations on V. For the statement for reductive groups, by definition a line invariant under a reductive group will have an invariant complement; this determines a projector from the ambient representation to the line, which is now an element of another representation. The reductive group can be defined as the one which preserves the projector.

The connected component of the identity of an algebraic group is not always an algebraic group. For example, the connected component of the identity of $GL_2(\mathbb{R})$ is the positive determinant matrices, and this is not an algebraic subgroup of $GL_2(\mathbb{R})$. Non-algebraic subgroups cannot arise as a stabilizer as in Theorem 2.1.

Cocycles. Suppose now that a group A has an ergodic probability measure-preserving action on a measure space (X, μ). A cocycle over this action is a vector bundle $V \to X$ with a lift of the action of A to V by linear transformations on the fibers. Below, the fiber of V above a point $x \in X$ will be denoted V_x.

Definition 2.2 (Algebraic Hull). The algebraic hull for a cocycle $V \to X$ over an A-action is the collection of all groups $G_x \subset \mathrm{GL}(V_x)$ such that G_x is the largest group preserving the fibers of all the A-invariant line subbundles of tensor power constructions on V. Similarly, the *reductive* algebraic hull is the collection of the largest groups preserving, in each fiber, the fibers of A-invariant sections of tensor power constructions on V.

The groups G_x are defined for μ-a.e. $x \in X$. The definition naturally extends to give measurable, continuous, real-analytic, etc., algebraic hulls, where the corresponding adjective is imposed on the line subbundles or sections defining the algebraic hull.

Remark 2.3.

(i) Definition 2.2 defines the algebraic hull to be a collection of groups $G_x \subset \mathrm{GL}(V_x)$ above μ-a.e point x. In Appendix B we show that the conjugacy class of G_x is independent of the choice of μ-generic point x and is equal to Zimmer's definition of algebraic hull.

(ii) Algebraic varieties satisfy the finite chain condition (i.e. the Noetherian property) so in the definition above only finitely many lines, resp. tensors, are required. In fact, by Chevalley's Theorem 2.1 a single one suffices.

(iii) From the results in [**AEM14**] the algebraic hull of the Kontsevich–Zorich cocycle on H^1 is automatically reductive.

3. Computing the Algebraic Hull

3.1. **Setup.** In the setting of the Kontsevich–Zorich cocycle, we have the action of $\mathrm{GL}_2^+(\mathbb{R})$ on a stratum of translation surfaces, and an affine invariant submanifold $\mathcal{M}$ equipped with a Lebesgue class probability measure, invariant and ergodic under $\mathrm{SL}_2(\mathbb{R})$. Each bundle in the short exact sequence

$$(3.1) \qquad 0 \to \ker(p) \to H_{rel}^1 \xrightarrow{p} H^1 \to 0$$

is equipped with a flat connection inducing the Kontsevich–Zorich cocycle for the action of $\mathrm{SL}_2(\mathbb{R})$. The bundles in (3.1) are taken to be real, i.e. the fibers are given by cohomology with real coefficients. (Parts of Section 5 and the first appendix are exceptions; when we must refer frequently to the Hodge decomposition it is more convenient to use the complexified bundles.) By passing to a finite cover of the ambient stratum (where zeros are labeled), assume that the bundle $\ker(p)$ is in fact trivial. Associated to the short exact sequence (3.1) is the sequence of groups denoted

$$(3.2) \qquad 0 \to \mathrm{Hom}\left(H^1, \ker(p)\right) \to \mathrm{Aut}\left(H_{rel}^1\right) \to \mathrm{Sp}(H^1) \to 0.$$

Here $\mathrm{Aut}(H_{rel}^1)$ is the group of automorphisms of H_{rel}^1 which preserve (i.e. act as the identity on) $\ker(p)$ and act symplectically on H^1. It can be written as a semi-direct product of the two other groups, but only after a choice of splitting $H^1 \to H_{rel}^1$.

An element $\xi \in \mathrm{Hom}(H^1, \ker(p))$ induces a unipotent linear automorphism of H_{rel}^1 defined by $v \mapsto v + \xi(p(v))$. The map $\xi \mapsto (v \mapsto v + \xi(p(v)))$ identifies $\mathrm{Hom}(H^1, \ker(p))$ with the subgroup of automorphisms of H_{rel}^1 acting trivially on H^1.

3.2. **Equivariant sections and bundles.** Suppose W is some bundle over $\mathcal{M}$, such that the $\mathrm{GL}_2^+(\mathbb{R})$-action on $\mathcal{M}$ lifts to W (e.g. by parallel transport, if W is a local system). Throughout, for a point $m \in \mathcal{M}$ the subscript $\bullet_m$ (e.g. W_m) will denote the fiber over this point.

A section ϕ of W defined over $\mathcal{M}$ is *equivariant* if for all $g \in \mathrm{GL}_2^+(\mathbb{R})$ and $m \in \mathcal{M}$ we have $\phi(gm) = g \cdot \phi(m)$.

Rigidity of equivariant sections. Recall that local period coordinates on a stratum $\mathcal{H}$ are given by the cohomology group H^1_{rel}. Similarly, local period coordinates on $\mathcal{M}$ are given by the sublocal system $T\mathcal{M} \subset H^1_{rel}$. These are local systems with complex coefficients that have a real structure, with local coordinates denoted by $(\mathbf{x}, \mathbf{y})$ where $\mathbf{x}$ and $\mathbf{y}$ are the real and imaginary parts.

Fix a basepoint $m_0 \in \mathcal{M}$. Monodromy refers to parallel transport along loops based at m_0. The local systems H^1, H^1_{rel}, etc., may be identified in a neighborhood of m_0 with their fibers at m_0.

Let $A(\mathbf{x}, \mathbf{y})$ denote the area of the flat surface with coordinates $(\mathbf{x}, \mathbf{y})$, which may be calculated as the symplectic pairing of $p(\mathbf{x}), p(\mathbf{y}) \in H^1$. This function is $\mathrm{SL}_2(\mathbb{R})$-invariant and scales quadratically under simultaneous scaling of both coordinates by $\mathbb{R}^\times$.

The next result restricts the local nature of equivariant sections of bundles obtained from H^1 by any tensor operations.

Theorem 3.3 ([**Fil15**, Thm. 7.7]). *Let $\mathbf{H}$ be some tensor construction on H^1 or H^1_{rel} and suppose ϕ is a $\mathrm{GL}_2^+(\mathbb{R})$-equivariant measurable section of $\mathbf{H}$ over $\mathcal{M}$. Then in a neighborhood of m_0 in $\mathcal{M}$, there exist flat local sections $\{s_l\}$ such that a.e. on the neighborhood*

$$(3.4) \qquad \phi(\mathbf{x}, \mathbf{y}) = \sum_l s_l \cdot \frac{P_l(\mathbf{x}, \mathbf{y})}{A(\mathbf{x}, \mathbf{y})^{k_l}},$$

where P_l are homogeneous polynomials of bidegree (k_l, k_l) in the variables $(\mathbf{x}, \mathbf{y})$. Moreover, the polynomials $P_l(\mathbf{x}, \mathbf{y})$ are invariant under the $\mathrm{SL}_2(\mathbb{R})$-action.

This is proven in [**Fil15**, Th. 7.7] for tensor constructions on H^1, and in Appendix A we show that tensor constructions on H^1_{rel} may be handled following the same outline.

Extending the section. Consider the Grassmannian $\mathrm{Gr}^\circ(2, T\mathcal{M}_{m_0})$ of real 2-planes in the real vector space $T\mathcal{M}_{m_0}$ whose projection to H^1 is a symplectically non-degenerate 2-plane. Note that this Grassmannian is an open subset of the full Grassmannian of real 2-planes in the real vector space $T\mathcal{M}_{m_0}$.

Consider the map $(X, \omega) \mapsto \operatorname{span}(\operatorname{Re}(\omega), \operatorname{Im}(\omega))$ from a simply connected neighborhood U of m_0 in $\mathcal{M}$ to $\operatorname{Gr}^\circ(2, T\mathcal{M}_{m_0})$. The fibers of this map are connected components of the intersection of $\operatorname{GL}_2(\mathbb{R})$ orbits with U. Thus we may say that the set of $\operatorname{GL}_2(\mathbb{R})$ orbits near m_0 is locally modeled on $\operatorname{Gr}^\circ(2, T\mathcal{M}_{m_0})$.

Since the polynomials defining ϕ are $\operatorname{SL}_2(\mathbb{R})$ invariant, ϕ defines a function on the image of U in $\operatorname{Gr}^\circ(2, T\mathcal{M}_{m_0})$. We can now see that this function can be extended to all of $\operatorname{Gr}^\circ(2, T\mathcal{M}_{m_0})$.

Proposition 3.5. *Let $\mathbf{H}_{m_0}$ denote the trivial bundle over $\operatorname{Gr}^\circ(2, T\mathcal{M}_{m_0})$. Then the expression in (3.4) defines an algebraic section ϕ_{ext, m_0} of $\mathbf{H}_{m_0}$ defined on the entire Grassmannian $\operatorname{Gr}^\circ(2, T\mathcal{M}_{m_0})$.*

Proof. The expression in Equation 3.4 defines a function on the space of points $(\mathbf{x}, \mathbf{y})$ in $T\mathcal{M}_{m_0} \otimes \mathbb{C}$ that satisfy $A(\mathbf{x}, \mathbf{y}) \neq 0$. This function is $\operatorname{GL}_2^+(\mathbb{R})$ invariant. The Grassmannian $\operatorname{Gr}^\circ(2, T\mathcal{M}_{m_0})$ is the quotient of the space by $\operatorname{GL}_2^+(\mathbb{R})$. □

The Grassmannian. Define $G_{T\mathcal{M}} \subset \operatorname{GL}(T\mathcal{M})$ to be the subgroup which acts as the identity on $(\ker p) \cap T\mathcal{M}$ and by symplectic transformations on $p(T\mathcal{M})$. (Later we will show that $G_{T\mathcal{M}}$ is the Zariski closure of monodromy of $T\mathcal{M}$, justifying the notation.)

The group $G_{T\mathcal{M}}$ acts transitively on $\operatorname{Gr}^\circ(2, T\mathcal{M}_{m_0})$, so

$$(3.6) \qquad \operatorname{Gr}^\circ(2, T\mathcal{M}_{m_0}) = G_{T\mathcal{M}} / \operatorname{Stab}_T$$

is a homogenous space, where T denotes a 2-plane $T \subset T\mathcal{M}_{m_0}$. To describe the stabilizer Stab_T, consider the symplectic-orthogonal decomposition $(p(T\mathcal{M}))_{m_0} = p(T) \oplus p(T)^\perp$. Then Stab_T surjects onto $\operatorname{Sp}(p(T)) \times \operatorname{Sp}(p(T)^\perp)$ with kernel a unipotent subgroup.

Compatibility with monodromy. Because the section ϕ_{ext, m_0} defined by equation (3.4) is polynomial, its equivariance properties extend to Zariski closures, as the next result shows.

To express it, first note that the Grassmannians $\operatorname{Gr}^\circ(2, T\mathcal{M}_m)$ fit into a "local system of real algebraic varieties" over $\mathcal{M}$, i.e. locally the bundle has a product structure, and the gluing maps are locally constant and algebraic, in fact linear. Denote by $\operatorname{Gr}^\circ(2, T\mathcal{M})$ the total space of this bundle. The monodromy then acts on the fibers of the projection $\operatorname{Gr}^\circ(2, T\mathcal{M}) \to \mathcal{M}$ via parallel transport.

Proposition 3.7. *Let ϕ be a $\operatorname{GL}_2^+(\mathbb{R})$-equivariant section of $\mathbf{H}$ over $\mathcal{M}$. By Proposition 3.5 it gives rise to an algebraic section ϕ_{ext, m_0} of the trivial bundle $\mathbf{H}_{m_0}$ over $\operatorname{Gr}^\circ(2, T\mathcal{M}_{m_0})$, which can be viewed as a map $\phi_{ext, m_0} : \operatorname{Gr}^\circ(2, T\mathcal{M}_{m_0}) \to \mathbf{H}_{m_0}$.*

Then ϕ_{ext,m_0} *is equivariant for the simultaneous action of the Zariski closure of the monodromy inside* $\mathrm{GL}\,(\mathbf{H}_{m_0}) \times \mathrm{GL}(T\mathcal{M}_{m_0})$ *(as given in* (3.8) *below, with* $\mathrm{GL}(T\mathcal{M}_0)$ *acting on the Grassmannian and* $\mathrm{GL}(\mathbf{H}_{m_0})$ *acting on* $\mathbf{H}_{m_0}$*).*

Proof. The $\mathrm{GL}_2\,\mathbb{R}$-equivariant section $\phi : \mathcal{M} \to \mathbf{H}$ extends, using Prop. 3.5, to a section $\phi_{ext} \colon \mathrm{Gr}^{\circ}(2, T\mathcal{M}) \to \mathbf{H}$. Note that the fiber of $\mathbf{H}$ only depends on $m \in \mathcal{M}$ and not the point of the Grassmannian (hence the abuse of notation). In a local trivialization of $\mathrm{Gr}^{\circ}(T, \mathcal{M}) \to \mathcal{M}$ as a product of a Grassmannian and an open set in $\mathcal{M}$, the section $\phi_{ext} : \mathrm{Gr}^{\circ}(2, T\mathcal{M}) \to \mathbf{H}$ is polynomial in the Grassmannian direction and flat in the direction of $\mathcal{M}$.

From the local description of an equivariant section ϕ in Eq. (3.4), the section ϕ_{ext} is given by the same formula, but now viewed over the entire Grassmannian bundle. Consider its behavior under a change of chart. Both the sections s_l and the coordinates $\mathbf{x}, \mathbf{y}$ will change according to the change of coordinates map. Going around a loop γ in the affine manifold $\mathcal{M}$ and comparing results, it follows that

$$(3.8) \quad \sum_l s_l \cdot \frac{P_l(\mathbf{x}, \mathbf{y})}{A(\mathbf{x}, \mathbf{y})^{k_l}} = \sum_l (\rho_{\mathbf{H}}(\gamma)s_l) \cdot \frac{P_l(\rho_{T\mathcal{M}}(\gamma)\mathbf{x}, \rho_{T\mathcal{M}}(\gamma)\mathbf{y})}{A(\rho_{T\mathcal{M}}(\gamma)\mathbf{x}, \rho_{T\mathcal{M}}(\gamma)\mathbf{y})^{k_l}}.$$

Here $\rho_{\bullet}(\gamma)$ denotes the monodromy matrix corresponding to γ. On s_l it acts by the appropriate representation on $\mathbf{H}_{m_0}$ (denoted $\rho_{\mathbf{H}}$ for brevity); on $(\mathbf{x}, \mathbf{y})$ it acts via the representation on $T\mathcal{M}$.

The equality of rational functions in (3.8) holds for all γ in the monodromy, therefore it holds for all elements of the Zariski closure of monodromy. $\qquad\square$

We can now prove the main result, Theorem 1.1.

Proof of Theorem 1.1. Note that, by Chevalley's Theorem, the Zariski closure of monodromy is the largest subgroup that fixes any flat subbundle of any tensor construction.

From the definition of algebraic hull, it is clear that it is contained in the Zariski closure of monodromy. Indeed, any flat tensor is automatically $\mathrm{GL}_2^+(\mathbb{R})$-invariant. Additionally, the tautological plane (in either H^1 or H^1_{rel}) is also invariant under the $\mathrm{GL}_2^+(\mathbb{R})$-action, so the algebraic hull must be contained in its stabilizer. In particular, the algebraic hull is contained in the intersection of these two groups.

We need to show that conversely, the algebraic hull contains the intersection of the Zariski closure of monodromy and the stabilizer of the tautological plane. Suppose therefore that ϕ is a global tensor in

H (some tensor construction on H^1 or H^1_{rel}) and that ϕ is $\mathrm{GL}_2^+(\mathbb{R})$-equivariant.

By Equation 3.3, ϕ has locally on $\mathcal{M}$ a representation in coordinates as a ratio of polynomials. We therefore consider the value of ϕ at every, and not just almost every, point.

Fix a basepoint $m_0 \in \mathcal{M}$. By Proposition 3.5, ϕ extends to an algebraic section $\phi_{m_0,ext}$ of the trivial bundle $\mathbf{H}_{m_0} \to \mathrm{Gr}^\circ(2, T\mathcal{M}_{m_0})$. Note that m_0 gives a point in $\mathrm{Gr}^\circ(2, T\mathcal{M}_{m_0})$ corresponding to its $\mathrm{GL}_2^+(\mathbb{R})$-orbit (i.e. the value of the tautological plane at m_0). By construction $\phi_{m_0,ext}(m_0) = \phi(m_0)$.

By Proposition 3.7 the section $\phi_{m_0,ext}$ is equivariant for the Zariski closure of monodromy acting simultaneously on the Grassmannian and on the bundle over it, i.e. for any such γ we have

$$\phi_{m_0,ext}(\gamma \cdot p) = \gamma \cdot \phi_{m_0,ext}(p).$$

Now if γ is also in the stabilizer of the tautological plane, the above equation becomes (for $p = m_0$) simply $\phi_{m_0,ext}(m_0) = \gamma \cdot \phi_{m_0,ext}(m_0)$.

Using that $\phi_{m_0,ext}(m_0) = \phi(m_0)$, this implies $\phi(m_0) = \gamma \cdot \phi(m_0)$. This invariance of ϕ implies that the algebraic hull contains the intersection of the Zariski closure of monodromy and the stabilizer of the tautological plane. $\qquad\square$

4. Monodromy

4.1. **Setup.** We keep the notation for bundles as in the beginning of Section 3, in particular we shall use the short exact sequence of cohomology bundles (3.1).

Consider a fixed affine invariant manifold $\mathcal{M}$ in a stratum of flat surfaces $\mathcal{H}$. Define $\ker(p)_{\mathcal{M}}$ to be $\ker(p) \cap T\mathcal{M}$. We have the short exact sequences

$$(4.1) \qquad 0 \to \ker(p)_{\mathcal{M}} \to T\mathcal{M} \xrightarrow{p} p(T\mathcal{M}) \to 0$$

of bundles and

$$(4.2) \qquad 0 \to U \to G_{T\mathcal{M}} \xrightarrow{p} G_{p(T\mathcal{M})} \to 0$$

of monodromies. The groups $G_{T\mathcal{M}}$ and $G_{p(T\mathcal{M})}$ are the Zariski closures of monodromies on the corresponding bundles, while U denotes the kernel of the projection. Note that U is naturally a subgroup of transformations that act by the identity when projected to $p(T\mathcal{M})$. This last group is naturally identified with $\mathrm{Hom}(p(T\mathcal{M}), \ker(p)_{\mathcal{M}})$, which will be abbreviated $U_{\mathcal{M}}$.

In Section 4.2 we shall compute the Zariski closure of the two monodromy groups. Namely, we will see that on absolute cohomology we

have $G_{p(T\mathcal{M})} = \mathrm{Sp}(p(T\mathcal{M}))$, i.e. we get the full symplectic group. On relative cohomology, we will see that the group is as large as it can be, namely the kernel U is all of $U_\mathcal{M}$.

Additionally, we will see that on bundles other than those coming from $\mathcal{M}$, the monodromy is "decoupled" from that on the tangent bundle to $\mathcal{M}$. A precise statement is Proposition 4.8.

Remark 4.3 (On Zariski closures). Throughout this paper, Zariski closures are taken with respect to $\mathbb{R}$, as opposed to $\mathbb{Q}$ (which could be larger). Concretely, the Zariski closure of a subset of $GL_n(\mathbb{R}) \subset \mathbb{R}^{n^2}$ is the intersection of $GL_n(\mathbb{R})$ and the zero locus in $\mathbb{R}^{n^2}$ of all *real* polynomials vanishing on the set.

4.2. **Monodromy of the Kontsevich–Zorich cocycle. Monodromy and absolute cohomology.**

It follows from [**AEM14**] that the Kontsevich–Zorich cocycle on H^1 or any of its tensor powers is reductive, i.e. any invariant bundle has a complement. This gives the decomposition of H^1 over $\mathcal{M}$

$$(4.4) \qquad H^1 = p(T\mathcal{M}) \oplus \left(\bigoplus_\iota p(T\mathcal{M})_\iota \right) \oplus V.$$

The bundles $p(T\mathcal{M})_\iota$ are Galois-conjugates of $p(T\mathcal{M})$ and V is the symplectic (as well as Hodge) orthogonal to the other spaces. There is one $p(T\mathcal{M})_\iota$ for each non-identity embedding of the field of affine definition of $\mathcal{M}$ to $\mathbb{R}$; see [**Wri14**] for more details. The list of possible monodromy groups, up to compact factors is given in [**Fil14**, Thm 1.2]. In particular, the next result holds.

Theorem 4.5 ([**Fil14**, Corollary 1.7]). *The Zariski closure of monodromy on $p(T\mathcal{M})$ or any of its Galois conjugates $p(T\mathcal{M})_\iota$ is the full symplectic group $\mathrm{Sp}(p(T\mathcal{M}))_\iota$.*

Remark 4.6. This particular consequence of [**Fil14**] can also be derived as follows, assuming familiarity with [**Wri15a**]. By [**MW**, Lemma 4.6], $\mathcal{M}$ contains a surface with an equivalence class of $\mathcal{M}$-parallel cylinders all of whose moduli are rationally related. Let α_i be the core curves of the cylinders in the given equivalence class. The twist on this equivalence class gives a closed loop in $\mathcal{M}$, whose monodromy on H_1 is a composition of powers of the Dehn twists $\gamma \mapsto \gamma + \alpha_i\langle\alpha_i, \gamma\rangle$. On $p(T\mathcal{M})^\vee$, which may be viewed as a subbundle of H_1, all the α_i are collinear by the definition of $\mathcal{M}$-parallel, hence the monodromy has the form $\gamma \mapsto \gamma + C\alpha_1\langle\alpha_1, \gamma\rangle$. A short argument of Kazhdan-Margulis gives that any subgroup of the symplectic group that is totally irreducible

and contains such a transformation must be Zariski dense, compare to [**FK88**, page 250]. By [**Wri14**], the monodromy of $p(T\mathcal{M})$ is totally irreducible.

Monodromy and relative cohomology. The monodromy on $T\mathcal{M}$ surjects onto monodromy on $p(T\mathcal{M})$ and we would like to understand the unipotent kernel U (see (4.2)). Recall also that the kernel sits inside the linear transformations that act by identity on $p(T\mathcal{M})$, which is naturally identified with $U_{\mathcal{M}} := \mathrm{Hom}(p(T\mathcal{M}), \ker(p)_{\mathcal{M}})$. For the Galois-conjugate bundles, we can similarly define $U_{\mathcal{M},\iota} := \mathrm{Hom}(p(T\mathcal{M})_\iota, \ker(p)_{\mathcal{M}})$.

Proposition 4.7. *The Zariski closure of monodromy on $T\mathcal{M}$ is $U_{\mathcal{M}} \rtimes \mathrm{Sp}(p(T\mathcal{M}))$. In other words, the unipotent part is as large as it can be.*

The same statement holds for the Galois-conjugate bundles: the Zariski closure of monodromy is $U_{\mathcal{M},\iota} \rtimes \mathrm{Sp}(p(T\mathcal{M})_\iota)$.

Proof. First, observe that $U_{\mathcal{M}}$ has a natural action of the monodromy $\mathrm{Sp}(p(T\mathcal{M}))$. The kernel $U \subset U_{\mathcal{M}}$ is invariant under this action, therefore $U = \mathrm{Hom}(p(T\mathcal{M}), S)$ for some subspace $S \subset \ker(p)_{\mathcal{M}}$. We will now see that S in fact equals $\ker(p)_{\mathcal{M}}$, thus establishing the claim.

Suppose therefore that S is such that the unipotent part of the monodromy is contained in $\mathrm{Hom}(p(T\mathcal{M}), S)$. We will construct a flat subbundle $E_S \subset T\mathcal{M}$ as follows.

Fix a point $m_0 \in \mathcal{M}$ and consider a lift L of $p(T\mathcal{M})_{m_0}$ to $T\mathcal{M}_{m_0}$, i.e. a subspace $L \subset T\mathcal{M}_{m_0}$ such that p is an isomorphism from L to $p(T\mathcal{M})$. Define the fiber of E_S at m_0 to be $(E_S)_{m_0} = \mathrm{span}(L, S)$.

Extend now E_S by parallel transport of the fiber at m_0 to all of $\mathcal{M}$. By the assumption on monodromy that its unipotent part lies in $\mathrm{Hom}(p(T\mathcal{M}), S)$, we see that this gives a well-defined extension. Note also that $E_S \cap \ker(p) = S$ by construction and E_S is not contained in $\ker(p)$.

However, by [**Wri14**, Theorem 7.4] there are no proper flat subbundles of $T\mathcal{M}$ other than those contained in $\ker(p)$, therefore $E_S = T\mathcal{M}$ and thus $S = \ker(p) \cap T\mathcal{M}$.

The statement for the Galois-conjugate bundles now follows by noting that the dimension of the unipotent part does not change under Galois conjugation, and it is maximal. $\qquad\square$

Decoupling monodromies. Let Γ denote the (orbifold) fundamental group of $\mathcal{M}$. Monodromy representations are denoted $\rho_V : \Gamma \to \mathrm{GL}(V)$, where V is an appropriate bundle (e.g. $T\mathcal{M}, p(T\mathcal{M}), H^1$, etc) and their Zariski closures are denoted G_V.

Proposition 4.8. *Suppose that W is a flat irreducible subbundle of H^1 other than $p(T\mathcal{M})$, and let G_W be the Zariski closure of the monodromy $\rho_W(\Gamma)$.*

> *(i) The Zariski closure of monodromy on $p(T\mathcal{M})\oplus W$ is $\mathrm{Sp}(p(T\mathcal{M}))\times G_W$.*
> *(ii) The Zariski closure of monodromy on $T\mathcal{M}\oplus W$ is $G_{T\mathcal{M}}\times G_W$.*
> *(iii) Let $\widetilde{W}:=p^{-1}(W)$ denote the preimage of W in relative cohomology, and assume $G_{\widetilde{W}}$ is the Zariski closure of monodromy on it. Then the Zariski closure of monodromy on $T\mathcal{M}\oplus\widetilde{W}$ is $G_{T\mathcal{M}}\times G_{\widetilde{W}}$.*

Proof. To prove (i), let $H:=\overline{\rho_{p(T\mathcal{M})\oplus W}(\Gamma)}$ be the Zariski closure of the monodromy. By assumption $H\subset\mathrm{Sp}(p(T\mathcal{M}))\times G_W$ surjects when projected to either component.

Consider the kernel $K\subset H$ of the surjection $H\twoheadrightarrow G_W$. Note that projecting K to $\mathrm{Sp}(p(T\mathcal{M}))$ embeds it as a normal subgroup. Indeed, K is normal in H, and H surjects onto $\mathrm{Sp}(p(T\mathcal{M}))$.

Since the symplectic group is simple, K is either the full group (in which case we are done) or trivial. Suppose that K is trivial.

Then H is the graph of an isomorphism $\mathrm{Sp}(p(T\mathcal{M}))\to G_W$. Therefore G_W is isomorphic to a symplectic group, and by the classification in [**Fil14**, Thm. 1.2] a symplectic group can only occur in the standard representation. Thus the isomorphism of monodromy groups gives also an isomorphism of the flat bundles $p(T\mathcal{M})$ and W. It follows that $p(T\mathcal{M})$ and W have the same Lyapunov exponents. This is a contradiction, since $p(T\mathcal{M})$ has top Lyapunov exponent 1, and all other exponents in H^1 are strictly smaller by a result of Forni [**For02**]. (It is also possible to derive a similar contradiction by directly using the representation theory of the symplectic group instead of invoking [**Fil14**, Thm. 1.2].)

To prove (ii), let again $H\subset G_{T\mathcal{M}}\times G_W$ be the Zariski closure of the monodromy. Recall the unipotent radical of $G_{T\mathcal{M}}\times G_W$ is $G_{T\mathcal{M}}^u$. Letting K be the kernel of the map $H\to\mathrm{Sp}(p(T\mathcal{M}))\times G_W$, it follows that $K\subset G_{T\mathcal{M}}^u$. Since H surjects onto $G_{T\mathcal{M}}$ by Proposition 4.7, it follows that K surjects onto $G_{T\mathcal{M}}^u$. Using part (i), it now follows that H is all of $G_{T\mathcal{M}}\times G_W$.

To prove (iii), let again $H\subset G_{T\mathcal{M}}\times G_{\widetilde{W}}$ be the Zariski closure of monodromy; we want to show H is the entire product. By assumption H surjects onto both $G_{\widetilde{W}}$ and $G_{T\mathcal{M}}$. Let $K\subset H$ be the kernel of the surjection to $G_{\widetilde{W}}$. As before, K viewed as a subgroup in $G_{T\mathcal{M}}$ is normal and surjects onto $\mathrm{Sp}(p(T\mathcal{M}))$. Such subgroups are in bijection with flat bundles $E\subset T\mathcal{M}$ which surject onto $p(T\mathcal{M})$, but from

[**Wri14**, Thm 5.1] we must have $E = T\mathcal{M}$, i.e. $K = G_{T\mathcal{M}}$. To see the bijection between subgroups and subbundles, note that the group is given by automorphism of the fibers which preserve the subbundle (and act symplectically on the quotient). $\square$

5. Algebraic Hulls along limits

Recall that by the results of [**EM13, EMM15**], given any infinite sequence of affine invariant manifolds $\{\mathcal{M}_i\}$ in a fixed stratum, there exists an affine manifold $\mathcal{M}$ and a subsequence $\{\mathcal{M}_{n_i}\}$ such that $\mathcal{M}_{n_i} \subseteq \mathcal{M}$, and Lebesgue measure on $\mathcal{M}_{n_i}$ tends to Lebesgue measure on $\mathcal{M}$. In particular, the $\mathcal{M}_{n_i}$ become dense in $\mathcal{M}$.

This section will establish that the algebraic hulls of the manifolds $\mathcal{M}_{n_i}$ and the algebraic hull of $\mathcal{M}$ eventually agree, up to finite index and compact factors. We give two proofs – one of a general ergodic-theoretic flavor, and a second one, based on Hodge-Teichmüller tensors. Both proofs imply that outside a finite collection of affine submanifolds $\mathcal{B} \subset \mathcal{M}$, any other affine submanifold $\mathcal{M}' \subset \mathcal{M}$ has the same algebraic hull as $\mathcal{M}$, up to compact factors (assuming $\mathcal{B} \cap \mathcal{M}' = \emptyset$). The proof via Hodge-Teichmüller tensors gives further information on the locus $\mathcal{B}$, whereas the ergodic-theoretic one only implies its existence.

Finite index, compact factors, and Forni subspaces. For the purposes of this section, two groups G_1, G_2 (contained in the same ambient GL_n) *agree up to finite index* if their connected components of identity are the same. The two groups agree *up to compact factors* if there is a common normal subgroup N such that each of the quotients G_i/N is compact.

Remark 5.1.

(1) Compact factors in the monodromy arise in particular when a sub-VHS V of H^1 may be written as the tensor product of a weight 1 VHS with Zariski closure of monodromy G and a weight 0 VHS with Zariski closure of monodromy K, which must be compact. In this case the monodromy of V is contained in $K \times G$, which may act via an irreducible representation (some examples can be found in [**FFM15**]). In the case where the weight 1 VHS is trivial one calls V a Forni subspace [**FMZ14b, For02**]. The results below allow for the possibility that $\mathcal{M}$ has Zariski closure of monodromy $G \times K$ but $\mathcal{M}_i$ has monodromy contained in $G \times K'$ with $K' \subsetneq K$.

(2) For an algebraic group G, denote by $G^\circledast$ the minimal normal algebraic subgroup of G such that $G/G^\circledast$ is compact. Note that

in particular $G^\oplus$ is connected in the Zariski topology. To prove that two algebraic groups G_1, G_2 agree up to finite index and compact factors, it suffices to check that $G_1^\oplus$ and $G_2^\oplus$ agree.

The algebraic hull of any factor of the Kontsevich–Zorich cocycle is semisimple, up to compact factors. Indeed, from [**AEM14**] it follows that the hull is reductive and from the classification in [**Fil14**, Thm. 1.2] it follows that all non-compact factors have to be semisimple. In particular, the algebraic hull cannot contain any non-compact abelian factors.

It then follows that for the algebraic hull G of any factor of the Kontsevich–Zorich cocycle, the group $G^\oplus$ is semisimple.

5.1. **The case of absolute cohomology.**

Theorem 5.2. *Suppose $\{\mathcal{M}_i\}$ is an infinite sequence of affine submanifolds of $\mathcal{M}$ that equidistribute towards another affine manifold $\mathcal{M}$.*

Let G_i and G be the algebraic hulls of $\mathcal{M}_i$ and $\mathcal{M}$ respectively, for the $\mathrm{GL}_2^+(\mathbb{R})$-action on the absolute cohomology bundle H^1.

- *(i) We have $G_i \subseteq G$ for all sufficiently large i.*
- *(ii) There exists $N \geq 0$ such that for all $i \geq N$, the groups G_i and G agree up to finite index and compact factors.*

Remark 5.3. By Theorem 4.5, the Zariski closure of monodromy for $p(T\mathcal{M})$ and its Galois conjugates do not have compact factors, and so the algebraic hull of any of these bundles for $\mathcal{M}_i$ and $\mathcal{M}$ are exactly equal for i sufficiently large. (Later this will also apply to $T\mathcal{M}$.)

5.2. **First proof of Theorem 5.2.** We begin by recalling some useful preliminaries from ergodic theory.

Suppose we have a bundle $P_F \to X$ over some space X, with fiber F and structure group G. In other words, locally on X we have an isomorphism $P_F|_{U_i} \cong U_i \times F$ (where $U_i \subset X$) and the gluing maps on overlaps are given by maps $U_i \cap U_j \to G$.

Suppose now that X carries an action of a group A, and the action lifts to P_F by G-maps, i.e. after local trivialization of the bundle, the maps between fibers are in G. (This is independent of the trivialization, since gluings are in G as well.) Suppose next that $s : X \to P_F$ is an A-equivariant section, i.e. $s(a \cdot x) = a \cdot s(x)$. Then s descends to a *map*

$$(5.4) \qquad\qquad s : X \to F/G$$

where F/G is the space of G-orbits on F.

The next results, due to Borel–Serre in the algebraic case, and Margulis and Zimmer for measures, give control over spaces of G-orbits

(see [**Zim84**, Sec. 3.2]). Throughout, we consider the real points of the corresponding algebraic varieties.

Proposition 5.5. *Let G be a real-algebraic group, and V an algebraic variety with a G-action.*

*Then the space of G-orbits on V, with its induced topology, is countably separated ([**Zim84**, Def. 2.1.8]). Moreover, any G-orbit in V is locally closed ([**Zim84**, Thm. 3.1.1]).*

*The same separability properties hold for the space of probability measures on V with the weak topology, for the induced action of G ([**Zim84**, Thm. 3.2.6, Cor. 3.2.12]).*

First proof of Theorem 5.2. Part (i) is immediate from [**Fil15**, Thm. 1.5] – the measurable and continuous (in fact, real-analytic) algebraic hulls of $\mathcal{M}$ coincide. Indeed, the cited result implies that any measurable invariant tensor is necessarily continuous, and is thus well defined on $\mathcal{M}_i \subset \mathcal{M}$.

Part (ii) is proved by contradiction. First, by passing to a finite cover of $\mathcal{M}$, we can assume that G is Zariski-connected, in particular an irreducible representation of G is strongly irreducible, i.e. it does not contain a proper finite collection of subspaces permuted by G. Suppose now that there exists a subsequence of the $\mathcal{M}_i$ such that we have the strict inclusion $G_i^{\circledast} \subsetneq G^{\circledast}$, where for a group H the subgroup $H^{\circledast}$ was defined in Rmk. 5.1(2) as the smallest normal subgroup such that $H/H^{\circledast}$ is compact.

Chevalley's Theorem associates to the groups $G_i^{\circledast} \subset G$ lines l_i inside G-representations V_i such that $G_i^{\circledast}$ is the stabilizer of l_i. There are only finitely many isomorphism types of groups $G_i^{\circledast}$, since they are semisimple and connected (see Rmk. 5.1(2)). By [**GP11**, Thm. XXIV 7.3.1(i)] the homomorphisms $\mathrm{Hom}(G_i^{\circledast}, G^{\circledast})$ form a variety of finite type, so in particular $\mathrm{Hom}(G_i^{\circledast}, G^{\circledast})$ has finitely many components. The proof of Chevalley's Theorem shows that all subgroups parameterized by a given component can be obtained as stabilizers of lines in a fixed representation V. So passing to a further subsequence (still denoted $G_i^{\circledast}$), we may assume that all the l_i occur inside one single representation V.

To get a contradiction to $G_i^{\circledast} \subsetneq G^{\circledast}$ for all sufficiently large i, it suffices to prove the following. Let $V = V' \oplus V''$ where V' is G-irreducible and V'' is G-invariant. If l_i is contained in V'' for all sufficiently large i, then we apply the reasoning to V''. Therefore, suppose that l_i projects non-trivially to V' along a subsequence. We will show G acts on V' via a compact group. If G acts on each irreducible $V' \subset V$ via a compact group, then G acts on V via a compact group, and so we will be able

to conclude that $G_i^\varphi = G^\varphi$. (We must also have that G_i acts on V via a compact group, since $G_i \subset G$.)

Assume that l_i projects non-trivially to V' for a subsequence, which we take again to be l_i. From now on, identify all the groups with their images in $\mathrm{GL}(V')$ and assume that $l_i \subset V'$. Then the orbit $G_i \cdot l_i \subset \mathbb{P}(V')$ is identified with G_i/G_i^φ and carries a natural G_i-invariant probability measure $\eta_{i,mod}$, since the quotient G_i/G_i^φ is compact.

Associate to the G-representation V' the vector bundle $E' \to \mathcal{M}$; since V arises as a subrepresentation in tensor construction on the natural representation of G on H^1, E' is itself a subbundle of such a natural tensor construction. Over each $\mathcal{M}_i$ define the measure η_i on $\mathbb{P}(E')$ which is the product of Lebesgue measure on $\mathcal{M}_i$ with the model measure $\eta_{i,mod}$ in the fiber direction. By construction, the probability measure η_i is invariant under $\mathrm{SL}_2(\mathbb{R})$.

Let now η be any weak limit of the η_i; it will still be invariant under $\mathrm{SL}_2(\mathbb{R})$ and now project to Lebesgue measure on $\mathcal{M}$. Denote by $\mathscr{P}(\mathbb{P}(E'))$ the bundle of probability measures on the fibers of $\mathbb{P}(E')$. Then the measure η, via its disintegration, gives a section $s : \mathcal{M} \to \mathscr{P}(\mathbb{P}(E'))$.

By (5.4), the section s descends to a map $[s] : \mathcal{M} \to G\backslash\mathscr{P}(\mathbb{P}(V'))$ where $\mathscr{P}(\mathbb{P}(V'))$ is the space of probability measures on the projectivization of the G-representation V'. Because the space of probability measures divided by the G-action is countably separated (Proposition 5.5), by [**Zim84**, Proposition 2.1.10] it follows that s takes values in a single G-orbit, Lebesgue-a.e. on $\mathcal{M}$. Therefore, for Lebeague a.e. $x \in \mathcal{M}$, the measure η_x given by the disintegration of η along the fiber $E'_x \approx V'$ is given by

$$\eta_x = \psi(x)_*\eta_0,$$

where $\psi(x) \in G$, and η_0 is some fixed measure on V'. (Here we are choosing some measurable trivialization of the bundle E'). Let $H \subset G$ denote the stabilizer of η_0.

If H is a proper subgroup of G, we can reduce the algebraic hull over $\mathcal{M}$ to H as follows. First, by [**Zim84**, Corollary 3.2.23] stabilizers of measures are algebraic subgroups, so there exists a tensor construction $T(V')$ on the representation V', and a line $l' \subset T(V')$ such that H is the stabilizer of l'. Moreover, for a.e. $x \in \mathcal{M}$ the linear map $\psi(x)^{-1}$ yields an isomorphism $E'_x \to V'$ taking η_x to η_0; such isomorphisms are parametrized by H acting by postcomposition on V'. A linear isomorphism $E'_x \to V'$ induces one on tensor constructions $T(E'_x) \to T(V')$ and so we can pull back the line $l \subset T(V')$ to $l_x \subset T(E'_x)$. The linear isomorphism was well-defined up to postcomposition with H,

but l is H-invariant so l_x is well-defined. The collection of lines l_x gives a further reduction of the algebraic hull over $\mathcal{M}$, which is not possible. Thus $G = H$, so G leaves invariant a nontrivial measure $[\eta]$ on $\mathbb{P}(V')$.

According to [**Zim84**, Corollary 3.2.2] either the stabilizer of the measure $[\eta]$ is compact, or there is a proper subspace of positive $[\eta]$-mass which is left invariant by a finite index subgroup of G. But we assumed at the start that G is connected (by passing to a finite cover) and V' is G-irreducible, so it must be the case that G acts on V' via a compact group. □

5.3. **Second proof of Theorem 5.2.** We now give the second approach to Theorem 5.2, which is related to [**MW15**]. In this section we assume all bundles are complexified.

Any tensor construction on H^1, denoted **H**, will admit a Hodge decomposition $\mathbf{H} = \oplus \mathbf{H}^{p,q}$. For establishing properties of the algebraic hull using Hodge-Teichmüller tensors, the following concepts will be useful. Throughout, $m \in \mathcal{M}$ is some point, and $\mathbf{H}_m$ denotes the fiber of **H** at m.

Definition 5.6 (Hodge-Teichmüller tensor)**.** A pure Hodge-Teichmüller tensor at m is an element of $\mathbf{H}_m$ of pure Hodge type (p, q) for some p, q such that the parallel transport along any path in the $\mathrm{GL}_2^+(\mathbb{R})$ orbit remains pure Hodge type (p, q). A Hodge-Teichmüller tensor at m is a linear combination of pure Hodge-Teichmüller tensors.

Proposition 5.7. *Let* **H** *denote some fixed tensor construction on H^1.*

(i) *The fiber of any $\mathrm{GL}_2^+(\mathbb{R})$equivariant line subbundle of* **H** *is the span of a Hodge-Teichmüller tensor.*

(ii) *For any affine invariant submanifold $\mathcal{M}$ there is a finite union $\mathcal{B}$ of proper affine invariant submanifold of $\mathcal{M}$ such that the Hodge-Teichmüller tensors in* **H** *form a continuous equivariant subbundle of* **H** *over $\mathcal{M} \setminus \mathcal{B}$.*

(iii) *The algebraic hull acts via isometries on any fiber of the bundle of Hodge-Teichmüller tensors.*

Proof. Part (i) is a direct consequence of [**Fil15**, Thm. 1.2].

For (ii), the set of Hodge-Teichmüller tensors by definition is a $\mathrm{GL}_2^+(\mathbb{R})$ invariant subset of **H**. Because the Hodge decomposition and $\mathrm{GL}_2^+(\mathbb{R})$-action are continuous, it is also a closed subset of **H**.

Let d be the minimal dimension of the space of Hodge-Teichmüller tensors for any point $m \in \mathcal{M}$. Define $\mathcal{B}$ to be the subset of $\mathcal{M}$ where the dimension of the space of Hodge-Teichmüller tensors is strictly greater than d. Note that $\mathcal{B}$ is automatically a closed proper subset of $\mathcal{M}$.

Over $\mathcal{M} \setminus \mathcal{B}$, the set of Hodge-Teichmüller tensors is vector subbundle of $\mathbf{H}$.

For (iii) note that the bundle of Hodge-Teichmüller tensors of pure type (p, q) forms an equivariant subbundle. The full bundle of Hodge-Teichmüller tensors is the direct sum of these subbundles, so it suffices to prove the result for the bundle of pure (p, q) Hodge-Teichmüller tensors.

The symplectic form on H^1 gives rise to a locally constant bilinear form B on $\mathbf{H}$. One can obtain a positive definite inner product from B called the Hodge inner product by scaling B by different signs on the different pieces of the Hodge decomposition. (This is part of the definition of a variation of Hodge structure.)

The tensor defining the bilinear form B is flat and hence in particular equivariant, so by definition the algebraic hull preserves B. Hence the algebraic hull acts via isometries for the Hodge norm on a fiber of the (p, q) Hodge-Teichmüller tensors, since on this space the Hodge inner product and B are proportional.   □

Second proof of Theorem 5.2. The proof of the inclusion in part (i) is the same.

For part (ii), suppose that $\mathcal{M}' \subset \mathcal{M}$ is an affine manifold outside the locus $\mathcal{B}$ defined in Proposition 5.7. By definition of the algebraic hull and using Chevalley's Theorem 2.1, the identity component of the algebraic hull of $\mathcal{M}'$ is the stabilizer of an equivariant polynomial line subbundle ℓ of the bundle of Hodge-Teichmüller tensors in some tensor construction $\mathbf{H}$. As in the first proof, only finitely many tensor constructions need to be considered.

By part (iii) of Proposition 5.7, the algebraic hull of $\mathcal{M}$ acts by isometries on the bundle of Hodge-Teichmüller tensors. Therefore the kernel of the action on Hodge-Teichmüller tensors is cocompact in both connected components of the identity of the algebraic hulls of $\mathcal{M}$ and $\mathcal{M}'$, showing again that the hulls agree up to finite index and compact factors.   □

5.4. **The relative case.** We will need several preliminaries before dealing with the algebraic hull in H^1_{rel}.

Multiplicities. The first step in controlling the algebraic hull of H^1_{rel} is to reduce to the case where bundles in H^1 have no multiplicity. (This is not necessary for subbundles of H^1_{rel} whose image in H^1 have no multiplicity, such as $T(\mathcal{M})$. In particular, this is not required for our finiteness applications. The reader willing to ignore multiplicities may proceed directly to Theorem 5.12.)

Let $\mathcal{M}$ be an affine invariant manifold and suppose that $E \subset H^1$ is a bundle with multiplicities, i.e. $E = E_{irr} \otimes W$ where E_{irr} is an irreducible bundle and W is a vector space parametrizing the isotypical components. Let $\tilde{E} := p^{-1}(E)$ denote the associated bundle in relative cohomology. So we have a short exact sequence

$$(5.8) \qquad 0 \to \ker p \to \tilde{E} \to E_{irr} \otimes W \to 0.$$

We would like to reduce to the case where there is no multiplicity in the pure weight 1 Hodge structure on the right. For this, take a tensor with the dual $W^\vee$ to obtain

$$0 \to \ker p \otimes W^\vee \to \tilde{E} \otimes W^\vee \xrightarrow{p} E_{irr} \otimes W \otimes W^\vee \to 0.$$

Identifying $W \otimes W^\vee = \text{End}(W)$ we have the direct sum decomposition $W \otimes W^\vee = Id \oplus (\text{trace } 0)$ where Id denotes multiples of the identity, and (trace 0) denotes the trace 0 endomorphisms. The factor $E_{irr} \otimes Id = E_{irr}$ is present on the right-hand side above, so we take its preimage to obtain

$$(5.9) \qquad 0 \to \ker p \otimes W^\vee \to \widetilde{E_{irr}} \to E_{irr} \to 0$$

where $\widetilde{E_{irr}} = p^{-1}(E_{irr} \otimes Id)$. The advantage is that now E_{irr} is irreducible.

Proposition 5.10. *Suppose that for an affine manifold $\mathcal{M}' \subset \mathcal{M}$ the algebraic hull of $\widetilde{E_{irr}}$ from (5.9) over $\mathcal{M}'$ agrees up to finite index and compact factors with that over $\mathcal{M}$. Then the same holds for $\tilde{E}$ from (5.8).*

Proof. It suffices to reconstruct the sequence (5.8) from (5.9) by natural operations. For this, take a tensor with W in (5.9) to obtain

$$0 \to \ker p \otimes W^\vee \otimes W \to \widetilde{E_{irr}} \otimes W \to E_{irr} \otimes W \to 0.$$

Consider the commutative diagram below, which involves the natural quotient map $q : \widetilde{E_{irr}} \twoheadrightarrow E_{irr}$, the inclusion $i : \widetilde{E_{irr}} \to \tilde{E} \otimes W^\vee$, and for a bundle X the identity homomorphism $\mathbf{1}_X$ (to be distinguished from id viewed as a vector in $X \otimes X^\vee$).

$$
\begin{array}{ccccccccc}
0 & \to & \ker p \otimes W^\vee \otimes W & \to & \widetilde{E_{irr}} \otimes W & \xrightarrow{q \otimes \mathbf{1}_W} & E_{irr} \otimes W & \to & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle i \otimes \mathbf{1}_W} & & \downarrow{\scriptstyle \mathbf{1}_{E_{irr}} \otimes id \otimes \mathbf{1}_W} & & \\
0 & \to & \ker p \otimes W^\vee \otimes W & \to & \tilde{E} \otimes W^\vee \otimes W & \xrightarrow{p \otimes \mathbf{1}_{W^\vee \otimes W}} & E_{irr} \otimes W \otimes W^\vee \otimes W & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \ker p & \to & \tilde{E} & \to & E_{irr} \otimes W & \to & 0
\end{array}
$$

The map from the second row to the last is simply quotienting out by the trace 0 part of $W^\vee \otimes W$. The commutativity of the upper-right corner of the diagram follows from the construction on $\widetilde{E_{irr}}$ by tensoring with W the corresponding maps. The composition in the last column from top to bottom is an isomorphism (as can be checked by selecting a basis of W) and the middle column is a surjection with kernel $\ker p \otimes (\text{trace } 0)$.

We thus obtain that (5.8) can be obtain from (5.9) by first tensoring with W and then quotienting the left and middle terms by $\ker p \otimes (\text{trace } 0)$. $\qquad\square$

Remark 5.11. In the exact sequence (5.9) the term $\ker p \otimes W^\vee$ is still a trivial vector space, not a bundle. So in all arguments below, $\ker p$ can still be treated as a trivial bundle, even if we are in the case with multiplicities.

To show containment of algebraic hulls in H^1_{rel} when $\mathcal{M}' \subset \mathcal{M}$, we will need an analyticity result similar to the one for H^1. It is established in the appendix, in Proposition A.3.

Theorem 5.12. *Suppose $\mathcal{M}_i$ is a sequence of affine invariant submanifolds of $\mathcal{M}$ that equidistribute towards $\mathcal{M}$.*

Let G_i and G be the algebraic hulls of $\mathcal{M}_i$ and $\mathcal{M}$ respectively, for the $\mathrm{GL}_2^+(\mathbb{R})$-action on the relative cohomology bundle H^1_{rel}.

  (i) We have $G_i \subseteq G$ for all i.
  (ii) There exists $N \geq 0$ such that for all $i \geq N$, the groups G_i and G agree up to finite index and compact factors.

Proof. Part (i) follows from Proposition A.3 since any reduction of the algebraic hull of $\mathcal{M}$ is necessarily real-analytic, so it descends to $\mathcal{M}_i \subset \mathcal{M}$.

To prove part (ii), note that by Theorem 5.2, in absolute cohomology the algebraic hull stabilizes (up to compact factors and finite index) on $\mathcal{M}_i$ for $i \gg 0$.

Assume therefore that $E \subset H^1$ is an irreducible piece for the action of the reductive part of the algebraic hull; by Proposition 5.10 we can assume irreducibility of E. It suffices to show that in the short exact sequence of bundles

$$(5.13) \qquad\qquad 0 \to \ker(p) \to p^{-1}(E) \to E \to 0$$

the unipotent part of the algebraic hull stabilizes. By Lemma A.2(i)&(ii) the unipotent part is a subbundle $S_i \subset \mathrm{Hom}(E, \ker p)$, which is invariant under the algebraic hull of E.

Since E is irreducible under the action of the algebraic hull, $S_i = \mathrm{Hom}(E, W_i)$ for some subspace $W_i \subset \ker p$. Passing to a subsequence we obtain an accumulation point W_∞ of W_i. Define $S_\infty = \mathrm{Hom}(E, W_\infty) \subset \mathrm{Hom}(E, \ker p)$.

Note that the bundle of holomorphic 1-forms $H^{1,0} \subset H^1$ lifts to a subbundle denoted $\widetilde{H^{1,0}} \subset H^1_{rel}$, where lifting means that $p : \widetilde{H^{1,0}} \to H^{1,0}$ is an isomorphism.

Lemma A.2(iii) shows that the S_i have the following description. Moving the bundle of 1-forms $\widetilde{E^{1,0}} \oplus \widetilde{E^{0,1}} \subset H^1_{rel}$ by parallel transport along $\mathrm{GL}_2^+(\mathbb{R})$-orbits in $\mathcal{M}_i$, it can be taken to its value at the new point by transformations in S_i.

It follows that S_∞ has the same property in $\mathcal{M}$, thus S_∞ contains the unipotent part of the algebraic hull of $\mathcal{M}$; they must agree since the S_i are contained in the algebraic hull of $\mathcal{M}$. $\qquad\square$

Remark 5.14. The above proof gives a $\mathrm{GL}_2^+(\mathbb{R})$-invariant closed locus $\mathcal{B}_{rel} \subset \mathcal{M}$ such that as soon as an affine submanifold $\mathcal{M}' \subset \mathcal{M}$ is disjoint from $\mathcal{B}_{rel}$, the unipotent parts of the algebraic hulls agree.

Indeed, above a fixed point $x \in \mathcal{M}$ we have a closed subset of the Grassmannian consisting of subspaces $S_x \subset \mathrm{Hom}(E_x, \ker p)$ for which the defining property in Lemma A.2(iii) holds in a neighborhood of x inside its $\mathrm{GL}_2^+(\mathbb{R})$-orbit. The subset of the total Grassmannian bundle is closed and $\mathrm{GL}_2^+(\mathbb{R})$-invariant, and $\mathcal{M}$ is stratified by the minimal possible dimension of an S. Away from a proper closed subset, this subspace is unique over the entire $\mathcal{M}$.

6. Finiteness and abundance results

In this section we prove Theorems 1.5 and 1.7. Recall that $A_V(\mathcal{M})$ and $G_V(\mathcal{M})$ denote the algebraic hull, and the Zariski closure of monodromy, on a flat bundle V in relative or absolute cohomology over $\mathcal{M}$.

6.1. **Finiteness.** If $\mathcal{M}' \subset \mathcal{M}$, the restriction any flat bundle on $\mathcal{M}$ gives a flat bundle on $\mathcal{M}'$.

Lemma 6.1. *Suppose $\mathcal{M}' \subsetneq \mathcal{M}$ both have the same algebraic hull for $T\mathcal{M}$ and its Galois conjugates. Then either*

- *$\mathcal{M}'$ is rank 1 degree 1, or*
- *$\mathcal{M}'$ is rank 1 degree 2 and $\mathcal{M}$ is rank 2 degree 1 and $\ker(p) \cap T\mathcal{M} = \ker(p) \cap T\mathcal{M}'$.*

Proof. Theorem 1.2 implies that

(O1) the only proper subspaces of $p(T\mathcal{M})$ invariant under $A_{p(T\mathcal{M})}(\mathcal{M})$ are the tautological plane and its complement,

(O2) the only proper subspace of $T\mathcal{M}$ invariant under $A_{T\mathcal{M}}(\mathcal{M})$ and not contained in p^{-1} of the tautological plane is p^{-1} of the complement of the tautological plane in $p(T\mathcal{M})$,

(O3) there are no proper subspaces of a nontrivial Galois conjugate $p(T\mathcal{M})_\iota$ invariant under $A_{p(T\mathcal{M})_\iota}(\mathcal{M})$,

(O4) there is no proper subspace of $(T\mathcal{M})_\iota$ that is not contained in $\ker(p)$ and is invariant under $A_{(T\mathcal{M})_\iota}(\mathcal{M})$.

Since the algebraic hulls of $\mathcal{M}$ and $\mathcal{M}'$ are the same, the algebraic hull of $\mathcal{M}$ must preserve flat subbundles over $\mathcal{M}'$. (Technically in such statements we should refer to fibers of bundles at points of $\mathcal{M}'$, but throughout this proof we omit this specification.) In particular, $p(T\mathcal{M}')$ must be stabilized by $A_{p(T\mathcal{M})}(\mathcal{M})$. By (O1), $p(T\mathcal{M}')$ is either the tautological plane or $p(T\mathcal{M})$.

Suppose $p(T\mathcal{M}')$ is not the tautological plane. So $p(T\mathcal{M}') = p(T\mathcal{M})$, and $\mathcal{M}'$ has rank greater than 1. Since $T\mathcal{M}'$ must be stabilized by $A_{T\mathcal{M}}(\mathcal{M})$, (O2) implies $T\mathcal{M}' = T\mathcal{M}$. This contradicts $\mathcal{M}' \neq \mathcal{M}$.

Hence $p(T\mathcal{M}')$ is the tautological plane. So $\mathcal{M}'$ is rank 1. Note that any Galois conjugate of $p(T\mathcal{M}')$ must be contained in some Galois conjugate of $p(T\mathcal{M})$. (This is a triviality about subspaces of vector spaces.) Any Galois conjugate of $p(T\mathcal{M}')$ has dimension 2 and is stabilized by the algebraic hull of a Galois conjugate of $p(T\mathcal{M})$.

If $\mathcal{M}$ has rank at least 3, (O1) and (O3) imply that $p(T\mathcal{M})$ and its Galois conjugates contain only one subspace invariant under algebraic hull of dimension 2, namely the tautological plane. Hence $\mathcal{M}'$ has degree 1.

If $\mathcal{M}$ has rank 2, (O1) and (O3) imply that $p(T\mathcal{M})$ and its Galois conjugates contain only two subspaces invariant under algebraic hull of dimension 2, namely the tautological plane and its complement in $p(T\mathcal{M})$. Hence $\mathcal{M}'$ has degree 1 or 2. If $\mathcal{M}'$ has degree 2, then the Galois conjugate of $p(T\mathcal{M}')$ must be the complement of the tautological plane in $p(T\mathcal{M})$. Hence $p(T\mathcal{M})$ is the sum of $p(T\mathcal{M}')$ and its Galois conjugate, and so $\mathcal{M}$ has degree 1. The Galois conjugate of $T\mathcal{M}'$ must be stabilized by $A_{T(\mathcal{M})}(\mathcal{M})$, so (O2) implies that it is p^{-1} of the complement of the tautological plane. In particular, $\ker(p) \cap T\mathcal{M} = \ker(p) \cap T\mathcal{M}$.

If $\mathcal{M}$ has rank 1 and is not degree 1, (O4) implies that a Galois conjugate of $T(\mathcal{M}')$ must be equal to a Galois conjugate of $T(\mathcal{M})$. This implies $\mathcal{M} = \mathcal{M}'$, a contradiction. □

Proof of Theorem 1.5. Let $\mathcal{M}_1, \mathcal{M}_2, \ldots$ be an infinite sequence of affine invariant submanifolds in some stratum that have fixed rank, degree, and dimension. The closure of their union is a finite union of affine invariant submanifolds, so passing to a subsequence we may assume that the $\mathcal{M}_i$ are contained in and equidistribute to a single affine invariant submanifold $\mathcal{M}$. By Theorem 1.3, removing finitely many of the $\mathcal{M}_i$ if necessary, we may assume that the algebraic hull of $T(\mathcal{M})$ and its Galois conjugates on $\mathcal{M}$ are equal to the algebraic hull of their restrictions to $\mathcal{M}_i$.

Lemma 6.1 now gives the result. $\qquad\square$

6.2. **Abundance.** Let $\mathcal{O}$ be an order in a real quadratic field, for example $\mathcal{O} = \mathbb{Z}[\sqrt{D}]$, where D is a positive integer. Let $\mathcal{M}$ be an affine invariant submanifold of rank 2 and degree 1. Say that (X, ω) is an eigenform for real multiplication by $\mathcal{O}$ if there is a proper action of $\mathcal{O}$ on $p(T\mathcal{M})_{(X,\omega)}$ by linear transformations that are self-adjoint with respect to the symplectic form, preserve the Hodge structure and integer lattice, and act on $p(\omega)$ via scalars.

Lemma 6.2. *The locus of eigenforms in $\mathcal{M}$ for real multiplication by $\mathcal{O}$ is a finite (possibly empty) union of codimension 2 degree 2 rank 1 affine invariant submanifolds.*

The proof is omitted and is almost identical to that of [**Wri15b**, Proposition 2.5], [**McM03**, Theorem 7.2], and [**McM06**, Theorem 3.2], see also [**MMW16**, Section 7].

The following proof arose from a conversation with Ronen Mukamel.

Proof of Theorem 1.7. Let U be an open subset of $\mathcal{M}$, and pick $m \in U$. Using period coordinates, we identify U with an open set in $(T\mathcal{M})_m$.

Pick any order $\mathcal{O}$ in any real quadratic field K, and consider any action of $\mathcal{O}$ of $p(T\mathcal{M})_m$ by self-adjoint transformations that preserve the integer lattice. Since the transformations in $\mathcal{O}$ are self-adjoint, they are diagonalizable, and since $\mathcal{O}$ is abelian they preserve each others eigenspaces.

For the next argument, see also [**EMMW**, Prop. 7.1]. Let v be an eigenvector for the action of $\mathcal{O}$, with $v \in p(T\mathcal{M})_m \otimes \mathbb{C}$ and $S \subset p(T\mathcal{M})_m$ the span of $\operatorname{Re} v, \operatorname{Im} v$. By construction the symplectic form on S is non-degenerate. Since $\mathcal{M}$ is degree 1, we may define $\operatorname{Sp}(p(T\mathcal{M})_m, \mathbb{Q})$ to be the group of symplectic transformations of $p(T\mathcal{M})_m$ preserving the set of rational points. Note $\operatorname{Sp}(p(T\mathcal{M})_m, \mathbb{Q})$ is isomorphic to $\operatorname{Sp}(4, \mathbb{Q})$, which is dense in $\operatorname{Sp}(4, \mathbb{R})$. Note that $\operatorname{Sp}(4, \mathbb{R})$ acts transitively on the 2-dimensional subspaces $S' \subset \mathbb{R}^4$ on which the symplectic form is non-degenerate, and the stabilizer of a fixed S'

takes any non-degenerate basis of S' to any other basis, up to scaling. By density of $\mathbb{Q}$ in $\mathbb{R}$, there exists $\gamma \in \mathrm{Sp}(p(T\mathcal{M})_m, \mathbb{Q})$ such that $v' := \gamma v \in p(U)$, after perhaps rescaling v by a rational scalar.

We can define a new action of $\mathcal{O}$ on $p(T\mathcal{M})_m$ by conjugating the original action by γ. The resulting action is via self-adjoint transformations that act via a scalar on v'. There exists N so that γ is in the set $Sp(4, \frac{1}{N}\mathbb{Z})$. The restriction of the resulting action to $\mathcal{O}' = N\mathcal{O}$ preserves the integer lattice.

Hence if we pick $(X, \omega) \in U$ with $p(\omega) = v'$, we get that (X, ω) is an eigenform for real multiplication by $\mathcal{O}'$. By Lemma 6.2, this gives the result. $\qquad\square$

Appendix A. Analyticity and polynomiality of measurable bundles in relative cohomology

Throughout this appendix, we work over a fixed affine invariant submanifold $\mathcal{M}$. Over $\mathcal{M}$ we have the exact sequence of bundles

$$(A.1) \qquad\qquad 0 \to \ker p \to H^1_{rel} \xrightarrow{p} H^1 \to 0$$

and we assume that $\ker p$ is trivialized (e.g. by passing to a finite cover). The bundles above are real, but their complexifications contain holomorphic subbundles $H^{1,0}$ and $\widetilde{H^{1,0}}$ of holomorphic 1-forms, inducing variations of Hodge structures.

A.1. **Analyticity.** To handle the relative cohomology bundle, the first step is to show that any measurable $GL_2^+(\mathbb{R})$-equivariant subbundle in H^1_{rel} must in fact be real-analytic. This extends [**Fil15**, Thm. 7.7] and was used in the proof of Theorem 5.12.

Let $E \subset H^1$ be an irreducible bundle over a fixed affine manifold $\mathcal{M}$, and let $\widetilde{E} := p^{-1}(E)$. Recall that we have the bundles of holomorphic 1-forms $E^{1,0} \subset E_{\mathbb{C}}$ and $\widetilde{E^{1,0}} \subset \widetilde{E_{\mathbb{C}}}$, and the forgetful map p is an isomorphism from $\widetilde{E^{1,0}}$ to $E^{1,0}$.

Recall that $\xi \in \mathrm{Hom}(E, \ker p)$ defines a unipotent automorphism $v \mapsto v + \xi(p(v))$ of $\widetilde{E}$, and all automorphisms of $\widetilde{E}$ which act as the identity on $\ker p$ and E are of this form. Hence the unipotent part of the algebraic hull of $\widetilde{E}$, denoted S, is naturally contained in $\mathrm{Hom}(E, \ker p)$. Moreover, since E carries a polarized weight one variation of Hodge structures, so does $\mathrm{Hom}(E, \ker p) = E^{\vee} \otimes \ker p$, where $\ker p$ is equipped with the trivial Hodge structure.

Lemma A.2. *With notation as above, we have:*

(i) The bundle $S \subset \mathrm{Hom}(E, \ker p)$ respects the Hodge structure, i.e.
has a Hodge decomposition compatible with that of $\mathrm{Hom}(E, \ker p)$.
Moreover S is invariant by the algebraic hull of E.

(ii) The bundle S can be alternatively described as the smallest bun-
dle with the following property. Consider the bundle $\widetilde{E_x^{1,0}}$ and
its complex conjugate denoted $\widetilde{E_x^{0,1}}$, where $x \in \mathcal{M}$. Then for
any $g \in \mathrm{GL}_2^+(\mathbb{R})$ small, using parallel transport for flat identifi-
cations, the bundle $\widetilde{E_x^{1,0}} \oplus \widetilde{E_x^{0,1}}$ can be taken to $\widetilde{E_{gx}^{1,0}} \oplus \widetilde{E_{gx}^{0,1}}$ using
a transformation in S.

Proof. For part (i), note that the algebraic hull is $\mathrm{GL}_2^+(\mathbb{R})$-invariant
(viewed as a group above each point) so in particular S is an invari-
ant bundle. Therefore S must also respect the Hodge structure, by
the semisimplicity results established in [**Fil15**], which apply to any
$\mathrm{GL}_2^+(\mathbb{R})$-invariant subbundle of a weight 1 variation of Hodge struc-
ture over $\mathcal{M}$.

To establish part (ii) let S' denote the bundle described in it. First
we show $S \subset S'$. Indeed, at each point we may pick a basis for $\tilde{E}$
which consists of a fixed basis for $\ker(p) \cap \tilde{E}$ together with any basis of
$\widetilde{E_x^{1,0}} \oplus \widetilde{E_x^{0,1}}$. In this basis we may consider the "the unipotent part" of
an element in $\mathrm{End}\,\widetilde{E}$ preserving $\ker p$, and by definition the unipotent
part of the cocycle is contained in S'.

Suppose therefore that the algebraic hull could have been reduced
to have unipotent part $S \subset S'$. The reduction to algebraic hull with
unipotent part S means the following. At every point $x \in \mathcal{M}$, we can
pick a subspace $\widetilde{E}'_x \subset \widetilde{E}$, projecting isomorphically to E. Moreover,
$\widetilde{E}'_x$ is well-defined up to the action of S on $\widetilde{E}$, and these choices and
ambiguities are $\mathrm{GL}_2^+(\mathbb{R})$-invariant.

So locally on a $\mathrm{GL}_2^+(\mathbb{R})$-orbit we have a map $\sigma_{hull} : E \to \widetilde{E}$ giving a
section of the projection (i.e. a map which, when composed with the
projection, gives the identity). The section σ_{hull} is well-defined up to
the action of S, and can be viewed as a flat section of $\mathrm{Hom}(E, \widetilde{E})/S$.

We also have the canonical section $\sigma_{hol} : E \to \widetilde{E}$, which is defined
as the inverse of p restricted to $\widetilde{E^{1,0}}$ plus its complex conjugate. The
difference $\sigma_{hull} - \sigma_{hol}$ is an element of $\mathrm{Hom}(E, \ker p)$, well-defined up
to the action of S, since σ_{hull} is. Now the image of $\sigma_{hull} - \sigma_{hol}$ in
$Q := \mathrm{Hom}(E, \ker p)/S$ is well-defined, and moreover Q carries a weight
1 variation of Hodge structure, since S is compatible with the Hodge
structure.

To finish, note that [**Fil16**, Thm. 4.2] applies here (although stated only for certain parts of H^1 and H^1_{rel}, the proof works in the present context). It implies that $\sigma_{hull} - \sigma_{hol}$ vanishes in Q, in particular the subspaces we started with $\widetilde{E}'_x$ can be taken to $\sigma_{hol}(E)$ by elements of S; therefore we can assume that they are, in fact, equal. By the definition of S', this implies that $S' = S$. $\qquad\square$

Using the above result as a preliminary step, we can now establish the analyticity of the algebraic hull in relative cohomology. Note that although the algebraic hull of H^1 and H^1_{rel} differ in a unipotent part only, the key difference is in the *lift* of the semisimple part from H^1 to H^1_{rel}.

Proposition A.3. *Let G be the (measurable) algebraic hull of the Kontsevich–Zorich cocycle over an affine manifold $\mathcal{M}$ for the $\mathrm{GL}_2^+(\mathbb{R})$-action on the relative cohomology bundle H^1_{rel}. Then the reduction to G can be made real-analytic.*

Proof. Decompose the algebraic hull G according to the short exact sequence A.1 into a unipotent and reductive part:

$$(A.4) \qquad 0 \to G^u \to G \to G^{ss} \to 0.$$

According to [**Fil15**], and as discussed in the proof of Theorem 5.2 the tensors defining G^{ss} can be picked real-analytically. Note that G^u is a subgroup of $\mathrm{Hom}(H^1, \ker(p))$ and hence decomposes according to the action of G^{ss} on H^1.

Let $E \subset H^1$ be an irreducible piece for the action of G^{ss} (by Proposition 5.10 we can reduce to this case). It suffices to check that G^u is defined by real-analytic bundles on the piece $\mathrm{Hom}(E, \ker(p))$. Indeed, G is of the form $\prod_E G^{ss}_E \ltimes G^u_E$ where G^{ss}_E is the algebraic hull of an irreducible piece $E \subset H^1$ and $G^u_E \subset \mathrm{Hom}(E, \ker(p))$.

The algebraic hull G_E on $p^{-1}(E)$ sits in the exact sequence

$$(A.5) \qquad 0 \to G^u_E \to G_E \to G^{ss}_E \to 0.$$

The unipotent part $S_E := G^u_E \subset \mathrm{Hom}(E, \ker p)$ respects the Hodge structure and is invariant under G^{ss}_E, by Lemma A.2(ii). Finally, part (iii) of the same Lemma shows how to real-analytically reduce the algebraic hull to have unipotent part contained in S_E. Indeed, the bundles $\widetilde{E^{1,0}}$ and its complex-conjugate vary real-analytically, and give a real-analytic splitting $G_E \simeq G^{ss}_E \ltimes S_E$, where G^{ss}_E is viewed as acting via the splitting of $p^{-1}(E) \to E$ coming from $\widetilde{E^{1,0}}$. $\qquad\square$

A.2. **Polynomiality.** We can now establish polynomiality of the algebraic hull, where polynomiality is understood in the following sense. Consider $\mathrm{GL}_2^+(\mathbb{R})$-invariant bundles $\tilde{E} \subset H_{rel}^1$, or perhaps some tensor powers thereof. Note that because H^1 is a symplectic bundle, and $\ker p$ is trivial, the bundle H_{rel}^1 is equipped with a natural volume form. Any bundle $\widetilde{E}$ obtained by tensor constructions from subbundles in H_{rel}^1 will be also filtered by bundles with volume forms, and so we can assume $\widetilde{E}$ has a volume form. Therefore, its top exterior power $\Lambda^{(\dim \tilde{E})}\tilde{E}$ carries a canonical trivializing vector, which is $\mathrm{GL}_2^+(\mathbb{R})$-equivariant and denoted $v_{\tilde{E}}$. The coordinates of $v_{\tilde{E}}$ in $\Lambda^{\dim \tilde{E}}(H_{rel}^1)$, in local flat trivializations of the bundles, give functions on $\mathcal{M}$ (these are just the Plücker coordinates on a Grassmannian). We will show that these functions are polynomial, when viewed in period coordinates on $\mathcal{M}$. This is meant in the sense of (3.4), i.e. as polynomials divided by the area function to some power.

Proposition A.6. *Suppose $\tilde{E} \subset H_{rel}^1$ is a measurable $\mathrm{GL}_2^+(\mathbb{R})$-invariant subbundle of the relative cohomology bundle, or some tensor power of H_{rel}^1. Then $\tilde{E}$ is in fact polynomial.*

Proof. By Prop. A.3, the bundle $\tilde{E}$ is at least real-analytic. Next, recall that polynomiality in H_{rel}^1 or its tensor powers is understood in terms of Plücker coordinates of the bundle. In other words, we have an invariant section of some tensor power of H_{rel}^1 given by the top exterior power $\Lambda^{\dim \tilde{E}}\tilde{E}$ (normalized by a fixed volume form).

But now, the same proof as in Prop. 7.5 and Thm. 7.7 of [**Fil15**] applies to give that the section must necessarily be polynomial in period coordinates.

Let us recall a sketch of proof. Let ϕ be some real-analytic, $\mathrm{GL}_2^+\,\mathbb{R}$-equivariant section of some tensor power of the Kontsevich–Zorich cocycle (on H_{rel}^1). The first part, following the proof of Prop. 7.5 in loc.cit., is to show that ϕ is polynomial on each stable, or each unstable, leaf. The joint polynomiality is then established as in Prop. 7.6 and Thm. 7.7, as those are simply statements about polynomials.

To establish polynomiality on a stable leaf, in a local chart around a point $x \in \mathcal{M}$ define $\tilde{\phi}(x, v) := \phi(x + v)$ where v is a (sufficiently small) tangent vector in the unstable direction. We then have a Taylor expansion

$$\tilde{\phi}(x, v) = \sum_\alpha c_\alpha(x)v^\alpha$$

where α denotes a multi-index and $c_\alpha(x)$ are vectors in the same bundle as ϕ. From the equivariance under the Teichmüller geodesic flow, for any large time t such that $g_t(x)$ returns to the same chart, we have:

$$\sum_\alpha c_\alpha(x)v^\alpha = \sum_\alpha g_{-t}c_\alpha(g_t x)(dg_t v)^\alpha$$

where to obtain the right-hand side, we have pulled back by g_{-t} the expansion near $g_t x$. Note that dg_t denotes the cocycle on the tangent space, which by Forni's spectral gap result for the Lyapunov spectrum has a definite contraction, linear in time. Thus if c_α must vanish for α sufficiently large depending on the spectral gap and the Lyapunov exponents of the bundle in which ϕ lives. $\square$

Appendix B. Algebraic Hulls and Bundles

B.1. **Algebraic Hulls following Zimmer. Setup.** Recall that Zimmer [**Zim84**, Sec. 4.2] works in the following setup. We have a group action on a space $A \curvearrowright X$.

Definition B.1. A cocycle for the action of A on X is a map $\alpha : A \times X \to \mathrm{GL}_n(\mathbb{R})$ satisfying the cocycle relation

$$\alpha(a_1, a_2 x) \cdot \alpha(a_2, x) = \alpha(a_1 a_2, x).$$

This induces an action of A on the trivial vector bundle $X \times \mathbb{R}^n$ by linear transformations on the fiber.

Moreover, there is a notion of equivalence (or cohomology) of cocycles. Namely, two cocycles α and β as above are equivalent if there exists $C : X \to \mathrm{GL}_n(\mathbb{R})$ such that $C(ax) \circ \alpha(a, x) = \beta(a, x) \circ C(x)$.

To have a more intrinsic view, one can work with general bundles $V \to X$, with a lift of the action of A to V by linear transformations on the fiber. Then a description as in Definition B.1 is obtained by trivializing the bundle so that $V \simeq X \times \mathbb{R}^n$. Different trivializations give cohomologous cocycles.

Recall also Zimmer's definition of the algebraic hull.

Definition B.2. The Zimmer algebraic hull of a cocycle α is the smallest algebraic group $H \subset \mathrm{GL}_n(\mathbb{R})$ such that α is cohomologous to a cocycle β taking values in H.

Note that with this definition, the algebraic hull is well-defined only up to conjugacy, and is not a priori clear why it is even well-defined. Namely, one has to check that if the cocycle can be conjugated to take values in H_1 and H_2, then there is a conjugation with values in $H_1 \cap g H_2 g^{-1}$, for some $g \in \mathrm{GL}_n(\mathbb{R})$.

Proposition B.3. *The algebraic hull as in Definition 2.2 of the main text (call it the tensor algebraic hull), and Zimmer's Definition B.2 are equivalent.*

Recall that by Chevalley's Theorem 2.1, to define an algebraic subgroup of $\mathrm{GL}_n(\mathbb{R})$ is equivalent to specifying a line in some tensor construction on $\mathbb{R}^n$, with the group being the stabilizer of the line.

Proof. First, we check that the tensor algebraic hull contains the Zimmer algebraic hull. Suppose that we have a cocycle $V \to X$ and its algebraic hull is defined by some line subbundle $l \subset \mathbf{V}$ in some tensor construction on V. Fix a model line $l_m \subset \mathbf{T}(\mathbb{R}^n)$ in a corresponding tensor construction on $\mathbb{R}^n$. Then we pick a measurable trivialization of V such that under the identification of each fiber $V_x \to \mathbb{R}^n$, the lines $l_x \subset \mathbf{V}$ are identified with $l_m \subset \mathbf{T}(\mathbb{R}^n)$. Thus we have reduced the cocycle in the sense of Zimmer to have algebraic hull contained in the tensor algebraic hull.

Conversely, suppose that under some trivialization $V \simeq X \times \mathbb{R}^n$, all the cocycle linear transformations are in some group $H \subset \mathrm{GL}_n(\mathbb{R})$. Then H is the stabilizer of some line $l_m \subset \mathbf{T}(\mathbb{R}^n)$ in some tensor construction. The line l_m pulls back to give a line subbundle $l \subset \mathbf{V}$ in the corresponding tensor construction on V. By definition, the cocycle preserves the line bundle l, so the tensor algebraic hull is contained in Zimmer's. $\square$

B.2. **Irreducible and absolutely irreducible bundles.** To end, we clarify a point regarding absolute irreducibility of bundles. It is not used in the main text, but shows that the analyticity results apply to both irreducible and strongly irreducible bundles.

Recall that a cocycle on a vector bundle V is irreducible if it has no invariant subbundle. Similarly, a cocycle is strongly irreducible if it does not admit a finite invariant collection of subbundles $W_i \subset V$. For instance, monodromy could permute a finite number of subspaces. Thus a cocycle can be irreducible without being strongly irreducible.

All the results above and in [**Fil15**] about analyticity and polynomiality of bundles refer to irreducible bundles. They also apply to non-irreducible bundles, by simply decomposing them into irreducible pieces. Below, we establish the same (local) property for a collection of invariant subspaces.

Proposition B.4. *Suppose that V is some irreducible piece of a tensor power of H^1 over $\mathcal{M}$, and suppose that V is not strongly irreducible for the $\mathrm{GL}_2^+(\mathbb{R})$-action. By assumption, there exists a finite collection of bundles $W_i \subset V$ which are permuted by the $\mathrm{GL}_2^+(\mathbb{R})$-action.*

Then locally (after a renumbering), each of the bundles varies polynomially in period coordinates.

Note that the numbering of the bundles can be pathological – take any (measurable!) function from $\mathcal{M}$ to permutations of the indices and relabel the bundles. Part of the statement is that there is locally a relabeling for which the bundles vary polynomially.

Proof. Consider the projectivizations $\mathbb{P}(W_i) \subset \mathbb{P}(V)$. Their union $X := \cup \mathbb{P}(W_i)$ is in each fiber a collection of linear spaces, i.e. an algebraic variety. Let $I^\bullet$ be (fiberwise) ideal of homogeneous polynomials which vanish on X.

Note that each homogeneous component I^k of $I^\bullet$ is a subbundle of some tensor power of the dual $V^\vee$. Therefore, each homogeneous component varies polynomially in period coordinates.

Next, any sufficiently high homogeneous component of $I^\bullet$ determines the variety X. Therefore each of the bundles W_i must (locally) vary polynomially in period coordinates, after an appropriate relabeling. $\square$

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